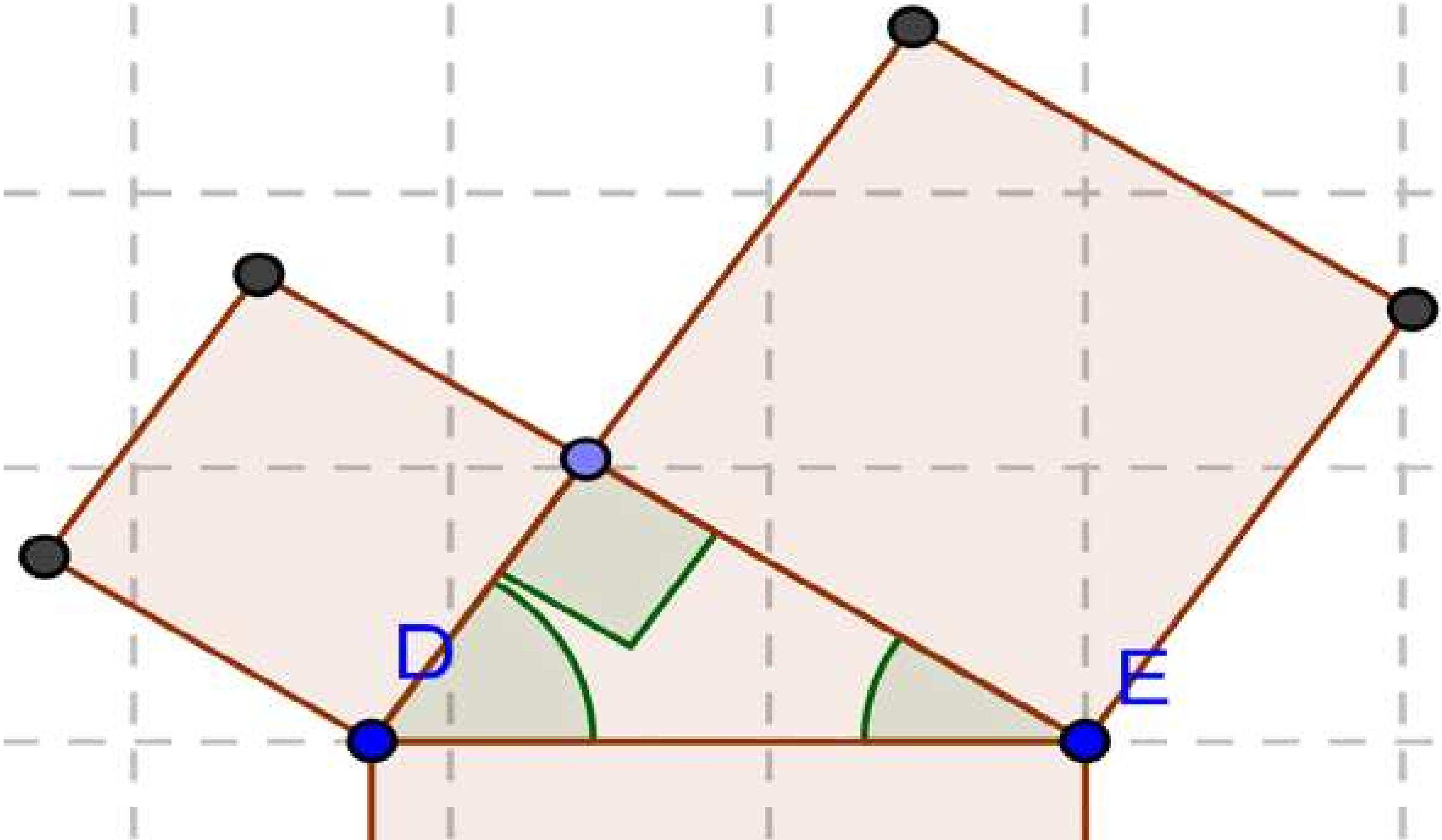

Handbook of

Dimensional Analysis & Physical Quantities

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Chapter- 1

Physical Quantity

A **physical quantity** is a physical property that can be quantified by measurement.

Formally, the *International Vocabulary of Metrology'*, 3rd edition (VIM) defines quantity as:

property of a phenomenon, body, or substance, where the property has a magnitude that can be expressed as a number and a reference

Hence the value of a physical quantity Q is expressed as the product of a numerical value $\{Q\}$ and a unit of measurement $[Q]$.

$$Q = \{Q\} \times [Q]$$

Quantity calculus describes how to do math with quantities.

Examples

If the temperature T of a body is quantified (measured) as 25 degrees Celsius this is written

$$T = 25 \times {}^\circ\mathrm{C} = 25\ {}^\circ\mathrm{C}$$

where T is the symbol of the physical quantity "tem

If a person weighs 120 pounds, then "120" is the numerical value and "pound" is the unit. This physical quantity mass would be written as "120 lbs", or

$$m = 120 \text{ lbs}$$

An example employing SI units and scientific notation for the number, might be a measurement of power written as

$$P = 42.3 \times 10^3 \text{ W},$$

Here, *P* represents the physical quantity of power, 42.3 x 10^3 is the numerical value {P}, and W is the symbol for the unit of power [P], the watt

Symbols for physical quantities

Usually, the symbols for physical quantities are chosen to be a single letter of the Latin or Greek alphabet written in italic type. Often, the symbols are modified by subscripts and superscripts, in order to specify what they pertain to — for instance E_k is usually used to denote kinetic energy and c_p heat capacity at constant pressure. (Note the difference in the style of the subscripts: “k” is the abbreviation of the word “kinetic”, whereas “*p*” is the symbol of the physical quantity “pressure” rather than the abbreviation of the word “pressure”.)

Symbols for quantities should be chosen according to the international recommendations from ISO 31, the IUPAP red book and the IUPAC green book. For example, the recommended symbol for the physical quantity 'mass' is *m*, and the recommended symbol for the quantity 'charge' is *Q*.

Symbols for physical quantities that are vectors are bold italic type. If, e.g., *u* is the speed of a particle, then the straightforward notation for its velocity is ***u***.

Note that concrete numbers, even those denoted by letters, are always roman (upright) type, e.g.: 1, 2, e (for the base of natural logarithm), i (for the imaginary unit) or π (for 3.14...). Symbols of concrete functions such as sin *α* must be roman type too.

Units of physical quantities

Most physical quantities *Q* include a unit [*Q*] (where [*Q*] means "unit of *Q*"). Neither the name of a physical quantity, nor the symbol used to denote it, implies a particular choice of unit. For example, a quantity of mass might be represented by the symbol *m*, and could be expressed in the units kilograms (kg), pounds (lb), or Daltons (Da). SI units are usually preferred today.

Base quantities, derived quantities and dimensions

The notion of *physical dimension* of a physical quantity was introduced by Fourier in 1822. By convention, physical quantities are organized in a dimensional system built upon base quantities, each of which is regarded as having its own dimension. The seven base quantities of the International System of Quantities (ISQ) and their corresponding SI units are listed in the following table. Other conventions may have a different number of fundamental units (e.g. the CGS and MKS systems of units).

International System of Units base quantities

Name	Symbol for quantity	Symbol for dimension	SI base unit	Symbol for unit
Length	l, x, r, etc.	L	meter	m
Time	t	T	second	s
Mass	m	M	kilogram	kg
Electric current	I, i	I	ampere	A
Thermodynamic temperature	T	θ	kelvin	K
Amount of substance	n	N	mole	mol
Luminous intensity	I_v	J	candela	cd

All other quantities are derived quantities since their dimensions are derived from those of base quantities by multiplication and division. For example, the physical quantity velocity is derived from base quantities length and time and has dimension L/T. Some derived physical quantities have dimension 1 and are said to be dimensionless quantities.

Extensive and intensive quantities

A quantity is called:

- ***extensive*** when its magnitude is additive for subsystems (volume, mass, etc.)
- ***intensive*** when the magnitude is independent of the extent of the system (temperature, pressure, etc.)

Some physical quantities are prefixed in order to further qualify their meaning:

- ***specific*** is added to refer to a quantity which is expressed per unit mass (such as specific heat capacity)
- ***molar*** is added to refer to a quantity which is expressed per unit amount of substance (such as molar volume)

There are also physical quantities that can be classified as neither extensive nor intensive, for example angular momentum, area, force, length, and time.

Physical quantities as *coordinates* over spaces of physical *qualities*

The meaning of the term physical *quantity* is generally well understood (everyone understands what is meant by *the frequency of a periodic phenomenon*, or *the resistance of an electric wire*). It is clear that behind a set of quantities like temperature − inverse temperature − logarithmic temperature, there is a qualitative notion: the *cold−hot* quality. Over this one-dimensional quality space, we may choose different *coordinates*: the temperature, the inverse temperature, etc. Other quality spaces are multidimensional. For instance, to represent the properties of an ideal elastic medium we need 21 coefficients, that can be the 21 components of the elastic stiffness tensor c_{ijkl}, or the 21 components of the elastic compliance tensor (inverse of the stiffness tensor), or the proper elements (six eigenvalues and 15 angles) of any of the two tensors, etc. Again, we are selecting coordinates over a 21-dimensional quality space. On this space, each point represents a particular elastic medium.

It is always possible to define the distance between two points of any quality space, and this distance is —inside a given theoretical context— uniquely defined. For instance, two periodic phenomena can be characterized by their periods, T_1 and T_2, or by their frequencies, ν_1 and ν_2. The only definition of distance that respects some clearly defined invariances is $D = |\log(T_2 / T_1)| = |\log(\nu_2 / \nu_1)|$.

These notions have implications in physics. As soon as we accept that behind the usual physical quantities there are quality spaces, that usual quantities are only special *coordinates* over these quality spaces, and that there is a metric in each space, the following question arises: Can we do physics intrinsically, i.e., can we develop physics using directly the notion of physical quality, and of metric, and without using particular coordinates (i.e., without any particular choice of physical quantities)? In fact, physics can (and must?) be developed independently of any particular choice of coordinates over the quality spaces, i.e., independently of any particular choice of physical quantities to represent the measurable physical qualities.

Chapter- 2

Units of Measurement

The former Weights and Measures office in Seven Sisters, London

A **unit of measurement** is a definite magnitude of a physical quantity, defined and adopted by convention and/or by law, that is used as a standard for measurement of the same physical quantity. Any other value of the physical quantity can be expressed as a simple multiple of the unit of measurement.

For example, length is a physical quantity. The metre is a unit of length that represents a definite predetermined length. When we say 10 metres (or 10 m), we actually mean 10 times the definite predetermined length called "metre".

The definition, agreement, and practical use of units of measurement have played a crucial role in human endeavour from early ages up to this day. Disparate systems of units used to be very common. Now there is a global standard, the International System of Units (SI), the modern form of the metric system.

In trade, **weights and measures** is often a subject of governmental regulation, to ensure fairness and transparency. The Bureau international des poids et mesures (BIPM) is tasked with ensuring worldwide uniformity of measurements and their traceability to the International System of Units (SI). Metrology is the science for developing nationally and internationally accepted units of weights and measures.

In physics and metrology, units are standards for measurement of physical quantities that need clear definitions to be useful. Reproducibility of experimental results is central to the scientific method. A standard system of units facilitates this. Scientific systems of units are a refinement of the concept of weights and measures developed long ago for commercial purposes.

Science, medicine, and engineering often use larger and smaller units of measurement than those used in everyday life and indicate them more precisely. The judicious selection of the units of measurement can aid researchers in problem solving (see, for example, dimensional analysis).

In the social sciences, there are no standard units of measurement and the theory and practice of measurement is studied in psychometrics and the theory of conjoint measurement.

History

A unit of measurement is a standardised quantity of a physical property, used as a factor to express occurring quantities of that property. Units of measurement were among the earliest tools invented by humans. Primitive societies needed rudimentary measures for many tasks: constructing dwellings of an appropriate size and shape, fashioning clothing, or bartering food or raw materials.

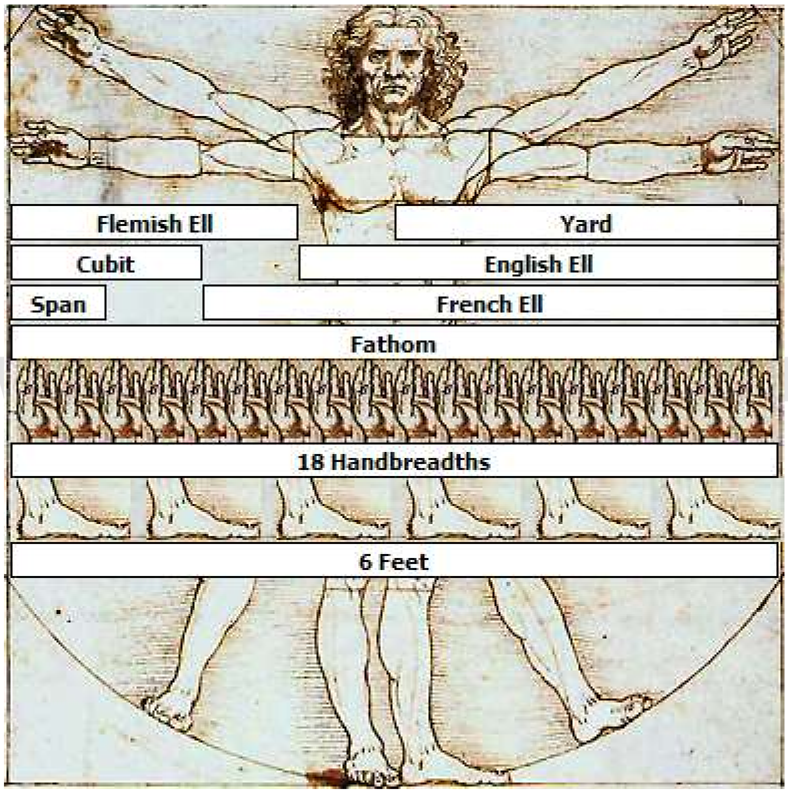
The earliest known uniform systems of weights and measures seem to have all been created sometime in the 4th and 3rd millennia BC among the ancient peoples of Mesopotamia, Egypt and the Indus Valley, and perhaps also Elam in Persia as well.

In "The Magna Carta" of 1215 (The Great Charter) with the seal of King John, put before him by the Barons of England, King John agreed in Clause 35 "There shall be one measure of wine throughout our whole realm, and one measure of ale and one measure of corn--namely, the London quart;--and one width of dyed and russet and hauberk cloths--namely, two ells below the selvage….". The Magna Carta helped lay the foundations of freedom codified in English Law and subsequently American Law.

Many systems were based on the use of parts of the body and the natural surroundings as measuring instruments. Our present knowledge of early weights and measures comes from many sources.

Systems of units

Traditional systems



This derivation of the Vitruvian Man by Leonardo da Vinci, depicts nine historical units of measurement: the yard, the span, the cubit, the Flemish ell, the English ell, the French ell, the fathom, the hand, and the foot. The Vitruvian Man was drawn to scale, so the units depicted are displayed with their proper historical ratios.

Prior to the near global adoption of the metric system many different systems of measurement had been in use. Many of these were related to some extent or other. Often they were based on the dimensions of the human body according to the proportions described by Marcus Vitruvius Pollio. As a result, units of measure could vary not only from location to location, but from person to person.

Metric systems

A number of metric systems of units have evolved since the adoption of the original metric system in France in 1791. The current international standard metric system is the

International system of units. An important feature of modern systems is standardization. Each unit has a universally recognized size.

Both the Imperial units and US customary units derive from earlier English units. Imperial units were mostly used in the British Commonwealth and the former British Empire. US customary units are still the main system of measurement used in the United States despite Congress having legally authorized metric measure on 28 July 1866. Some steps towards US metrication have been made, particularly the redefinition of basic US units to derive exactly from SI units, so that in the US the inch is now defined as 0.0254 m (exactly), and the avoirdupois pound is now defined as 453.59237 g (exactly)

Natural systems

While the above systems of units are based on arbitrary unit values, formalised as standards, some unit values occur naturally in science. Systems of units based on these are called natural units. Similar to natural units, atomic units (au) are a convenient system of units of measurement used in atomic physics.

Also a great number of unusual and non-standard units may be encountered. These may include the Solar mass, the Megaton (1,000,000 tons of TNT).

Legal control of weights and measures

To reduce the incidence of retail fraud, many national statutes have standard definitions of weights and measures that may be used (hence "statute measure"), and these are verified by legal officers.

Base and derived units

Different systems of units are based on different choices of a set of fundamental units. The most widely used system of units is the International System of Units, or SI. There are seven SI base units. All other SI units can be derived from these base units.

For most quantities a unit is absolutely necessary to communicate values of that physical quantity. For example, conveying to someone a particular length without using some sort of unit is impossible, because a length cannot be described without a reference used to make sense of the value given.

But not all quantities require a unit of their own. Using physical laws, units of quantities can be expressed as combinations of units of other quantities. Thus only a small set of units is required. These units are taken as the *base units*. Other units are *derived units*. Derived units are a matter of convenience, as they can be expressed in terms of basic units. Which units are considered base units is a matter of choice.

The base units of SI are actually not the smallest set possible. Smaller sets have been defined. For example, there are unit sets in which the electric and magnetic field have the same unit. This is based on physical laws that show that electric and magnetic field are actually different manifestations of the same phenomenon.

Calculations with units

Units as dimensions

Any value of a physical quantity is expressed as a comparison to a unit of that quantity. For example, the value of a physical quantity *Z* is expressed as the product of a unit [Z] and a numerical factor:

$$Z = n \times [Z] = n[Z].$$

The multiplication sign is usually left out, just as it is left out between variables in scientific notation of formulas. The conventions used to express quantities is referred to as quantity calculus. In formulas the unit [Z] can be treated as if it were a specific magnitude of a kind of physical dimension.

A distinction should be made between units and standards. A unit is fixed by its definition, and is independent of physical conditions such as temperature. By contrast, a standard is a physical realization of a unit, and realizes that unit only under certain physical conditions. For example, the metre is a unit, while a metal bar is a standard. One metre is the same length regardless of temperature, but a metal bar will be one metre long only at a certain temperature.

Guidelines

- Treat units algebraically. Only add like terms. When a unit is divided by itself, the division yields a unitless one. When two different units are multiplied, the result is a new unit, referred to by the combination of the units. For instance, in SI, the unit of speed is metres per second (m/s). A unit can be multiplied by itself, creating a unit with an exponent (e.g. m^2/s^2). Put simply, units obey the laws of indices.
- Some units have special names, however these should be treated like their equivalents. For example, one newton (N) is equivalent to one $kg{\cdot}m/s^2$. Thus a quantity may have several unit designations, for example: the unit for surface tension can be referred to as either N/m (newtons per metre) or kg/s^2 (kilograms per second squared). Whether these designations are equivalent is disputed amongst metrologists.

Expressing a physical value in terms of another unit

Conversion of units involves comparison of different standard physical values, either of a single physical quantity or of a physical quantity and a combination of other physical quantities.

Starting with:

$$Z = n_i \times [Z]_i$$

just replace the original unit $[Z]_i$ with its meaning in terms of the desired unit $[Z]_j$, e.g. if $[Z]_i = c_{ij} \times [Z]_j$, then:

$$Z = n_i \times (c_{ij} \times [Z]_j) = (n_i \times c_{ij}) \times [Z]_j$$

Now n_i and c_{ij} are both numerical values, so just calculate their product.

Or, which is just mathematically the same thing, multiply Z by unity, the product is still Z:

$$Z = n_i \times [Z]_i \times (c_{ij} \times [Z]_j/[Z]_i)$$

For example, you have an expression for a physical value Z involving the unit *feet per second* ($[Z]_i$) and you want it in terms of the unit *miles per hour* ($[Z]_j$):

1. Find facts relating the original unit to the desired unit:

   1 mile = 5280 feet and 1 hour = 3600 seconds

2. Next use the above equations to construct a fraction that has a value of unity and that contains units such that, when it is multiplied with the original physical value, will cancel the original units:

$$1 = \frac{1\,\mathrm{mi}}{5280\,\mathrm{ft}} \quad \text{and} \quad 1 = \frac{3600\,\mathrm{s}}{1\,\mathrm{h}}$$

3. Last,multiply the original expression of the physical value by the fraction, called a *conversion factor*, to obtain the same physical value expressed in terms of a different unit. Note: since valid conversion factors are dimensionless and have a numerical value of one, multiplying any physical quantity by such a conversion factor (which is 1) does not change that physical quantity.

$$52.8\,\frac{\mathrm{ft}}{\mathrm{s}} = 52.8\,\frac{\mathrm{ft}}{\mathrm{s}}\,\frac{1\,\mathrm{mi}}{5280\,\mathrm{ft}}\,\frac{3600\,\mathrm{s}}{1\,\mathrm{h}} = \frac{52.8 \times 3600}{5280}\,\mathrm{mi/h} = 36\,\mathrm{mi/h}$$

Or as an example using the metric system, you have a value of fuel economy in the unit *litres per 100 kilometres* and you want it in terms of the unit *microlitres per metre*:

$$\frac{9\,\mathrm{L}}{100\,\mathrm{km}} = \frac{9\,\mathrm{L}}{100\,\mathrm{km}}\,\frac{1000000\,\mu\mathrm{L}}{1\,\mathrm{L}}\,\frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = \frac{9 \times 1000000}{100 \times 1000}\,\mu\mathrm{L/m} = 90\,\mu\mathrm{L/m}$$

Real-world implications

One example of the importance of agreed units is the failure of the NASA Mars Climate Orbiter, which was accidentally destroyed on a mission to the planet Mars in September 1999 instead of entering orbit, due to miscommunications about the value of forces: different computer programs used different units of measurement (newton versus pound force). Enormous amounts of effort, time, and money were wasted.

On April 15, 1999 Korean Air cargo flight 6316 from Shanghai to Seoul was lost due to the crew confusing tower instructions (in metres) and altimeter readings (in feet). Three crew and five people on the ground were killed. Thirty seven were injured.

In 1983, a Boeing 767 (which came to be known as the Gimli Glider) ran out of fuel in mid-flight because of two mistakes in figuring the fuel supply of Air Canada's first aircraft to use metric measurements. This accident is apparently the result of confusion both due to the simultaneous use of metric & Imperial measures as well as mass & volume measures.

Chapter- 3

Heat Capacity

Heat capacity (usually denoted by a capital *C*, often with subscripts) is the measurable physical quantity that characterizes the amount of heat required to change a body's temperature by a given amount. In the International System of Units, heat capacity is expressed in units of joules per kelvin.

Derived quantities include the **molar heat capacity**, which is the heat capacity per mole of a pure substance. Similarly, **specific heat capacity** (also called more properly "mass-specific heat capacity" or more loosely "specific heat"), is the heat capacity per unit mass of a body. These quantities are "intensive quantities". That is, they are not dependent on amount of material, but directly reflect the type of material, as well as the physical conditions of heating.

Temperature reflects the average total kinetic energy of particles in matter. Heat is transfer of thermal energy; it flows from regions of high temperature to regions of low temperature. Thermal energy is stored as kinetic energy and, in molecules and solids, also as potential energy in the modes of vibration or phonons. These represent degrees of freedom of movement for atoms. These degrees of freedom, and sometimes others, contribute to the heat capacity of a thermodynamic system. As the temperature approaches absolute zero, the specific heat capacity of a system also approaches zero. Quantum theory can be used to quantitatively predict specific heat capacities in simple systems.

Background

Before the development of modern thermodynamics, it was thought that heat was a fluid, the so-called *caloric*. Bodies were capable of holding a certain amount of this fluid, hence the term *heat capacity*, named and first investigated by Joseph Black in the 1750s. Today one instead discusses the internal energy of a system. This is made up of its microscopic kinetic and potential energy. Heat is no longer considered a fluid. Rather, it is a transfer of disordered energy at the microscopic level. Nevertheless, at least in English, the term "heat capacity" survives. Some other languages prefer the term **thermal capacity**, which is also sometimes used in English.

Extensive and intensive quantities

The **heat capacity** indicates how much thermal energy ΔQ a physical body can absorb for a change in temperature ΔT. It refers to a specific body, and gives no indication of the amount of substance or composition of the body.

$$C = \frac{\Delta Q}{\Delta T}$$

C is the heat capacity of the body, and in the International System of Units (SI) has the units $[C] = \frac{\mathrm{J}}{\mathrm{K}}$(joule per kelvin).

If the heat capacity is related to a certain amount of substance, or a volume, we can distinguish

- the **specific heat capacity** c, the heat capacity per unit mass, which has SI units $[c] = \frac{\mathrm{J}}{\mathrm{kg{\cdot}K}}$
- the **molar heat capacity** C_{mol} (also **molar specific heat**), the heat capacity per mole of chemical substance, which has SI units $[C_{\mathrm{mol}}] = \frac{\mathrm{J}}{\mathrm{mol{\cdot}K}}$
- the volumetric heat capacity s, the heat capacity per unit volume, which has SI units $[s] = \frac{\mathrm{J}}{\mathrm{m^3{\cdot}K}}$

The heat capacity itself is an extensive quantity, meaning one that is a property of the whole body, and is therefore sensitive to the size of the object (for example, a bathtub of water has a greater heat capacity than a cup of water). The three derived quantities are intensive quantities, meaning they are no longer dependent on amount of material, but capture more directly the dependence on the type of material, given the particular physical conditions of heating.

Metrology

The heat capacity of most systems is not a constant. Rather, it depends on the state variables of the thermodynamic system under study. In particular it is dependent on temperature itself, as well as on the pressure and the volume of the system.

Different measurements of heat capacity can therefore be performed, most commonly at constant pressure and constant volume. The values thus measured are usually subscripted (by p and V, respectively) to indicate the definition. Gases and liquids are typically also measured at constant volume. Measurements under constant pressure produce larger values than those at constant volume because the constant pressure values also include heat energy that is used to do work to expand the substance against the constant pressure as its temperature increases. This difference is particularly notable in gases where values under constant pressure are typically 30% to 66.7% greater than those at constant volume.

The specific heat capacities of substances comprising molecules (as distinct from monatomic gases) are not fixed constants and vary somewhat depending on temperature. Accordingly, the temperature at which the measurement is made is usually also specified. Examples of two common ways to cite the specific heat of a substance are as follows:

- Water (liquid): c_p = 4.1855 [J/(g·K)] (15 °C, 101.325 kPa)
- Water (liquid): C_vH = 74.539 J/(mol·K) (25 °C)

For liquids and gases, it is important to know the pressure for which given heat-capacity data refer. Most published data are given for standard pressure. However, quite different standard conditions for temperature and pressure have been defined by different organizations. The International Union of Pure and Applied Chemistry (IUPAC) changed its recommendation from one Atmosphere to the round value 100 kPa (≈750.062 Torr).

Calculation

The path integral Monte Carlo method is a numerical approach for determining the values of heat capacity, based on quantum dynamical principles.

Alternative units

An older unit of heat is the calorie, originally defined as the energy required to raise the temperature of one kilogram of water by one degree centigrade, typically from 15°C to 16°C. The specific heat capacity of water on this scale would therefore be exactly 1 Cal/(K·kg). However, due to the temperature-dependence of the specific heat, a large number of different definitions of the calorie came into being. Whilst once it was very prevalent, especially its smaller cgs variant the gram-calorie (cal), defined so that the specific heat of water would be 1 cal/(K·g), in most fields the use of the calorie is now archaic.

In the United States other units of measure for heat capacity may be quoted in disciplines such as construction, civil engineering, and chemical engineering. A still common system is the English Engineering Units in which the mass reference is pound mass and the temperature is specified in degrees Fahrenheit or Rankine. One (rare) unit of heat is the pound calorie (lb-cal), defined as the amount of heat required to raise the temperature of one pound of water by one degree centigrade. On this scale the specific heat of water would be 1 lb-cal/(K·lb). More common is the British thermal unit, the standard unit of heat in the U.S. construction industry. This is defined such that the specific heat of water is 1 BTU/(°F·lb).

Definitions and formal properties

Definition of heat capacity

Heat capacity is defined more precisely as a differential quantity, the ratio of a small amount of heat δQ added to the body, to the corresponding small increase in its temperature dT. This can also be related to the increase in its entropy S:

$$C = \frac{\delta Q}{dT} = T\frac{dS}{dT}$$

For thermodynamic systems with more than one state variable or physical property, the above definition does not give a single, unique quantity unless a particular infinitesimal path has been defined through the system's thermodynamic phase space, defining not just the change in temperature, but also the change in pressure, and volume, and number of particles, and any other relevant macroscopic variables, in each part of the system if the system has more than just a simple homogenous structure. This information is used to account for different ways that heat can be stored as kinetic energy (energy of motion) and potential energy (energy stored in force fields), as an object expands or contracts. For all real systems, the path through these changes must be explicitly defined, since the value of heat capacity depends on which path from one temperature to another, is chosen. Of particular usefulness in this context are the values of heat capacity for constant volume, C_V, and constant pressure, C_P. These will be defined below.

Thermodynamic relation to internal energy

The state of a simple compressible thermodynamic system with fixed mass may be described by three thermodynamic parameters, temperature T, pressure p, and volume V, linked together in an equation of state,

$$f(T, p, V) = 0.$$

The internal energy of the system is a function of these state variables only.

Most measurable thermodynamic properties are the second derivatives of the thermodynamic potentials. Heat capacity, however, may also be defined in terms of first derivatives of the internal energy, if the potential is expanded into terms of the state variables, T, p, and V.

When heat, dQ, is introduced into the system, the change of its internal energy, dU, is:

$$dU = dQ - pdV.$$

If the process is performed at constant volume, i.e. when the second term of this relation vanishes, one readily obtains

$$\left(\frac{\delta Q}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V = C_V$$

This is defined as the *heat capacity at constant volume*.

Most experimental conditions, especially for the condensed phases, involve working at constant pressure, rather than constant volume of the sample, and therefore the volume will change in general as heat is introduced into a system. Some of this heat is returned to the environment in performance of the work of expanding the sample and the change of the internal energy is reduced.

The corresponding *heat capacity at constant pressure* is

$$C_p = \left(\frac{\delta Q}{dT}\right)_p = \left(\frac{\partial U + p\partial V}{\partial T}\right)_p.$$

This introduces the enthalpy into the formalism, which is defined as $H = U + pV$. Enthalpy, like internal energy is another state function.

$$C_p = \left(\frac{\partial H}{\partial T}\right)_p.$$

The heat capacity at constant pressure is the change of the enthalpy of the sample with respect to temperature.

Relation between heat capacities

Measuring the heat capacity, sometimes referred to as specific heat, at constant volume can be prohibitively difficult for liquids and solids. That is, small temperature changes typically require large pressures to maintain a liquid or solid at constant volume implying the containing vessel must be nearly rigid or at least very strong. Instead it is easier to measure the heat capacity at constant pressure (allowing the material to expand or contract as it wishes) and solve for the heat capacity at constant volume using mathematical relationships derived from the basic thermodynamic laws. Starting from the fundamental Thermodynamic Relation one can show

$$C_p - C_V = T\left(\frac{\partial p}{\partial T}\right)_{V,N}\left(\frac{\partial V}{\partial T}\right)_{p,N}$$

where the partial derivatives are taken at constant volume and constant number of particles; and constant pressure and constant number of particles, respectively.

This can also be rewritten

$$C_p - C_V = VT\frac{\alpha^2}{\beta_T}$$

where

> α is the coefficient of thermal expansion,
> β_T is the isothermal compressibility.

The heat capacity ratio or adiabatic index is the ratio of the heat capacity at constant pressure to heat capacity at constant volume. It is sometimes also known as the isentropic expansion factor.

Ideal gas

For an ideal gas, evaluating the partial derivatives above according to the equation of state

$$pV = nRT$$

the relation can be found to reduce to

$$C_p - C_V = nR$$

where *n* is number of moles of gas in the thermodynamic system under consideration, and R is the universal gas constant. Dividing through by *n*, this equation reduces simply to Mayer's relation,

$$C_{p,m} - C_{v,m} = R$$

where $C_{p,m}$ and $C_{v,m}$ are intensive property heat capacities expressed on a per mole basis at constant pressure and constant volume, respectively.

Specific heat capacity

The specific heat capacity of a material on a per mass basis is

$$c = \frac{\partial C}{\partial m},$$

which in the absence of phase transitions is equivalent to

$$c = E_m = \frac{C}{m} = \frac{C}{\rho V},$$

where

C is the heat capacity of a body made of the material in question,
m is the mass of the body,
V is the volume of the body, and
$\rho = \frac{m}{V}$ is the density of the material.

For gases, and also for other materials under high pressures, there is need to distinguish between different boundary conditions for the processes under consideration (since values differ significantly between different conditions). Typical processes for which a heat capacity may be defined include isobaric (constant pressure, $dp = 0$) or isochoric (constant volume, $dV = 0$) processes. The corresponding specific heat capacities are expressed as

$$c_p = \left(\frac{\partial C}{\partial m}\right)_p,$$
$$c_V = \left(\frac{\partial C}{\partial m}\right)_V.$$

From the results of the previous section, dividing through by the mass gives the relation

$$c_p - c_V = \frac{\alpha^2 T}{\rho \beta_T}$$

A related parameter to c is CV^{-1}, the volumetric heat capacity. In engineering practice, c_V for solids or liquids often signifies a volumetric heat capacity, rather than a constant-volume one. In such cases, the mass-specific heat capacity (specific heat) is often explicitly written with the subscript m, as c_m. Of course, from the above relationships, for solids one writes

$$c_m = \frac{C}{m} = \frac{c_{volumetric}}{\rho}.$$

For pure homogeneous chemical compounds with established molecular or molar mass or a molar quantity is established, heat capacity as an intensive property can be expressed on a per mole basis instead of a per mass basis by the following equations analogous to the per mass equations:

$C_{p,m} = \left(\frac{\partial C}{\partial n}\right)_p$ = molar heat capacity at constant pressure

$$C_{V,m} = \left(\frac{\partial C}{\partial n}\right)_V$$ = molar heat capacity at constant volume

where n = number of moles in the body or thermodynamic system. One may refer to such a *per mole* quantity as molar heat capacity to distinguish it from specific heat capacity on a per mass basis.

Polytropic heat capacity

The polytropic heat capacity is calculated at processes if all the thermodynamic properties (pressure, volume, temperature) change

$$C_{i,m} = \left(\frac{\partial C}{\partial n}\right)$$ = molar heat capacity at polytropic process

The most important polytropic processes run between the adiabatic and the isotherm functions, the polytropic index is between **1** and the adiabatic exponent (γ or κ)

Dimensionless heat capacity

The dimensionless heat capacity of a material is

$$C^* = \frac{C}{nR} = \frac{C}{Nk}$$

where

C is the heat capacity of a body made of the material in question (J/K)
n is the amount of substance in the body (mol)
R is the gas constant (J/(K·mol)
N is the number of molecules in the body. (dimensionless)
k is Boltzmann's constant (J/(K·molecule)

In the ideal gas article, dimensionless heat capacity C^* is expressed as $\hat{c}$, and is related there directly to half the number of degrees of freedom per particle. This holds true for quadratic degrees of freedom, a consequence of the equipartition theorem.

More generally, the dimensionless heat capacity relates the logarithmic increase in temperature to the increase in the dimensionless entropy per particle $S^* = S / Nk$, measured in nats.

$$C^* = \frac{dS^*}{d \ln T}$$

Alternatively, using base 2 logarithms, C^* relates the base-2 logarithmic increase in temperature to the increase in the dimensionless entropy measured in bits.

Heat capacity at absolute zero

From the definition of entropy

$$T\,dS = \delta Q$$

the absolute entropy can be calculated by integrating from zero kelvins temperature to the final temperature T_f

$$S(T_f) = \int_{T=0}^{T_f} \frac{\delta Q}{T} = \int_0^{T_f} \frac{\delta Q}{dT}\frac{dT}{T} = \int_0^{T_f} C(T)\,\frac{dT}{T}.$$

The heat capacity must be zero at zero temperature in order for the above integral not to yield an infinite absolute entropy, which would violate the third law of thermodynamics. One of the strengths of the Debye model is that (unlike the preceding Einstein model) it predicts the proper mathematical form of the approach of heat capacity toward zero, as absolute zero temperature is approached.

Negative heat capacity (stars)

Most physical systems exhibit a positive heat capacity. However, even though it can seem paradoxical at first, there are some systems for which the heat capacity is *negative*. These include gravitating objects such as stars; and also sometimes some nano-scale clusters of a few tens of atoms, close to a phase transition. A negative heat capacity can result in a negative temperature.

According to the virial theorem, for a self-gravitating body like a star or an interstellar gas cloud, the average potential energy U_{Pot} and the average kinetic energy U_{Kin} are locked together in the relation

$$U_{\text{Pot}} = -2U_{\text{Kin}},$$

The total energy U (= $U_{\text{Pot}} + U_{\text{Kin}}$) therefore obeys

$$U = -U_{\text{Kin}},$$

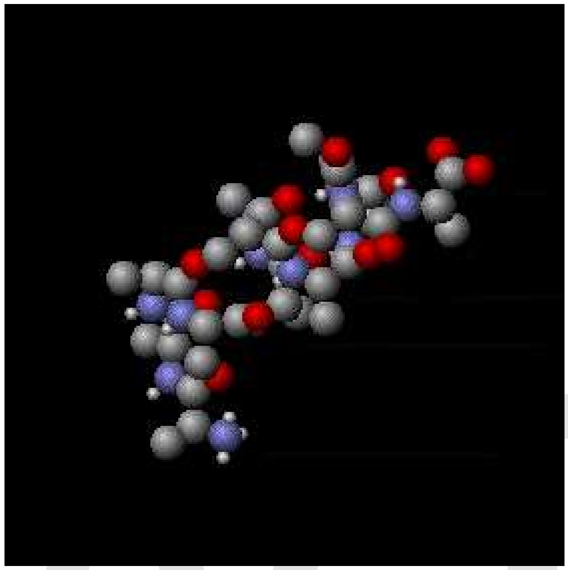
If the system loses energy, for example by radiating energy away into space, the average kinetic energy and with it the average temperature actually *increases*. The system therefore can be said to have a negative heat capacity.

A more extreme version of this occurs with black holes. According to black hole thermodynamics, the more mass and energy a black hole absorbs, the colder it becomes.

On the other hand if it is a net emitter of energy, through Hawking radiation, it will become hotter and hotter until it literally boils away.

Theory of heat capacity

Factors that affect specific heat capacity



Molecules undergo many characteristic internal vibrations. Potential energy stored in these internal degrees of freedom contributes to a sample's energy content, but not to its temperature. More internal degrees of freedom tend to increase a substance's specific heat capacity, so long as temperatures are high enough to overcome quantum effects.

For any given substance, the heat capacity of a body is directly proportional to the amount of substance it contains (measured in terms of mass or moles or volume). Doubling the amount of substance in a body doubles its heat capacity, etc.

However, when this effect has been corrected for, by dividing the heat capacity by the quantity of substance in a body, the resulting specific heat capacity is a function of the structure of the substance itself. In particular, it depends on the number of degrees of freedom that are available to the particles in the substance, each of which type of freedom allows substance particles to store thermal energy. The kinetic energy of substance particles is the only one of the many possible degrees of freedom which manifests as *temperature change*, and thus the larger the number of degrees of freedom available to the particles of a substance *other* than kinetic energy, the larger will be the specific heat capacity for the substance.

In addition, quantum effects require that whenever energy be stored in any mechanism associated with a bound system which confers a degree of freedom, it must be stored in certain minimal-sized deposits (quanta) of energy, or else not stored at all. Such effects limit the full ability of some degrees of freedom to store energy when their lowest energy storage quantum amount is not easily supplied at the average energy of particles at a given temperature. In general, for this reason, specific heat capacities tend to fall at lower temperatures where the average thermal energy available to each particle degree of freedom is smaller, and thermal energy storage begins to be limited by these quantum effects.

Degrees of freedom

Molecules are quite different from the monatomic gases like helium and argon. With monatomic gases, thermal energy comprises only translational motions. Translational motions are ordinary, whole-body movements in 3D space whereby particles move about and exchange energy in collisions—like rubber balls in a vigorously shaken container. These simple movements in the three X, Y, and Z–axis dimensions of space means individual atoms have three translational degrees of freedom. A degree of freedom is any form of energy in which heat transferred into an object can be stored. This can be in translational kinetic energy, rotational kinetic energy, or other forms such as potential energy in vibrational modes. Only three translational degrees of freedom are essential for individual atoms as vibrations don't apply to them.

As to rotation about an atom's axis, its energy is proportional to the moment of inertia for the atom, which is extremely small because almost all of the mass of the atom is concentrated in its nucleus, which has radius too small as to give an insignificant moment of inertia. In contrast, however, the *spacing* of quantum energy levels for a rotating object is inversely proportional to its moment of inertia, and this spacing becomes very large for objects with small moments of inertia. For these reasons, the contribution from rotation of atoms on their axes can be neglected in monatomic gases, because the energy-spacing of the associated quantum levels is too large for any thermal energy to be stored in rotation of systems with moments of inertia so small.

In molecules, however, rotational modes may become active due to much higher moments of inertia about certain axes, and thus the narrowing of quantum mechanically-determined energy spacing between rotational states. Furthermore (although at generally higher temperatures yet) internal vibrational degrees of freedom also may become active; molecules are complex objects their population of atoms that can move about within the molecule in different ways. The heat capacity of molecules (on a per-atom basis) does not exceed the heat capacity of monatomic gases, unless vibrational modes are brought into play. The reason for this is that vibrational modes allow energy to be stored as potential energy in intra-atomic bonds in a molecule, which are not available to atoms in monatomic gases.

Up to about twice as much energy (on a per-atom basis) per unit of temperature increase can be stored in a solid as in a monatomic gas, by this mechanism of storing energy in the potentials of interatomic bonds. This gives most solids about twice the (atomic) molar heat capacity of monatomic gases. However, quantum effects heavily affect the actual ratio at lower temperatures-- especially in solids with light and tightly-bound atoms (eg., beryllium metal). Smaller polyatomic gases store intermediate amounts of energy, giving them a *per-atom* heat capacity that is between that of monatomic gases ($\frac{3}{2}$ *R* per mole, where *R* is the ideal gas constant), and the maximum of fully-excited warmer solids (3 *R* per mole). For gases, heat capacity never falls below the minumum of $\frac{3}{2}$ *R* per mole, since the kinetic energy of gas molecules is always available to store this much heat energy. However, for cryogenic temperatures in solids, heat capacity falls toward zero, as temperature approaches absolute zero.

Example of temperature-dependent specific heat capacity, in a diatomic gas

For instance, nitrogen, which is a diatomic molecule, has five active degrees of freedom at room temperature: the three comprising translational motion plus two rotational degrees of freedom internally. Although the constant-volume molar heat capacity of nitrogen at this temperature is five-thirds that of monatomic gases, on a per-mole of atoms basis, it is only five-sixths that of a monatomic gas. Two separate nitrogen atoms would have a total of six degrees of freedom—the three translational degrees of freedom of each atom. When the atoms are bonded the molecule will still only have three translational degrees of freedom, as the two atoms in the molecule move as one. However, the molecule cannot be treated as a point object, and the moment of inertia has increased sufficiently about two axes to allow two rotational degrees of freedom to be active at room temperature to give five degrees of freedom. The moment of inertia about the third axis remains small, as this is the axis passing through the centres of the two atoms, and so is similar to that of a monatomic gas. Thus, this degree of freedom does not act to store heat, and does not contribute to the heat capacity of nitrogen. The heat capacity *per atom* for nitrogen is therefore less than for a monatomic gas, if vibrational degrees of freedom are not activated.

At higher temperatures, however, nitrogen gas gains two more degrees of internal freedom, as the molecule is excited into higher vibrational modes which store thermal energy, and then the heat capacity per volume or mole of molecules approaches seven-thirds that of monatomic gases, or seven-sixths of monatomic, on a mole-of-atoms basis. This is now a *higher* heat capacity per atom than the monatomic figure, because the vibrational mode enables an extra degree of potential energy freedom per pair of atoms, which monatomic gases cannot possess.

However, even at these large temperatures where gaseous nitrogen stores 7/6 $^{\text{ths}}$ of the energy *per atom* of a monatomic gas (making it more efficient at storing energy on an atomic basis), it still only stores 7/12 $^{\text{ths}}$ of the maximal per-atom heat capacity of a solid, meaning it is not as efficient on an atomic basis, as substances can be. This is due to the fact that many of the potential bonds which might be storing potential energy in gaseous nitrogen (as opposed to solid nitrogen) are lacking, because only one of the spacial dimensions for each nitrogen atom offers a bond into which potential energy can be stored, without increasing the kinetic energy of the atom. In general, solids are most efficient, on an atomic basis, at storing thermal energy (that is, they have the highest per-atom or per-mole-of-atoms heat capacity).

Per mole of...

Per mole of molecules

When the specific heat capacity, *c*, of a material is measured (lowercase *c* means the unit quantity is in terms of mass), different values arise because different substances have different molar masses (essentially, the weight of the individual atoms or molecules). In solids, thermal energy arises due to the number of atoms that are vibrating. "Molar" heat capacity *per mole of molecules*, for both gases and solids, offer figures which are arbitrarily large, since molecules may be arbrarily large. Such heat capacities are thus not intensive equantities for this reason, since the quantity of mass being considered can be increased without limit.

Per mole of atoms

Conversely, for *molecular-based* substances (which also absorb heat into their internal degrees of freedom), massive, complex molecules with high atomic count—like octane—can store a great deal of energy per mole and yet are quite unremarkable on a mass basis, or on a per-atom basis. This is because, in fully excited systems, heat is stored independently by each atom in a substance, not primarily by the bulk motion of molecules.

Thus, it is the heat capacity per-mole-of-atoms, not per-mole-of-molecules, which is the intensive quantity, and which comes closest to being a constant for all substances at high temperatures. This relationship was noticed empirically in 1819, and is called the Dulong-Petit law, after its two discoverers. Historically, the fact that specific heat capacities are approximately equal when corrected by the presumed weight of the atoms of solids, was an important piece of data in favor of the atomic theory of matter.

Because of the connection of heat capacity to the number of atoms, some care should be taken to specify a mole-of-molecules basis vs. a mole-of-atoms basis, when comparing specific heat capacities of molecular solids and gases. Ideal gases have the same numbers of molecules per volume, so increasing molecular complexity adds heat capacity on a per-volume and per-mole-of-molecules basis, but may lower or raise heat capacity on a per-atom basis, depending on whether the temperature is sufficient to store energy as atomic vibration.

In solids, the quantitative limit of heat capacity in general is about 3 R per mole of atoms, where R is the ideal gas constant. This 3 R value is about 24.9 J/mole.K. Six degrees of freedom (three kinetic and three potential) are available to each atom. Each of these six contributes $\frac{1}{2}$R specific heat capacity per mole of atoms. This limit of 3 R per mole specific heat capacity is approached at room temperature for most solids, with significant departures at this temperature only for solids composed of the lightest atoms which are bound very strongly, such as beryllium (where the value is only of 66% of 3 R), or diamond (where it is only 24% of 3 R). These large departures are due to quantum effects which prevent full distribution of heat into all vibrational modes, when the energy difference between vibrational quantum states is very large compared to the average energy available to each atom from the ambient temperature.

For monatomic gases, the specific heat is only half of 3 R per mole, i.e. ($^3\!/_2$R per mole) due to loss of all potential energy degrees of freedom in these gases. For polyatomic gases, the heat capacity will be intermediate between these values on a per-mole-of-atoms basis, and (for heat-stable molecules) would approach the limit of 3 R per mole of atoms, for gases composed of complex molecules, and at higher temperatures at which all vibrational modes accept excitational energy. This is because very large and complex gas molecules may be thought of as relatively large blocks of solid matter which have lost only a relatively small fraction of degrees of freedom, as compared to a fully-integrated solid.

Corollaries of these considerations for solids (volume-specific heat capacity)

Since the bulk density of a solid chemical element is strongly related to its molar mass (usually about 3 R per mole, as noted above), there exists noticeable inverse correlation between a solid's density and its specific heat capacity on a per-mass basis. This is due to a very approximate tendency of atoms of most elements to be about the same size, despite much wider variations in density and atomic weight. These two factors (constancy of atomic volume and constancy of mole-specific heat capacity) result in a good correlation between the *volume* of any given solid chemical element and its total heat capacity. Another way of stating this, is that the volume-specific heat capacity (volumetric heat capacity) of solid elements is roughly a constant. The molar volume of solid elements is very roughly constant, and (even more reliably) so also is the molar heat capacity for most solid substances. These two factors determine the volumetric heat capacity, which as a bulk property may be striking in consistency. For example, the element uranium is a metal which has a density almost 36 times that of the metal lithium, but uranium's specific heat capacity on a volumetric basis (i.e. per given volume of metal) is only 18% larger than lithium's.

Since the volume-specific corollary of the Dulong-Petit specific heat capacity relationship requires that atoms of all elements take up (on average) the same volume in solids, there are many departures from it, with most of these due to variations in atomic size. For instance, arsenic, which is only 14.5% less dense than antimony, has nearly 59% more specific heat capacity on a mass basis. In other words; even though an ingot of arsenic is only about 17% larger than an antimony one of the same mass, it absorbs about 59% more heat for a given temperature rise. The heat capacity ratios of the two substances closely follows the ratios of their molar volumes (the ratios of numbers of atoms in the same volume of each substance); the departure from the correlation to simple volumes in this case is due to lighter arsenic atoms being significantly more closely-packed than antimony atoms, instead of similar size. In other words, similar-sized atoms would cause a mole of arsenic to be 63% larger than a mole of antimony, with a correspondingly lower density, allowing its volume to more closely mirror its heat capacity behavior.

Other factors

Hydrogen bonds

Hydrogen-containing polar molecules like ethanol, ammonia, and water have powerful, intermolecular hydrogen bonds when in their liquid phase. These bonds provide another place where heat may be stored as potential energy of vibration, even at comparatively low temperatures. Hydrogen bonds account for the fact that liquid water stores nearly the theoretical limit of 3 R per mole of atoms, even at relatively low temperatures (i.e. near the freezing point of water).

Impurities

In the case of alloys, there are several conditions in which small impurity concentrations can greatly affect the specific heat. Alloys may exhibit marked difference in behaviour even in the case of small amounts of impurities being one element of the alloy; for example impurities in semiconducting ferromagnetic alloys may lead to quite different specific heat properties.

The simple case of the monatomic gas

In the case of a monatomic gas such as helium under constant volume, if it assumed that no electronic or nuclear quantum excitations occur, each atom in the gas has only 3 degrees of freedom, all of a translational type. No energy dependence is associated with the degrees of freedom which define the position of the atoms. While, in fact, the degrees of freedom corresponding to the momenta of the atoms are quadratic, and thus contribute to the heat capacity. There are N atoms, each of which has 3 components of momentum, which leads to 3N total degrees of freedom. This gives:

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2} N\, k_B = \frac{3}{2} n\, R$$
$$C_{V,m} = \frac{C_V}{n} = \frac{3}{2} R$$

where

C_V is the heat capacity at constant volume of the gas
$C_{V,m}$ is the *molar heat capacity* at constant volume of the gas
N is the total number of atoms present in the container
n is the number of moles of atoms present in the container (n is the ratio of N and Avogadro's number)
R is the ideal gas constant, (8.314570[70] J/(mol·K). R is equal to the product of Boltzmann's constant k_B and Avogadro's number

The following table shows experimental molar constant volume heat capacity measurements taken for each noble monatomic gas (at 1 atm and 25 °C):

Monatomic gas	**$C_{V,m}$ (J/(mol·K))**	**$C_{V,m}/R$**
He	12.5	1.50
Ne	12.5	1.50
Ar	12.5	1.50
Kr	12.5	1.50
Xe	12.5	1.50

It is apparent from the table that the experimental heat capacities of the monatomic noble gases agrees with this simple application of statistical mechanics to a very high degree.

Diatomic gas

In the somewhat more complex case of an ideal gas of diatomic molecules, the presence of internal degrees of freedom are apparent. In addition to the three translational degrees of freedom, there are rotational and vibrational degrees of freedom. In general, the number of degrees of freedom, f, in a molecule with n_a atoms is $3n_a$:

$$f = 3n_a$$

Mathematically, there are a total of three rotational degrees of freedom, one corresponding to rotation about each of the axes of three dimensional space. However, in practice only the existence of two degrees of rotational freedom for linear molecules will be considered. This approximation is valid because the moment of inertia about the internuclear axis is vanishingly small with respect other moments of inertia in the molecule (this is due to the extremely small radii of the atomic nuclei, compared to the distance between them in a molecule). Quantum mechanically, it can be shown that the interval between successive rotational energy eigenstates is inversely proportional to the moment of inertia about that axis. Because the moment of inertia about the internuclear axis is vanishingly small relative to the other two rotational axes, the energy spacing can be considered so high that no excitations of the rotational state can possibly occur unless the temperature is extremely high. It is easy to calculate the expected number of vibrational degrees of freedom (or vibrational modes). There are three degrees of translational freedom, and two degrees of rotational freedom, therefore

$$f_{\text{vib}} = f - f_{\text{trans}} - f_{\text{rot}} = 6 - 3 - 2 = 1$$

Each rotational and translational degree of freedom will contribute $R/2$ in the total molar heat capacity of the gas. Each vibrational mode will contribute R to the total molar heat capacity, however. This is because for each vibrational mode, there is a potential and kinetic energy component. Both the potential and kinetic components will contribute $R/2$

to the total molar heat capacity of the gas. Therefore, a diatomic molecule would be expected to have a molar constant-volume heat capacity of

$$C_{V,m} = \frac{3R}{2} + R + R = \frac{7R}{2} = 3.5R$$

where the terms originate from the translational, rotational, and vibrational degrees of freedom, respectively.

The following is a table of some molar constant-volume heat capacities of various diatomic gases at standard temperature (25 °C = 298 K)

Diatomic gas	**$C_{V,m}$ (J/(mol·K))**	**$C_{V,m}$ / R**
H_2	20.18	2.427
CO	20.2	2.43
N_2	19.9	2.39
Cl_2	24.1	3.06
Br_2 (vapour)	28.2	3.39

From the above table, clearly there is a problem with the above theory. All of the diatomics examined have heat capacities that are lower than those predicted by the equipartition theorem, except Br_2. However, as the atoms composing the molecules become heavier, the heat capacities move closer to their expected values. One of the reasons for this phenomenon is the quantization of vibrational, and to a lesser extent, rotational states. In fact, if it is assumed that the molecules remain in their lowest energy vibrational state because the inter-level energy spacings for vibration-energies are large, the predicted molar constant volume heat capacity for a diatomic molecule becomes just that from the contributions of translation and rotation:

$$C_{V,m} = \frac{3R}{2} + R = \frac{5R}{2} = 2.5R$$

which is a fairly close approximation of the heat capacities of the lighter molecules in the above table. If the quantum harmonic oscillator approximation is made, it turns out that the quantum vibrational energy level spacings are actually inversely proportional to the square root of the reduced mass of the atoms composing the diatomic molecule. Therefore, in the case of the heavier diatomic molecules such as chlorine or bromine, the quantum vibrational energy level spacings become finer, which allows more excitations into higher vibrational levels at lower temperatures. This limit for storing heat capacity in vibrational modes, as discussed above, becomes 7"R" /2 = 3.5 R per mole, which is fairly consistent with the measured value for Br_2 at room temperature. As temperatures rise, all diatomic gases approach this value.

General gas phase

The specific heat of the gas is best conceptualized in terms of the degrees of freedom of an individual molecule. The different degrees of freedom correspond to the different ways in which the molecule may store energy. The molecule may store energy in its translational motion according to the formula:

$$E = \frac{1}{2} m \left(v_x^2 + v_y^2 + v_z^2 \right)$$

where m is the mass of the molecule and $[v_x, v_y, v_z]$ is velocity of the center of mass of the molecule. Each direction of motion constitutes a degree of freedom, so that there are three translational degrees of freedom.

In addition, a molecule may have rotational motion. The kinetic energy of rotational motion is generally expressed as

$$E = \frac{1}{2} \left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right)$$

where I is the moment of inertia tensor of the molecule, and $[\omega_1, \omega_2, \omega_3]$ is the angular velocity pseudo-vector (in a coordinate system aligned with the principle axes of the molecule). In general, then, there will be three additional degrees of freedom corresponding to the rotational motion of the molecule, (For linear molecules one of the inertia tensor terms vanishes and there are only two rotational degrees of freedom). The degrees of freedom corresponding to translations and rotations are called the rigid degrees of freedom, since they do not involve any deformation of the molecule.

The motions of the atoms in a molecule which are not part of its gross translational motion or rotation may be classified as vibrational motions. It can be shown that if there are n atoms in the molecule, there will be as many as $v = 3n - 3 - n_r$ vibrational degrees of freedom, where n_r is the number of rotational degrees of freedom. A vibrational degree of freedom corresponds to a specific way in which all the atoms of a molecule can vibrate. The actual number of possible vibrations may be less than this maximal one, due to various symmetries.

For example, triatomic nitrous oxide N_2O will have only 2 degrees of rotational freedom (since it is a linear molecule) and contains n=3 atoms: thus the number of possible vibrational degrees of freedom will be v = (3*3)-3-2 = 4. There are four ways or "modes" in which the three atoms can vibrate, corresponding to *1)* A mode in which an atom at each end of the molecule moves away from, or towards, the center atom at the same time, *2)* a mode in which either end atom moves asynchronously with regard to the other two, and *3)* and *4)* two modes in which the molecule bends out of line, from the center, in the two possible planar directions that are orthogonal to its axis. Each vibrational degree of freedom confers TWO total degrees of freedom, since vibrational energy mode partitions into 1 kinetic and 1 potential mode. This would give nitrous oxide 3 translational, 2 rotational, and 4 vibrational modes (but these last giving 8 vibrational degrees of freedom), for storing energy. This is a total of f = 3+2+8 = 13 total energy-storing degrees of freedom, for N_2O.

For a bent molecule like water H_2O, a similar calculation gives 9-3-3 = 3 modes of vibration, and 3 (translational) + 3 (rotational) + 6(vibratonal) = 12 degrees of freedom.

The storage of energy into degrees of freedom

If the molecule could be entirely described using classical mechanics, then the theorem of equipartition of energy could be used to predict that each degree of freedom would have an average energy in the amount of $(1/2)kT$ where k is Boltzmann's constant and T is the temperature. Our calculation of the constant-volume heat capacity would be straightforward. Each molecule would be holding, on average, an energy of $(f/2)kT$ where f is the total number of degrees of freedom in the molecule. Note that $Nk = R$ if N is Avogadro's number, which is the case in considering the heat capacity of a mole of molecules. Thus, the total internal energy of the gas would be $(f/2)\ NkT$ where N is the total number of molecules. The heat capacity (at constant volume) would then be a constant $(f/2)Nk$ the mole-specific heat capacity would be $(f/2)R$ the molecule-specific heat capacity would be $(f/2)k$ and the dimensionless heat capacity would be just $f/2$. Here again, each vibrational degree of freedom contributes 2f. Thus, a mole of nitrous oxide would have a total constant-volume heat capacity (including vibration) of $(13/2)R$ by this calculation.

In summary, the molar heat capacity (mole-specific heat capacity) of an ideal gas with f degrees of freedom is given by

$$C_{V,m} = \frac{f}{2}R$$

This equation applies to all polyatomic gases, if the degrees of freedom are known.

The constant-pressure heat capacity for any gas would exceed this by an extra factor of R. As example C_p would be a total of (15/2)R/mole for nitrous oxide.

The effect of quantum energy levels in storing energy in degrees of freedom

The various degrees of freedom cannot generally be considered to obey classical mechanics, however. Classically, the energy residing in each degree of freedom is assumed to be continuous—it can take on any positive value, depending on the temperature. In reality, the amount of energy that may reside in a particular degree of freedom is quantized: It may only be increased and decreased in finite amounts. A good estimate of the size of this minimum amount is the energy of the first excited state of that degree of freedom above its ground state. For example, the first vibrational state of the hydrogen chloride (HCl) molecule has an energy of about 5.74×10^{-20} joule. If this amount of energy were deposited in a classical degree of freedom, it would correspond to a temperature of about 4156 K.

If the temperature of the substance is so low that the equipartition energy of $(1/2)kT$ is much smaller than this excitation energy, then there will be little or no energy in this degree of freedom. This degree of freedom is then said to be “frozen out". As mentioned above, the temperature corresponding to the first excited vibrational state of HCl is about 4156 K. For temperatures well below this value, the vibrational degrees of freedom of the HCl molecule will be frozen out. They will contain little energy and will not contribute to the thermal energy or the heat capacity of HCl gas.

Energy storage mode "freeze-out" temperatures

It can be seen that for each degree of freedom there is a critical temperature at which the degree of freedom “unfreezes” and begins to accept energy in a classical way. In the case of translational degrees of freedom, this temperature is that temperature at which the thermal wavelength of the molecules is roughly equal to the size of the container. For a container of macroscopic size (e.g. 10 cm) this temperature is extremely small and has no significance, since the gas will certainly liquify or freeze before this low temperature is reached. For any real gas translational degrees of freedom may be considered to always be classical and contain an average energy of $(3/2)kT$ per molecule.

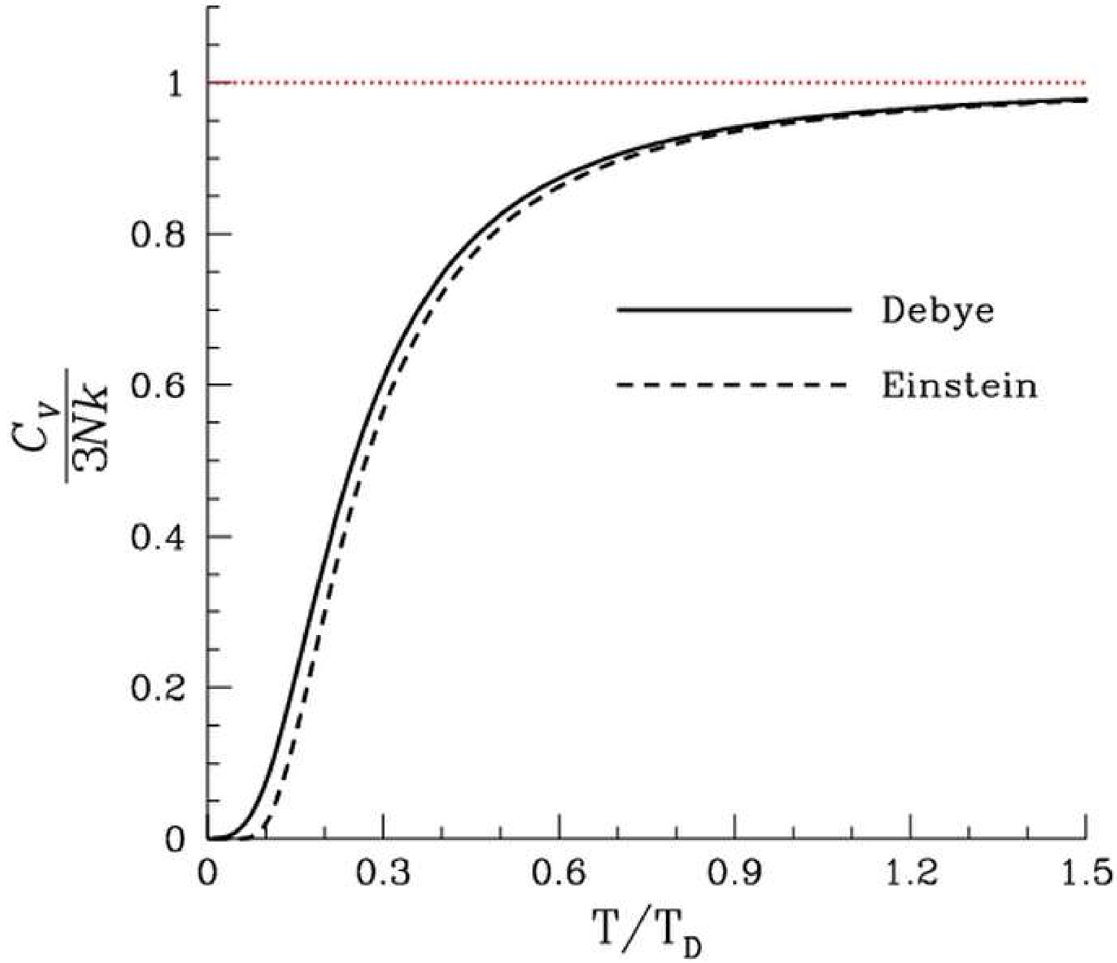
The rotational degrees of freedom are the next to “unfreeze". In a diatomic gas, for example, the critical temperature for this transition is usually a few tens of kelvins, although with a very light molecule such as hydrogen the rotational energy levels will be spaced so widely that rotational heat capacity may not completely "unfreeze" until considerably higher temperatures are reached. Finally, the vibrational degrees of freedom are generally the last to unfreeze. As an example, for diatomic gases, the critical temperature for the vibrational motion is usually a few thousands of kelvins, and thus for the nitrogen in our example at room temperature, no vibration modes would be exited, and the constant-volume heat capacity at room temperature is $(5/2)R$/mole, not $(7/2)R$/mole. As seen above, with some unusually heavy gases such as iodine gas Cl_2, or bromine gas Br_2, some vibrational heat capacity may be observed even at room temperatures.

It should be noted that it has been assumed that atoms have no rotational or internal degrees of freedom. This is in fact untrue. For example, atomic electrons can exist in excited states and even the atomic nucleus can have excited states as well. Each of these internal degrees of freedom are assumed to be frozen out due to their relatively high excitation energy. Nevertheless, for sufficiently high temperatures, these degrees of freedom cannot be ignored. In a few exceptional cases, such molecular electronic transitions are of sufficiently low energy that they contribute to heat capacity at room temperature, or even at cryogenic temperatures. One example of an electronic transition degree of freedom which contributes heat capacity at standard temperature is that of nitric oxide (NO), in which the single electron in an anti-bonding molecular orbital has energy transitions which contribute to the heat capacity of the gas even at room temperature.

An example of a nuclear magnetic transition degree of freedom which is of importance to heat capacity, is the transition which converts the spin isomers of hydrogen gas (H_2) into each other. At room temperature, the proton spins of hydrogen gas are aligned 75% of the time, resulting in *orthohydrogen* when they are. Thus, some thermal energy has been stored in the degree of freedom available when *parahydrogen* (in which spins are anti-aligned) absorbs energy, and is converted to the higher energy ortho form. However, at the temperature of liquid hydrogen, not enough heat energy is available to produce orthohydrogen (that is, the transition energy between forms is large enough to "freeze out" at this low temperature), and thus the parahydrogen form predominates. The heat capacity of the transition is sufficient to release enough heat, as orthohydrogen converts to the lower-energy parahydrogen, to boil the hydrogen liquid to gas again, if this evolved heat is not removed with a catalyst after the gas has been cooled and condensed. This example also illustrates the fact that some modes of storage of heat may not be in constant equilibrium with each other in substances, and heat absorbed or released from such phase changes may "catch up" with temperature changes of substances, only after a certain time. In other words, the heat evolved and absorbed from the ortho-para isomeric transition contributes to the heat capacity of hydrogen on long time-scales, but not on *short* time-scales. These time scales may also depend on the presence of a catalyst.

Less exotic phase-changes may contribute to the heat-capacity of substances and systems, as well, as (for example) when water is converted back and forth from solid to liquid or gas form. Phase changes store heat energy entirely in breaking the bonds of the potential energy interactions between molecules of a substance. As in the case of hydrogen, it is also possible for phase changes to be hindered as the temperature drops, so that they do not catch up and become apparent, without a catalyst. For example, it is possible to supercool liquid water to below the freezing point, and not observe the heat evolved when the water changes to ice, so long as the water remains liquid. This heat appears instantly when the water freezes.

Solid phase



The dimensionless heat capacity divided by three, as a function of temperature as predicted by the Debye model and by Einstein's earlier model. The horizontal axis is the temperature divided by the Debye temperature. Note that, as expected, the dimensionless heat capacity is zero at absolute zero, and rises to a value of three as the temperature becomes much larger than the Debye temperature. The red line corresponds to the classical limit of the Dulong-Petit law

For matter in a crystalline solid phase, the Dulong-Petit law, which was discovered empirically, states that the mole-specific heat capacity assumes the value 3 R. Indeed, for solid metallic chemical elements at room temperature, molar heat capacities range from about 2.8 R to 3.4 R. Large exceptions involve solids composed of light, tightly-bonded atoms such as beryllium at 2.0 R, and diamond at only 0.735 R. The latter conditions create large quantum vibrational energy spacing, so that many vibrational modes are not available (are frozen out) at room temperature.

The theoretical maximum heat capacity for larger and larger multi-atomic gases at higher temperatures, also approaches the Dulong-Petit limit of 3 R, so long as this is calculated per mole of atoms, not molecules. The reason is that gases with very large molecules, in theory have almost the same high-temperature heat capacity as solids, lacking only the (small) heat capacity contribution that comes from potential energy that cannot be stored between separate molecules in a gas.

The Dulong-Petit limit results from the equipartition theorem, and as such is only valid in the classical limit of a microstate continuum, which is a high temperature limit. For light and non-metallic elements, as well as most of the common molecular solids based on carbon compounds at standard ambient temperature, quantum effects may also play an important role, as they do in multi-atomic gases. These effects usually combine to give heat capacities lower than 3 R per mole of *atoms* in the solid, although in molecular solids, heat capacities calculated *per mole of molecules* in molecular solids may be more than 3 R. For example, the heat capacity of water ice at the melting point is about 4.6 R per mole of molecules, but only 1.5 R per mole of atoms. The lower than 3 R number "per atom" (as is the case with diamond and beryllium) results from the "freezing out" of possible vibration modes for light atoms at suitably low temperatures, just as in many low-mass-atom gases at room temperatures. Because of high crystal binding energies, these effects are seen in solids more often than liquids: for example the heat capacity of liquid water is twice that of ice at near the same temperature, and is again close to the 3 R per mole of atoms of the Dulong-Petit theoretical maximum.

For a more modern and precise analysis of the heat capacities of solids, especially at low temperatures, it is useful to use the idea of phonons.

The specific heat of amorphous materials has characteristic discontinuities at the glass transition temperature due to rearrangements that occur in the distribution of atoms. These discontinuities are frequently used to detect the glass transition temperature where a supercooled liquid transforms to a glass.

Table of specific heat capacities

Note that especially high values, as for paraffin, water and ammonia, result from calculating specific heats in terms of moles of molecules. If specific heat is expressed per mole of atoms for these substances, few constant-volume values exceed the theoretical Dulong-Petit limit of 25 J/(mol·K) = 3 R per mole of atoms.

Notable minima and maxima are shown in maroon

Table of specific heat capacities at 25 °C unless otherwise noted **Substance**	**Phase**	C_p **J·g^{-1}·K^{-1}**	$C_{p,m}$ **J·mol^{-1}·K^{-1}**	$C_{v,m}$ **J·mol^{-1}·K^{-1}**	**Volumetric heat capacity J·cm^{-3}·K^{-1}**
Air (Sea level, dry, 0 °C)	gas	1.0035	29.07	20.7643	0.001297
Air (typical room conditions[A])	gas	1.012	29.19	20.85	
Aluminium	solid	0.897	24.2		2.422
Ammonia	liquid	4.700	**80.08**		3.263

Animal (and human) tissue	mixed	3.5	—		3.7*
Antimony	solid	0.207	25.2		1.386
Argon	gas	0.5203	20.7862	12.4717	
Arsenic	solid	0.328	24.6		1.878
Beryllium	solid	1.82	16.4		3.367
Bismuth	solid	0.123	25.7		1.20
Cadmium	solid	0.231	—		—
Carbon dioxide CO_2	gas	0.839*	36.94	28.46	
Chromium	solid	0.449	—		—
Copper	solid	0.385	24.47		3.45
Diamond	solid	0.5091	**6.115**		1.782
Ethanol	liquid	2.44	112		1.925
Gasoline	liquid	2.22	228		1.64
Glass	solid	0.84			
Gold	solid	0.129	25.42		2.492
Granite	solid	0.790			2.17
Graphite	solid	0.710	8.53		1.534
Helium	gas	5.1932	20.7862	12.4717	
Hydrogen	gas	**14.30**	28.82		
Hydrogen sulfide H_2S	gas	1.015*	34.60		
Iron	solid	0.450	25.1		3.537
Lead	solid	0.129	26.4		1.44
Lithium	solid	3.58	24.8		1.912
Magnesium	solid	1.02	24.9		1.773
Mercury	liquid	0.1395	27.98		1.888
Methane at 2 °C	gas	2.191			
Nitrogen	gas	1.040	29.12	20.8	
Neon	gas	1.0301	20.7862	12.4717	
Oxygen	gas	0.918	29.38		
Paraffin wax	solid	2.5	**900**		2.325
Polyethylene (rotomolding grade)	solid	2.3027			
Polyethylene (rotomolding grade)	liquid	2.9308			
Silica (fused)	solid	0.703	42.2		1.547
Silver	solid	0.233	24.9		2.44
Tin	solid	0.227	—		—
Tungsten	solid	0.134	24.8		2.58
Uranium	solid	**0.116**	27.7		2.216
Water at 100 °C (steam)	gas	**2.080**	**37.47**	**28.03**	

Water at 25 °C	liquid	**4.1813**	**75.327**	**74.53**	**4.1796**
Water at 100 °C	liquid	**4.1813**	**75.327**	**74.53**	**4.2160**
Water at −10 °C (ice)	solid	**2.05**	**38.09**		**1.938**
Zinc	solid	0.387	25.2		2.76
Substance	**Phase**	**C_p J/(g·K)**	**$C_{p,m}$ J/(mol·K)**	**$C_{v,m}$ J/(mol·K)**	**Volumetric heat capacity J/(cm^3·K)**

[A] Assuming an altitude of 194 metres above mean sea level (the world–wide median altitude of human habitation), an indoor temperature of 23 °C, a dewpoint of 9 °C (40.85% relative humidity), and 760 mm–Hg sea level–corrected barometric pressure (molar water vapor content = 1.16%).
*Derived data by calculation. This is for water-rich tissues such as brain. The whole-body average figure for mammals is approximately 2.9 J/(cm^3·K)

Specific heat capacity of building materials

(Usually of interest to builders and solar designers)

Specific heat capacity of building materials

Substance	Phase	c_p J/(g·K)
Asphalt	solid	0.920
Brick	solid	0.840
Concrete	solid	0.880
Glass, silica	solid	0.840
Glass, crown	solid	0.670
Glass, flint	solid	0.503
Glass, pyrex	solid	0.753
Granite	solid	0.790
Gypsum	solid	1.090
Marble, mica	solid	0.880
Sand	solid	0.835
Soil	solid	0.800
Wood	solid	1.7 (1.2 to 2.3)
Substance	**Phase**	**c_p J/(g·K)**

Chapter- 4

Dimensional Analysis

In mathematics and science, **dimensional analysis** is a tool to understand the properties of physical quantities independent of the units used to measure them. Every physical quantity is some combination of mass, length, time, electric charge, and temperature, (denoted M, L, T, Q, and Θ (theta), respectively). For example, speed, which may be measured in meters per second (m/s) or miles per hour (mi/h), has the dimension L/T, or alternatively LT^{-1}.

Dimensional analysis is routinely used to check the plausibility of derived equations and computations. It is also used to form reasonable hypotheses about complex physical situations that can be tested by experiment or by more developed theories of the phenomena, and to categorize types of physical quantities and units based on their relations to or dependence on other units, or their dimensions if any.

The basic principle of dimensional analysis was known to Isaac Newton (1686) who referred to it as the "*Great Principle of Similitude*". The 19th-century French mathematician Joseph Fourier made important contributions based on the idea that physical laws like $F = ma$ should be independent of the units employed to measure the physical variables. This led to the conclusion that meaningful laws must be homogeneous equations in their various units of measurement, a result which was eventually formalized in the Buckingham π theorem. This theorem describes how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of $n - m$ dimensionless parameters, where m is the number of fundamental dimensions used. Furthermore, and most importantly, it provides a method for computing these dimensionless parameters from the given variables.

A dimensional equation can have the dimensions reduced or eliminated through nondimensionalization, which begins with dimensional analysis, and involves scaling quantities by characteristic units of a system or natural units of nature. This gives insight into the fundamental properties of the system, as illustrated in the examples below.

Introduction

Definition

The dimensions of a physical quantity are associated with combinations of mass, length, time, electric charge, and temperature, represented by sans-serif symbols **M**, **L**, **T**, **Q**, and **Θ**, respectively, each raised to rational powers.

The term *dimension* is more abstract than *scale* unit: *mass* is a dimension, while kilograms are a scale unit (choice of standard) in the mass dimension.

As examples, the dimension of the physical quantity speed is *distance/time* (**L**/**T** or $\mathbf{LT}^{-1}$), and the dimension of the physical quantity force is "mass × acceleration" or "mass×(distance/time)/time" ($\mathbf{ML}/\mathbf{T}^2$ or $\mathbf{MLT}^{-2}$). In principle, other dimensions of physical quantity could be defined as "fundamental" (such as momentum or energy or electric current) in lieu of some of those shown above. Most physicists do not recognize temperature, **Θ**, as a fundamental dimension of physical quantity since it essentially expresses the energy per particle per degree of freedom, which can be expressed in terms of energy (or mass, length, and time). Still others do not recognize electric charge, **Q**, as a separate fundamental dimension of physical quantity, since it has been expressed in terms of mass, length, and time in unit systems such as the cgs system. There are also physicists that have cast doubt on the very existence of incompatible fundamental dimensions of physical quantity.

The unit of a physical quantity and its dimension are related, but not identical concepts. The units of a physical quantity are defined by convention and related to some standard; e.g., length may have units of meters, feet, inches, miles or micrometres; but any length always has a dimension of **L**, independent of what units are arbitrarily chosen to measure it. Two different units of the same physical quantity have conversion factors that relate them. For example: 1 in = 2.54 cm; then (2.54 cm/in) is the conversion factor, and is itself dimensionless and equal to one. Dimensional symbols do not have conversion factors.

Mathematical properties

Dimensional symbols, such as **L**, form a group: The identity is defined as $\mathbf{L}^0 = 1$, and the inverse to **L** is 1/**L** or $\mathbf{L}^{-1}$. **L** raised to any rational power *p* is a member of the group, having an inverse of $\mathbf{L}^{-p}$ or $1/\mathbf{L}^{p}$. The operation of the group is multiplication, having the usual rules for handling exponents ($\mathbf{L}^n \times \mathbf{L}^m = \mathbf{L}^{n+m}$).

Dimensional symbols may be seen as a vector space over the rational numbers, with the coordinates of the vector being the powers of the exponents – expressing a dimensional symbol as $\mathbf{M}^i\mathbf{L}^j\mathbf{T}^k$ corresponds to the vector (i,j,k). A basis for a given space of dimensional symbols is called a set of fundamental quantities or fundamental dimensions.

When quantities, be they like-dimensioned or unlike-dimensioned, are multiplied or divided by each other, their dimensional symbols are likewise multiplied or divided; this corresponds to vector addition or subtraction (on the exponents). When dimensioned quantities are raised to a rational power, the same is done to the dimensional symbols attached to those quantities; this corresponds to scalar multiplication on the exponents.

As in any vector space, one may choose different bases, which yield different physical interpretations.

Mechanics

In mechanics, the dimension of any physical quantity can be expressed in terms of the fundamental dimensions (or *base dimensions*) **M**, **L**, and **T** – these form a 3-dimensional vector space. This is not the only possible choice, but it is the one most commonly used. For example, one might choose force, length and mass as the base dimensions (as some have done), with associated dimensions **F**, **L**, **M**; this corresponds to a different basis, and one may convert between these representations by a change of basis. The choice of the base set of dimensions is, thus, partly a convention, resulting in increased utility and familiarity. It is, however, important to note that the choice of the set of dimensions cannot be chosen arbitrarily – it is not *just* a convention – because the dimensions must form a basis: they must span the space, and be linearly independent.

For example, **F**, **L**, **M** form a set of fundamental dimensions because they form an equivalent basis to *M, L, T:* the former can be expressed as $[\mathbf{F}=\mathbf{ML}/\mathbf{T}^2]$,**L**,**M** while the latter can be expressed as **M**,**L**,$[\mathbf{T}=(\mathbf{ML}/\mathbf{F})^{1/2}]$.

On the other hand, using length, velocity and time (*L, V, T*) as base dimensions will not work well (they do not form a set of fundamental dimensions), for two reasons:

- There is no way to obtain mass — or anything derived from it, such as force — without introducing another base dimension (thus these do not *span the space*).
- Velocity, being derived from length and time (**V**=**L**/**T**), is redundant (the set is not *linearly independent*).

Other fields of physics and chemistry

Depending on the field of physics, it may be advantageous to choose one or another extended set of dimensional symbols. In electromagnetism, for example, it may be useful to use dimensions of **M**, **L**, **T**, and **Q**, where **Q** represents quantity of electric charge. In thermodynamics, the base set of dimensions is often extended to include a dimension for temperature, **Θ**. In chemistry the number of moles of substance (loosely, but not precisely, related to the number of molecules or atoms) is often involved and a dimension for this is used as well. The choice of the dimensions or even the number of dimensions to be used in different fields of physics is to some extent arbitrary, but consistency in use and ease of communications are very important.

Commensurability

The most basic consequence of dimensional analysis is:

Only *commensurable* quantities (quantities with the same dimensions) may be *compared, equated, added,* or *subtracted.*

However,

One may take *ratios* of *incommensurable* quantities (quantities with different dimensions), and *multiply* or *divide* them.

For example, it makes no sense to ask if 1 hour is more or less than 1 kilometer, as these have different dimensions, nor to add 1 hour to 1 kilometer. On the other hand, if an object travels 100 km in 2 hours, one may divide these and conclude that the object's average speed was 50 km/hour.

As a corollary of this requirement, it follows that in a physically meaningful *expression,* only quantities of the same dimension can be added, subtracted, or compared. For example, if m_{man}, m_{rat} and L_{man} denote, respectively, the mass of some man, the mass of a rat and the length of that man, the expression $m_{\mathrm{man}} + m_{\mathrm{rat}}$ is meaningful, but $m_{\mathrm{man}} + L_{\mathrm{man}}$ is meaningless. However, $m_{\mathrm{man}}/L^2_{\mathrm{man}}$ is fine. Thus, dimensional analysis may be used as a sanity check of physical equations: the two sides of any equation must be commensurable or have the same dimensions, i.e., the equation must be *dimensionally homogeneous.*

Even when two physical quantities have identical dimensions, it may be meaningless to compare or add them. For example, although torque and energy share the dimension $\mathrm{ML^2T^{-2}}$, they are fundamentally different physical quantities.

To compare, add, or subtract quantities with the same dimensions but expressed in different units, the standard procedure is to first convert them all to the same units. For example, to compare 32 metre with 35 yards, use 1 yard = 0.9144 m to convert 35 yards to 32.004 m.

Polynomials and transcendental functions

Scalar arguments to transcendental functions such as exponential, trigonometric and logarithmic functions, or to inhomogeneous polynomials, must be dimensionless quantities. (Note: this requirement is somewhat relaxed in Siano's orientational analysis described below, in which the square of certain dimensioned quantities are dimensionless)

This requirement is clear when one observes the Taylor expansions for these functions (a sum of various powers of the function argument). For example, the logarithm of 3 kg is undefined even though the logarithm of 3 is nearly 0.477. An attempt to compute ln 3 kg would produce, if one naively took ln 3 kg to mean the dimensionally meaningless "ln (1 + 2 kg)",

$$2\,\mathrm{kg} - \frac{4\,\mathrm{kg}^2}{2} + \cdots$$

which is dimensionally incompatible – the sum has no meaningful dimension – requiring the argument of transcendental functions to be dimensionless.

Another way to understand this problem is that the different coefficients *scale* differently under change of units – were one to reconsider this in grams as "ln 3000 g" instead of "ln 3 kg", one could compute ln 3000, but in terms of the Taylor series, the degree 1 term would scale by 1000, the degree-2 term would scale by 1000^2, and so forth – the overall output would not scale as a particular dimension.

While most mathematical identities about dimensionless numbers translate in a straightforward manner to dimensional quantities, care must be taken with logarithms of ratios: the identity log(a/b) = log a - log b, where the logarithm is taken in any base, holds for dimensionless numbers a and b, but it does *not* hold if a and b are dimensional, because in this case the left-hand side is well-defined but the right-hand side is not.

Similarly, while one can evaluate monomials (x^n) of dimensional quantities, one cannot evaluate polynomials of mixed degree with dimensionless coefficients on dimensional quantities: for x^2, the expression $(3\text{ m})^2 = 9\text{ m}^2$ makes sense (as an area), while for $x^2 + x$, the expression $(3\text{ m})^2 + 3\text{ m} = 9\text{ m}^2 + 3\text{ m}$ does not make sense.

However, polynomials of mixed degree can make sense if the coefficients are suitably chosen physical quantities that are not dimensionless. For example,

$$\frac{1}{2} \cdot \left(-32 \frac{\mathrm{foot}}{\mathrm{second}^2}\right) \cdot t^2 + \left(500 \frac{\mathrm{foot}}{\mathrm{second}}\right) \cdot t.$$

This is the height to which an object rises in time *t* if the acceleration of gravity is 32 feet per second per second and the initial upward speed is 500 feet per second. It is not even necessary for *t* to be in *seconds*. For example, suppose *t* = 0.01 minutes. Then the first term would be

$$\frac{1}{2} \cdot \left(-32 \frac{\text{foot}}{\text{second}^2}\right) \cdot (0.01 \text{ minute})^2$$

$$= \frac{1}{2} \cdot -32 \cdot (0.01^2) \left(\frac{\text{minute}}{\text{second}}\right)^2 \cdot \text{foot}$$

$$= \frac{1}{2} \cdot -32 \cdot (0.01^2) \cdot 60^2 \cdot \text{foot}.$$

Incorporating units

The value of a dimensional physical quantity *Z* is written as the product of a unit [*Z*] within the dimension and a dimensionless numerical factor, *n*.

$$Z = n \times [Z] = n[Z]$$

In a strict sense, when like-dimensioned quantities are added or subtracted or compared, these dimensioned quantities must be expressed in consistent units so that the numerical values of these quantities may be directly added or subtracted. But, in concept, there is no problem adding quantities of the same dimension expressed in different units. For example, 1 meter added to 1 foot *is* a length, but it would not be correct to add 1 to 1 to get the result. A conversion factor, which is a ratio of like-dimensioned quantities and is equal to the dimensionless unity, is needed:

$1 \text{ ft} = 0.3048 \text{ m}$ is identical to $1 = \frac{0.3048 \text{ m}}{1 \text{ ft}}$.

The factor $0.3048 \ \frac{\text{m}}{\text{ft}}$ is identical to the dimensionless 1, so multiplying by this conversion factor changes nothing. Then when adding two quantities of like dimension, but expressed in different units, the appropriate conversion factor, which is essentially the dimensionless 1, is used to convert the quantities to identical units so that their numerical values can be added or subtracted.

$$\begin{aligned} 1 \text{ m} + 1 \text{ ft} &= 1 \text{ m} + 1 \text{ ft} \times 0.3048 \ \frac{\text{m}}{\text{ft}} \\ &= 1 \text{ m} + 1 \cancel{\text{ft}} \times 0.3048 \ \frac{\text{m}}{\cancel{\text{ft}}} \\ &= 1 \text{ m} + 0.3048 \text{ m} \\ &= 1.3048 \text{ m} \end{aligned}$$

Only in this manner is it meaningful to speak of adding like-dimensioned quantities of differing units.

Position vs displacement

Consider position versus displacement (on a line, to avoid issues of higher dimension). These both have dimensions of length, but are not interchangeable:

- one may add two displacements, obtaining a new displacement,
- one may add a displacement to a position, obtaining a new position,
- one may subtract two positions, obtaining a displacement,
- but one may *not* add two positions.

This illustrates the subtle distinction between *affine* quantities (ones modeled by an affine space, such as position) and *vector* quantities (ones modeled by a vector space, such as displacement).

- Vector quantities may be added to each other, yielding a new vector quantity, and a vector quantity may be added to a suitable affine quantity (a vector space *acts on* an affine space), yielding a new affine quantity.
- Affine quantities cannot be added, but may be subtracted, yielding *relative* quantities which are vectors, and these *relative differences* may then be added to each other or to an affine quantity.

Properly then, positions have dimension of *affine* length, while displacements have dimension of *vector* length. To assign a number to an *affine* unit, one must not only choose a unit of measurement, but also a point of reference, while to assign a number to a *vector* unit only requires a unit of measurement.

Thus some physical quantities are better modeled by *affine* quantities than by *vector* quantities, and the distinction is reflected in their dimensional analysis.

Some discussions of dimensional analysis implicitly assume that all quantities are vectors (can be added, can be divided), or assume an implicit point of reference.

Orientation and frame of reference

Similar to the issue of a point of reference is the issue of orientation: a displacement in 2 or 3 dimensions is not just a length, but is a length together with a *direction.* (This issue does not arise in 1 dimension, or rather is equivalent to the distinction between positive and negative.) Thus, to compare or combine two dimensional quantities in a multi-dimensional space, one also needs an orientation: they need to be compared to a frame of reference.

This leads to the extensions discussed below, namely Huntley's directed dimensions and Siano's orientational analysis.

Other uses

Dimensional analysis is also used to derive relationships between the physical quantities that are involved in a particular phenomenon that one wishes to understand and characterize. It was used for the first time (Pesic, 2005) in this way in 1872 by Lord Rayleigh, who was trying to understand why the sky is blue.

Examples

A simple example: period of a harmonic oscillator

What is the period of oscillation T of a mass m attached to an ideal linear spring with spring constant k suspended in gravity of strength g? The four quantities have the following dimensions: T [T]; m [M]; $k[M / T^2]$; and $g[L / T^2]$. From these we can form only one dimensionless product of powers of our chosen variables, $G_1 = T^2k / m$. The dimensionless product of powers of variables is sometimes referred to as a dimensionless group of variables, but the group, G_1, referred to means "collection" rather than mathematical group. They are often called dimensionless numbers as well.

Note that no other dimensionless product of powers involving g with k, m, T, and g alone can be formed, because only g involves L. Dimensional analysis can sometimes yield strong statements about the *irrelevance* of some quantities in a problem, or the need for additional parameters. If we have chosen enough variables to properly describe the problem, then from this argument we can conclude that the period of the mass on the spring is independent of g: it is the same on the earth or the moon. The equation demonstrating the existence of a product of powers for our problem can be written in an entirely equivalent way: $T = \kappa\sqrt{m/k}$, for some dimensionless constant κ.

When faced with a case where our analysis rejects a variable (g, here) that we feel sure really belongs in a physical description of the situation, we might also consider the possibility that the rejected variable is in fact relevant, and that some other relevant variable has been omitted, which might combine with the rejected variable to form a dimensionless quantity. That is, however, not the case here.

When dimensional analysis yields a solution of problems where only one dimensionless product of powers is involved, as here, there are no unknown functions, and the solution is said to be "complete."

A more complex example: energy of a vibrating wire

Consider the case of a vibrating wire of length ℓ (L) vibrating with an amplitude A (L). The wire has a linear density ρ (M/L) and is under tension s (ML/T^2), and we want to know the energy E (ML^2/T^2) in the wire. Let π_1 and π_2 be two dimensionless products of powers of the variables chosen, given by

$$\pi_1 = E/As$$
$$\pi_2 = \ell/A.$$

The linear density of the wire is not involved. The two groups found can be combined into an equivalent form as an equation

$$F(E/As, \ell/A) = 0,$$

where F is some unknown function, or, equivalently as

$$E = Asf(\ell/A),$$

where f is some other unknown function. Here the unknown function implies that our solution is now incomplete, but dimensional analysis has given us something that may not have been obvious: the energy is proportional to the first power of the tension. Barring further analytical analysis, we might proceed to experiments to discover the form for the unknown function f. But our experiments are simpler than in the absence of dimensional analysis. We'd perform none to verify that the energy is proportional to the tension. Or perhaps we might guess that the energy is proportional to ℓ, and so infer that $E = \ell s$. The power of dimensional analysis as an aid to experiment and forming hypotheses becomes evident.

The power of dimensional analysis really becomes apparent when it is applied to situations, unlike those given above, that are more complicated, the set of variables involved are not apparent, and the underlying equations hopelessly complex. Consider, for example, a small pebble sitting on the bed of a river. If the river flows fast enough, it will actually raise the pebble and cause it to flow along with the water. At what critical velocity will this occur? Sorting out the guessed variables is not so easy as before. But dimensional analysis can be a powerful aid in understanding problems like this, and is usually the very first tool to be applied to complex problems where the underlying equations and constraints are poorly understood. In such cases, the answer may depend on a dimensionless number such as the Reynolds number, which may be interpreted by dimensional analysis.

Extensions

Huntley's extension: directed dimensions

Huntley (Huntley, 1967) has pointed out that it is sometimes productive to refine our concept of dimension. Two possible refinements are:

- The magnitude of the components of a vector are to be considered dimensionally distinct. For example, rather than an undifferentiated length unit L, we may have L_x represent length in the x direction, and so forth. This requirement stems ultimately from the requirement that each component of a physically meaningful equation (scalar, vector, or tensor) must be dimensionally consistent.

- Mass as a measure of quantity is to be considered dimensionally distinct from mass as a measure of inertia.

As an example of the usefulness of the first refinement, suppose we wish to calculate the distance a cannon ball travels when fired with a vertical velocity component V_y and a horizontal velocity component V_x, assuming it is fired on a flat surface. Assuming no use of directed lengths, the quantities of interest are then V_x, V_y, both dimensioned as L / T, R, the distance travelled, having dimension L, and g the downward acceleration of gravity, with dimension L / T^2

With these four quantities, we may conclude that the equation for the range R may be written:

$$R \propto V_x^a V_y^b g^c.$$

Or dimensionally

$$L = (L/T)^{a+b}(L/T^2)^c$$

from which we may deduce that $a + b + c = 1$ and $a + b + 2c = 0$, which leaves one exponent undetermined. This is to be expected since we have two fundamental quantities L and T and four parameters, with one equation.

If, however, we use directed length dimensions, then V_x will be dimensioned as L_x / T, V_y as L_y / T, R as L_x and g as L_y / T^2. The dimensional equation becomes:

$$L_x = (L_x/T)^a\,(L_y/T)^b(L_y/T^2)^c$$

and we may solve completely as $a = 1$, $b = 1$ and $c = -1$. The increase in deductive power gained by the use of directed length dimensions is apparent.

In a similar manner, it is sometimes found useful (e.g., in fluid mechanics and thermodynamics) to distinguish between mass as a measure of inertia (inertial mass), and mass as a measure of quantity (substantial mass). For example, consider the derivation of Poiseuille's Law. We wish to find the rate of mass flow of a viscous fluid through a circular pipe. Without drawing distinctions between inertial and substantial mass we may choose as the relevant variables

- $\dot{m}$ the mass flow rate with dimensions M / T
- p_x the pressure gradient along the pipe with dimensions M / L^2T^2
- ρ the density with dimensions M / L^3
- η the dynamic fluid viscosity with dimensions M / LT

- r the radius of the pipe with dimensions L

There are three fundamental variables so the above five equations will yield two dimensionless variables which we may take to be $\pi_1 = \dot{m}/\eta r$ and $\pi_2 = p_x \rho r^5/\dot{m}^2$ and we may express the dimensional equation as

$$C = \pi_1 \pi_2^a = \left(\frac{\dot{m}}{\eta r}\right)\left(\frac{p_x \rho r^5}{\dot{m}^2}\right)^a$$

where ***C*** and ***a*** are undetermined constants. If we draw a distinction between inertial mass with dimensions M_i and substantial mass with dimensions M_s, then mass flow rate and density will use substantial mass as the mass parameter, while the pressure gradient and coefficient of viscosity will use inertial mass. We now have four fundamental parameters, and one dimensionless constant, so that the dimensional equation may be written:

$$C = \frac{p_x \rho r^4}{\eta \dot{m}}$$

where now only C is an undetermined constant (found to be equal to π / 8 by methods outside of dimensional analysis). This equation may be solved for the mass flow rate to yield Poiseuille's law.

Siano's extension: orientational analysis

Huntley's extension has some serious drawbacks:

- It does not deal well with vector equations involving the *cross product,*
- nor does it handle well the use of *angles* as physical variables.

It also is often quite difficult to assign the L, L_x, L_y, L_z, symbols to the physical variables involved in the problem of interest. He invokes a procedure that involves the "symmetry" of the physical problem. This is often very difficult to apply reliably: It is unclear as to what parts of the problem that the notion of "symmetry" is being invoked. Is it the symmetry of the physical body that forces are acting upon, or to the points, lines or areas at which forces are being applied? What if more than one body is involved with different symmetries? Consider the spherical bubble attached to a cylindrical tube, where one wants the flow rate of air as a function of the pressure difference in the two parts. What are the Huntley extended dimensions of the viscosity of the air contained in the connected parts? What are the extended dimensions of the pressure of the two parts? Are they the same or different? These difficulties are responsible for the limited application of Huntley's addition to real problems.

Angles are, by convention, considered to be dimensionless variables, and so the use of angles as physical variables in dimensional analysis can give less meaningful results. As an example, consider the projectile problem mentioned above. Suppose that, instead of the *x*- and *y*-components of the initial velocity, we had chosen the magnitude of the velocity *v* and the angle θ at which the projectile was fired. The angle is, by convention, considered to be dimensionless, and the magnitude of a vector has no directional quality, so that no dimensionless variable can be composed of the four variables *g*, *v*, *R*, and θ. Conventional analysis will correctly give the powers of *g* and *v*, but will give no information concerning the dimensionless angle θ.

Siano (Siano, 1985-I, 1985-II) has suggested that the directed dimensions of Huntley be replaced by using *orientational symbols* 1_x 1_y 1_z to denote vector directions, and an orientationless symbol 1_0. Thus, Huntley's 1_x becomes L 1_x with L specifying the dimension of length, and 1_x specifying the orientation. Siano further shows that the orientational symbols have an algebra of their own. Along with the requirement that $1_i^{-1} = 1_i$, the following multiplication table for the orientation symbols results:

	$\mathbf{1_0}$	$\mathbf{1_x}$	$\mathbf{1_y}$	$\mathbf{1_z}$
$\mathbf{1_0}$	1_0	1_x	1_y	1_z
$\mathbf{1_x}$	1_x	1_0	1_z	1_y
$\mathbf{1_y}$	1_y	1_z	1_0	1_x
$\mathbf{1_z}$	1_z	1_y	1_x	1_0

Note that the orientational symbols form a group (the Klein four-group or "Viergruppe"). In this system, scalars always have the same orientation as the identity element, independent of the "symmetry of the problem." Physical quantities that are vectors have the orientation expected: a force or a velocity in the *z*-direction has the orientation of 1_z. For angles, consider an angle θ that lies in the *z*-plane. Form a right triangle in the z plane with θ being one of the acute angles. The side of the right triangle adjacent to the angle then has an orientation 1_x and the side opposite has an orientation 1_y. Then, since $\tan(\theta) = 1_y/1_x = \theta + \ldots$ we conclude that an angle in the xy plane must have an orientation $1_y/1_x = 1_z$, which is not unreasonable. Analogous reasoning forces the conclusion that $\sin(\theta)$ has orientation 1_z while $\cos(\theta)$ has orientation 1_0. These are different, so one concludes (correctly), for example, that there are no solutions of physical equations that are of the form a $\cos(\theta)$+b $\sin(\theta)$, where a and b are real scalars. Physical quantities may be expressed as complex numbers (e.g. $e^{i\theta}$) which imply that the complex quantity *i* has an orientation equal to that of the angle it is associated with (1_z in the above example).

The assignment of orientational symbols to physical quantities and the requirement that physical equations be orientationally homogeneous can actually be used in a way that is similar to dimensional analysis to derive a little more information about acceptable solutions of physical problems. In this approach one sets up the dimensional equation and solves it as far as one can. If the lowest power of a physical variable is fractional, both sides of the solution is raised to a power such that all powers are integral. This puts it into "normal form". The orientational equation is then solved to give a more restrictive condition on the unknown powers of the orientational symbols, arriving at a solution that is more complete than the one that dimensional analysis alone gives. Often the added information is that one of the powers of a certain variable is even or odd.

As an example, for the projectile problem, using orientational symbols, θ, being in the *xy*-plane will thus have dimension 1_z and the range of the projectile R will be of the form:

$$R = g^a\, v^b\, \theta^c \text{ which means } L\, 1_x \sim \left(\frac{L\, 1_y}{T^2}\right)^a \left(\frac{L}{T}\right)^b 1_z^c.$$

Dimensional homogeneity will now correctly yield $a = -1$ and $b = 2$, and orientational homogeneity requires that c be an odd integer. In fact the required function of theta will be $\sin(\theta)\cos(\theta)$ which is a series of odd powers of θ.

It is seen that the Taylor series of $\sin(\theta)$ and $\cos(\theta)$ are orientationally homogeneous using the above multiplication table, while expressions like $\cos(\theta) + \sin(\theta)$ and $\exp(\theta)$ are not, and are (correctly) deemed unphysical.

It should be clear that the multiplication rule used for the orientational symbols is not the same as that for the cross product of two vectors. The cross product of two identical vectors is zero, while the product of two identical orientational symbols is the identity element.

Percentages and derivatives

Percentages are dimensionless quantities, since they are ratios of two quantities with the same dimensions.

Derivatives with respect to a quantity add the dimensions of the variable one is differentiating with respect to on the denominator. Thus:

- position (x) has units of L (Length);
- derivative of position with respect to time (dx/dt, velocity) has units of L/T – Length from position, Time from the derivative;
- the second derivative (d^2x/dt^2, acceleration) has units of L/T^2.

In economics, one distinguishes between stocks and flows: a stock has units of "units" (say, widgets or dollars), while a flow is a derivative of a stock, and has units of "units/time" (say, dollars/year).

Beware that in some contexts, dimensional quantities are expressed as dimensionless quantities or percentages by omitting some dimensions. This may or may not be misleading. For example, Debt to GDP ratios are generally expressed as percentages: total debt outstanding (dimension of Currency) divided by annual GDP (dimension of Currency) – but one may argue that in comparing a stock to a flow, annual GDP should have dimensions of Currency/Time (Dollars/Year, for instance), and thus Debt to GDP should have units of years.

Dimensionless concepts

Constants

The dimensionless constants that arise in the results obtained, such as the C in the Poiseuille's Law problem and the κ in the spring problems discussed above come from a more detailed analysis of the underlying physics, and often arises from integrating some differential equation. Dimensional analysis itself has little to say about these constants, but it is useful to know that they very often have a magnitude of order unity. This observation can allow one to sometimes make "back of the envelope" calculations about the phenomenon of interest, and therefore be able to more efficiently design experiments to measure it, or to judge whether it is important, etc.

Formalisms

Paradoxically, dimensional analysis can be a useful tool even if all the parameters in the underlying theory are dimensionless, e.g., lattice models such as the Ising model can be used to study phase transitions and critical phenomena. Such models can be formulated in a purely dimensionless way. As we approach the critical point closer and closer, the distance over which the variables in the lattice model are correlated (the so-called correlation length, ξ) becomes larger and larger. Now, the correlation length is the relevant length scale related to critical phenomena, so one can, e.g., surmize on "dimensional grounds" that the non-analytical part of the free energy per lattice site should be ˜$1 / \xi^d$ where *d* is the dimension of the lattice.

It has been argued by some physicists, e.g., Michael Duff, that the laws of physics are inherently dimensionless. The fact that we have assigned incompatible dimensions to Length, Time and Mass is, according to this point of view, just a matter of convention, borne out of the fact that before the advent of modern physics, there was no way to relate mass, length, and time to each other. The three independent dimensionful constants: c, $\hbar$, and G, in the fundamental equations of physics must then be seen as mere conversion factors to convert Mass, Time and Length into each other.

Just as in the case of critical properties of lattice models, one can recover the results of dimensional analysis in the appropriate scaling limit; e.g., dimensional analysis in mechanics can be derived by reinserting the constants $\hbar$, c, and G (but we can now consider them to be dimensionless) and demanding that a nonsingular relation between quantities exists in the limit $c \rightarrow \infty, \hbar \rightarrow 0$ and $G \rightarrow 0$. In problems involving a gravitational field the latter limit should be taken such that the field stays finite.

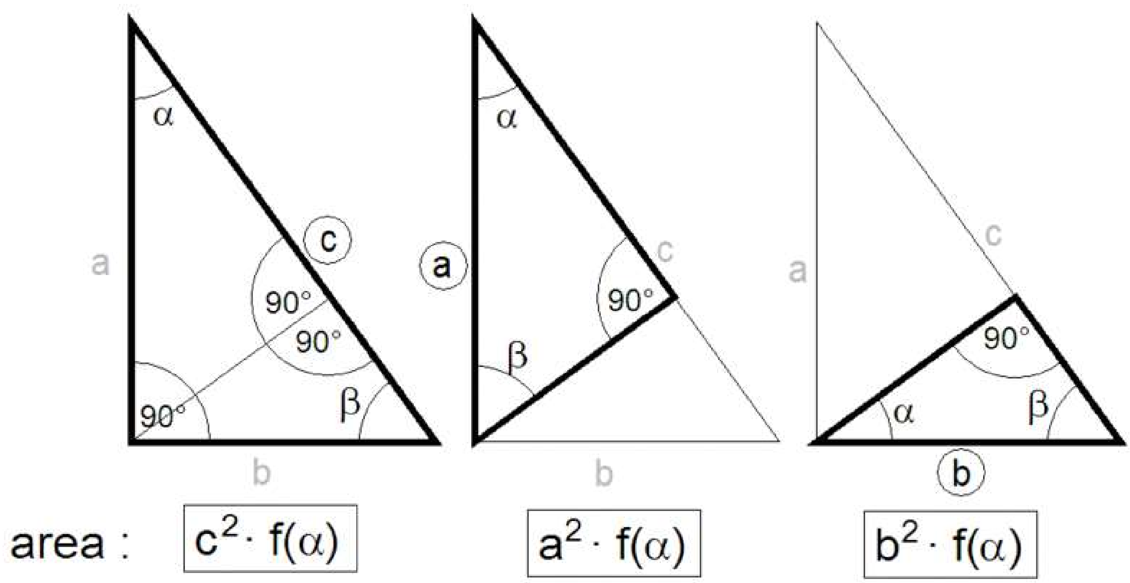
Applications

Dimensional analysis is most often used in physics and chemistry- and in the mathematics thereof- but finds some applications outside of those fields as well.

Mathematics

A simple application of dimensional analysis to mathematics is in computing the form of the volume of an n-ball (the solid ball in n-dimensions), or the area of its surface, the n-sphere: being an n-dimensional figure, the volume scales as x^n, while the surface area, being $(n - 1)$-dimensional, scales as x^{n-1}. Thus the volume of the n-ball in terms of the radius is $C_n r^n$, for some constant C_n. Determining the constant takes more involved mathematics, but the form can be deduced and checked by dimensional analysis alone.

Proof of the Pythagorean theorem



Dimensional proof of the Pythagorean theorem

A very simple proof of the Pythagorean theorem can be obtained by just dimensional reasoning.

The area of any triangle depends on its size and shape, which can be unambiguously identified by the length of one of its edges (for example, the largest) and by any two of its angles (the third being determined by the fact that the sum of all three is π). Thus, *recalling that an area has the dimensions of a length squared*, we can write:

area = largest_edge² • f (angle_1, angle_2),

where f is an adimensional function of the angles.

Now, referring to the figure at right, if we divide a right triangle in two smaller ones by tracing the segment perpendicular to its hypotenuse and passing by the opposite vertex, and express the obvious fact that the total area is the sum of the two smaller ones, by applying the previous equation we have:

$$c^2 \bullet f(\alpha, \pi/2) = a^2 \bullet f(\alpha, \pi/2) + b^2 \bullet f(\alpha, \pi/2).$$

And, eliminating *f*:

$$c^2 = a^2 + b^2, \text{ Q.E.D.}$$

Note that the result is obtained without specifying the form of the adimensional function *f*.

Finance, economics, and accounting

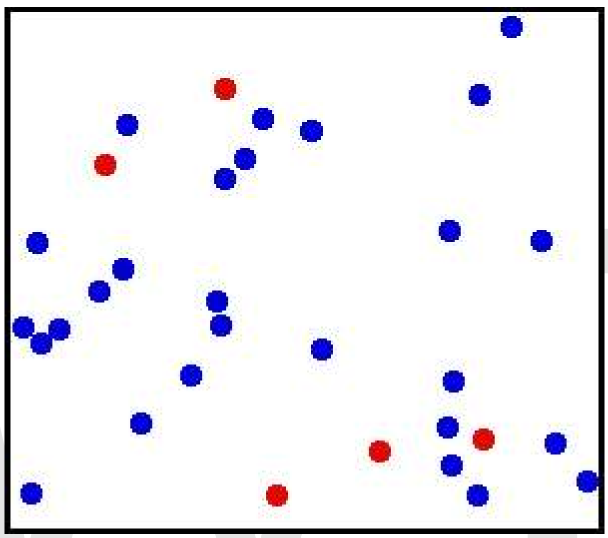
In finance, economics, and accounting, dimensional analysis is most commonly used in interpreting various financial ratios, economics ratios, and accounting ratios.

- For example, the P/E ratio has dimensions of time (units of years), and can be interpreted as "years of earnings to earn the price paid."
- In economics, debt-to-GDP ratio also has units of years (debt has units of currency, GDP has units of currency/year).
- More surprisingly, bond duration also has units of years, which can be shown by dimensional analysis, but takes some financial intuition to understand.
- Velocity of money has units of 1/Years (GDP/Money supply has units of Currency/Year over Currency): how often a unit of currency circulates per year.

Dimensional analysis is rarely used in (mainstream/neoclassical) economic modeling, and economic models are often dimensionally inconsistent. The equation of exchange is the most notable example of a dimensional equation in economic modeling, while the widely-used Cobb–Douglas model does not use dimensions in a meaningful way. This lack of dimensional consistency is criticized by heterodox economics, notably Austrian economics, while dimensional consistency is not considered necessary or desirable by mainstream economists.

Chapter- 5

Temperature



The temperature of an ideal monatomic gas is related to the average kinetic energy of its atoms. In this animation, the size of helium atoms relative to their spacing is shown to scale under 1950 atmospheres of pressure. These atoms have a certain, average speed (slowed down here two trillion fold from room temperature).

Temperature is a physical property that quantitatively expresses the common notions of hot and cold. Objects of low temperature are cold, while various degrees of higher temperatures are referred to as warm or hot.

Temperature is measured with thermometers, which may be calibrated to a variety of temperature scales. Much of the world uses the Celsius scale for most temperature measurements. The Kelvin scale is the temperature standard for scientific or engineering purposes. It has the same incremental scaling as the Celsius scale, but fixes its origin, or null point, at absolute zero (0K = −273.15°C). Few countries, most notably the United States, use the Fahrenheit scale for common purposes, a historical scale on which water freezes at 32°F and boils at 212°F.

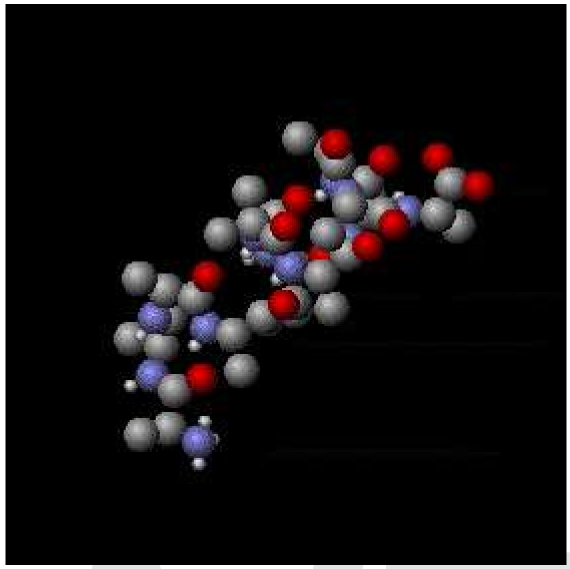
A thermometer is essentially a device which measures its own temperature. If two systems are in thermal equilibrium, they have the same temperature, by definition. This means that bringing a thermometer into thermal contact with another system, and allowing equilibrium to occur, the temperature of the other system will be equal to that indicated by the thermometer. The second law of thermodynamics then defines absolute temperature through the use of the concept of reversible heat engines operating between systems at different temperatures, but only to within a calibration constant. This allows the actual construction of a thermometer. The concept of a thermometer is further refined by the zeroth law of thermodynamics which guarantees that two equivalently calibrated thermometers will always agree when simultaneously measuring the temperature of a third body, assuring the uniqueness of the temperature of that body.

Historically, several equivalent scientific concepts of temperature have developed: the macroscopic thermodynamic description, the kinetic theory of gases which relates the macroscopic description to the probability distribution of the motions of gas particles, i.e. the kinetic energy, and a microscopic explanation based on statistical physics and quantum theory. The thermodynamic definition of temperature, first stated by Lord Kelvin, is based entirely on empirical variables that could be measured in a laboratory. Statistical physics provides a deeper understanding by describing the atomic behavior of matter, and derives macroscopic properties from statistical averages of microscopic states, including quantum states, in the thermodynamic limit. In the modern metric system of units, the macroscopic and the microscopic descriptions are interrelated by the Boltzmann constant, a proportionality factor that scales empirical temperature to the microscopic mean kinetic energy.

In the context of thermodynamics, the kinetic energy is also referred to as *thermal energy* and the transfer of thermal energy is commonly referred to as heat. Spontaneously, heat flows from regions of higher temperature to regions of lower temperature, as described by the second law of thermodynamics. If no net heat is transferred between two objects, the objects have the same temperature. The second law affords another rigorous definition of temperature as the variation of the internal energy with respect to changes in entropy of the system.

The statistical approach has shown that the temperature can be interpreted as a measure of the average energy of motion (kinetic energy) in each energetically available degree of freedom of the particles in a thermodynamic system. However, the definition of temperature is only meaningful for large ensembles of such elementary particles to fulfill the requirements of the statistical model. For a solid, this thermal energy is found primarily in the vibrations of its atoms or molecules about their equilibrium position. In an ideal monatomic gas, the kinetic energy is found exclusively in the purely translational motions of the particles. In other systems, vibrational and rotational motions also contribute degrees of freedom.

Overview



Thermal vibration of a segment of protein alpha helix. The amplitude of the vibrations increases with temperature.

Macroscopically, temperature is a measure of the thermal energy held by a body of matter. An immediate way of sensing this is by feeling the material and deciding whether it is hot, warm, or cold. A thermometer precisely measures the temperature and records a numerical value for the temperature.

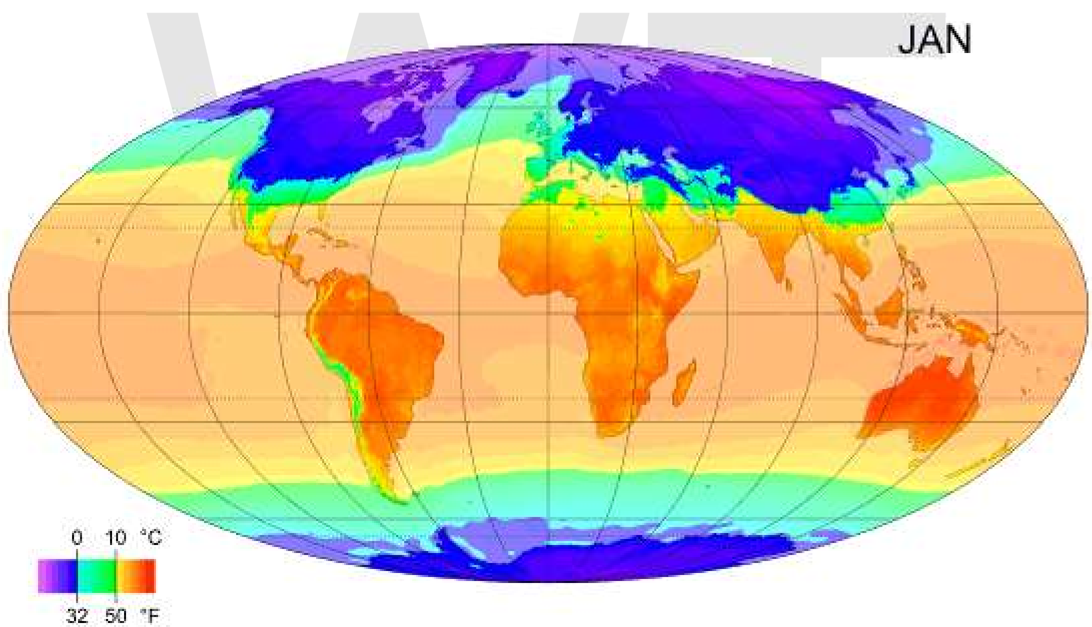
On the molecular level, temperature is the result of the motion of the particles that constitute the material. Moving particles carry kinetic energy. Temperature increases as this motion and the kinetic energy increase. The motion may be the translational motion of particles, or the energy of the particle due to molecular vibration or the excitation of an electron energy level. Although very specialized laboratory equipment is required to directly detect the translational thermal motions, thermal collisions by atoms or molecules with small particles suspended in a fluid produces Brownian motion that can be seen with an ordinary microscope. The thermal motions of atoms are very fast and temperatures close to absolute zero are required to directly observe them. For instance, when scientists at the NIST achieved a record-setting low temperature of 700 nK (1 nK = 10^{-9} K) in 1994, they used laser equipment to create an optical lattice to adiabatically cool caesium atoms. They then turned off the entrapment lasers and directly measured atom velocities of 7mm per second in order to calculate their temperature.

Molecules, such as oxygen (O_2), have more degrees of freedom than single spherical atoms: they undergo rotational and vibrational motions as well as translations. An increase in temperature will cause the average translational energy to increase. It will also cause, through equipartitioning, the energy associated with vibrational and rotational modes to increase. Thus a diatomic gas will require a higher energy input to increase its temperature by a certain amount, i.e. it will have a higher heat capacity than a monatomic gas.

The process of cooling involves removing thermal energy from a system. When no more energy can to be removed, the system is very close to absolute zero, which, however, cannot be achieved experimentally for quantum mechanical reasons. Absolute zero is the null point of the thermodynamic temperature scale, also called absolute temperature. If it were possible to cool a system to absolute zero, all macroscopic motion of the particles comprising matter would cease and they would be at complete rest in this *classical* sense. Microscopically in the description of quantum mechanics, however, matter still has zero-point energy even at absolute zero, because of the uncertainty principle.

Absolute zero is defined as a temperature of precisely 0 kelvins, which is equal to −273.15 °C or −459.68 °F.

Use in science



A map of global long term monthly average surface air temperatures in Mollweide projection.

Temperature plays an important role in all fields of natural science, including physics, geology, chemistry, atmospheric sciences and biology.

Many physical properties of materials including the phase (solid, liquid, gaseous or plasma), density, solubility, vapor pressure, and electrical conductivity depend on the temperature. Temperature also plays an important role in determining the rate and extent to which chemical reactions occur. This is one reason why the human body has several elaborate mechanisms for maintaining the temperature at 310 K, since temperatures only a few degrees higher can result in harmful reactions with serious consequences. Temperature also controls the thermal radiation emitted from a surface. One application of this effect is the incandescent light bulb, in which a tungsten filament is electrically heated to a temperature at which significant quantities of visible light are emitted.

For study and analysis, real world systems are usually spatially and temporally divided into cells of small size, in which classical thermodynamical equilibrium conditions for matter are fulfilled to good approximation (local thermodynamic equilibrium).

Foundation

The principles of thermodynamics and statistical mechanics provide a framework for the formal description of the properties of temperature. Temperature is defined empirically by the zeroth law of thermodynamics and has additional properties given in the second law. Statistical mechanics provides a detailed microscopic explanation based on the probabilistic properties of large ensembles of particles.

Temperature is a property determined by the average of a property of many particles and is defined in a state of at least local thermodynamic equilibrium, unlike other thermodynamic quantities, such as entropy and heat, whose microscopic definitions are valid even far away from equilibrium.

As a system receives heat, its temperature rises; similarly, a loss of heat from the system decreases its temperature. When two systems are at the same temperature, no heat transfer occurs between them. When a temperature difference does exist, heat will transfer from the warmer system to the colder system, until they are at thermal equilibrium. This transfer occurs via heat conduction.

Statistically, temperature (T) is a direct measure of the mean kinetic energy of the particles forming a sample of matter. For each degree of freedom that a particle possesses, the mean kinetic energy (E_k) of the particles is:

$$\overline{E}_{\mathrm{k}} = \tfrac{1}{2}kT,$$

where k is the Boltzmann constant, a fixed proportionality factor introduced by the system of units used to measure energy and temperature.

Macroscopically, temperature is related to the amount of internal energy and enthalpy of a system: the higher the temperature of a system, the higher its internal energy and enthalpy. For a system in thermal equilibrium at a constant volume, temperature is thermodynamically defined in terms of its energy (E) and entropy (S) as:

$$T \equiv \frac{\partial E}{\partial S}$$

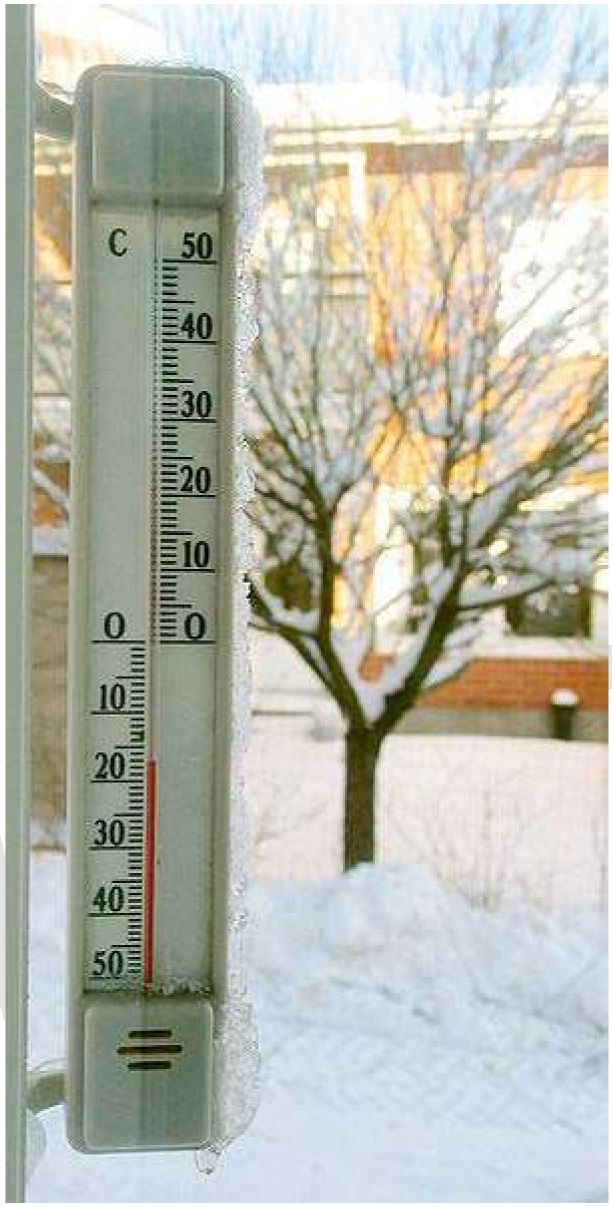
Temperature is an intensive property of a system, meaning that it does not depend on the system size, the amount or type of material in the system, the same as for the pressure and density. By contrast, mass, volume, and entropy are extensive properties, and depend on the amount of material in the system.

Heat capacity

When a sample is heated, meaning it receives thermal energy from an external source, some of the introduced heat is converted into kinetic energy, the rest to other forms of internal energy, specific to the material. The amount converted into kinetic energy causes the temperature of the material to rise. The introduced heat (ΔQ) divided by the observed temperature change is the heat capacity (C) of the material.

$$C = \frac{\Delta Q}{\Delta T}$$

If heat capacity is measured for a well defined amount of substance, the specific heat is the measure of the heat required to increase the temperature of such a unit quantity by one unit of temperature. For example, to raise the temperature of water by one kelvin (equal to one degree Celsius) requires 4186 joules per kilogram (J/kg).



Bulk water freezes at 0°C. The frost shown here is at -17 °C

Temperature measurement

Temperature measurement using modern scientific thermometers and temperature scales goes back at least as far as the early 18th century, when Gabriel Fahrenheit adapted a thermometer (switching to mercury) and a scale both developed by Ole Christensen Rømer. Fahrenheit's scale is still in use in the United States for non-scientific applications.

Temperature is measured with thermometers that may be calibrated to a variety of temperature scales. In most of the world (except for Belize, Myanmar, Liberia and the

United States), the Celsius scale is used for most temperature measuring purposes. Most scientist measures temperature using the Celsius scale and the thermodynamic temperature using the Kelvin scale, which is the Celsius scale offset so that its null point is 0K = −273.15°C, or absolute zero. Many engineering fields in the U.S., notably high-tech and US federal specifications (civil and military), also use the Kelvin and Celsius scales. Other engineering fields in the U.S. also rely upon the Rankine scale (a shifted Fahrenheit scale) when working in thermodynamic-related disciplines such as combustion.

Units

The basic unit of temperature in the International System of Units (SI) is the kelvin. It has the symbol K.

For everyday applications, it is often convenient to use the Celsius scale, in which 0°C corresponds to the freezing point of water and 100°C is its boiling point at sea level. Because liquid droplets commonly exist in clouds at sub-zero temperatures, 0°C is better defined as the freezing point of bulk water or the melting point of ice. In this scale a temperature difference of 1 degree is the same as a 1K temperature difference, so the Celsius scale is the same as the Kelvin scale, but is offset by the temperature at which water freezes (273.15 K).

By international agreement the Kelvin and Celsius scales are defined by two fixing points: absolute zero and the triple point of Vienna Standard Mean Ocean Water, which is water specially prepared with a specified blend of hydrogen and oxygen isotopes. Absolute zero is defined as precisely 0K and −273.15°C. It is the temperature at which all classical translational motion of the particles comprising matter ceases and they are at complete rest in the classical model. Quantum-mechanically, however, zero-point motion remains and has an associated energy, the zero-point energy. Matter is in its ground state, and contains no thermal energy. The triple point of water is defined as being precisely 273.16K and 0.01°C. This definition serves the following purposes: it fixes the magnitude of the kelvin as being precisely 1 part in 273.16 parts of the difference between absolute zero and the triple point of water; it establishes that one kelvin has precisely the same magnitude as one degree on the Celsius scale; and it establishes the difference between the null points of these scales as being 273.15K (0K = −273.15°C and 273.16K = 0.01°C).

In the United States, the Fahrenheit scale is widely used. On this scale the freezing point of water corresponds to 32 °F and the boiling point to 212 °F. The Rankine scale, still used in fields of chemical engineering in the U.S., is an absolute scale based on the Fahrenheit increment.

Conversion

The following table shows the temperature conversion formulas for conversions to and from the Celsius scale.

	from Celsius	to Celsius
Fahrenheit	[°F] = [°C] × $^9/_5$ + 32	[°C] = ([°F] − 32) × $^5/_9$
Kelvin	[K] = [°C] + 273.15	[°C] = [K] − 273.15
Rankine	[°R] = ([°C] + 273.15) × $^9/_5$	[°C] = ([°R] − 491.67) × $^5/_9$
Delisle	[°De] = (100 − [°C]) × $^3/_2$	[°C] = 100 − [°De] × $^2/_3$
Newton	[°N] = [°C] × $^{33}/_{100}$	[°C] = [°N] × $^{100}/_{33}$
Réaumur	[°Ré] = [°C] × $^4/_5$	[°C] = [°Ré] × $^5/_4$
Rømer	[°Rø] = [°C] × $^{21}/_{40}$ + 7.5	[°C] = ([°Rø] − 7.5) × $^{40}/_{21}$

Plasma physics

In the field of plasma physics, because of the high temperatures encountered and the electromagnetic nature of the phenomena involved, it is customary to express temperature in electronvolts (eV) or kiloelectronvolts (keV), where 1 eV = 11605K. In the study of QCD matter one routinely meets temperatures of the order of a few hundred MeV, equivalent to about 10^{12}K.

Theoretical foundation

Temperature in gases

For an ideal gas the kinetic theory of gases uses statistical mechanics to relate the temperature to the average kinetic energy of the atoms in the system. Temperature is related to the average kinetic energy of the particles in a gas, given by the relationship from classical mechanics:

$$E_{\mathrm{k}} = \tfrac{1}{2}mv^2,$$

where *m* is the particle mass and *v* its velocity. The distribution of energies (and thus speeds) of the particles in any gas are given by the Maxwell-Boltzmann distribution. The temperature of a classical ideal gas is related to its average kinetic energy via the equation:

$$\overline{E}_{\mathrm{k}} = \tfrac{1}{2}kT,$$

for each degree of freedom, where $k = R / n$ (n= Avogadro number, R= ideal gas constant). This relation is valid in the classical regime, i.e. when the particle density is much less than $1 / \Lambda^3$, where Λ is the thermal de Broglie wavelength. A monoatomic gas has only the three translational degrees of freedom.

The second law of thermodynamics states that any two given systems when interacting with each other will later reach the same average energy per particle (and hence the same temperature).

In a mixture of particles of various mass, the heaviest particles will move more slowly than lighter counterparts, but will still have the same average energy. A neon atom moves slower relative to a hydrogen molecule of the same kinetic energy; a pollen particle moves in a slow Brownian motion among fast moving water molecules. Particles with different mass have different velocity distributions, but the average kinetic energy is the same because of the ideal gas law.

Definition based on the zeroth law of thermodynamics

If two systems with fixed volumes are brought together in thermal contact, changes will most likely take place in the properties of both systems. These changes are caused by the transfer of heat between the systems. A state must be reached in which no further changes occur, to put the objects into thermal equilibrium.

A basis for the definition of temperature can therefore be obtained from the zeroth law of thermodynamics which empirically states that if two systems are each in thermal equilibrium with a third system, then they are also in thermal equilibrium with each other. Since all three are in thermal equilibrium, it is reasonable to postulate an empirical property that has the same value for all systems. This property is the temperature. The transitive relation of thermal equilibrium permits the construction of a thermometer to measure this property.

A temperature scale is based on the properties of some reference system to which other thermometers may be calibrated. One such reference system is a fixed quantity of gas. The ideal gas law indicates that the product of the pressure (*p*) and volume (*V*) of a gas is directly proportional to the temperature:

$$pV = nRT$$

where *T* is temperature, *n* is the number of moles of gas and R = 8.314472(15) $\mathrm{Jmol^{-1}K^{-1}}$ is the gas constant. Reformulating the pressure-volume term as the sum of classical mechanical particle energies in terms of particle mass, *m*, and root-mean-square particle speed *v*, the ideal gas law directly provides the relationship between kinetic energy and temperature:

$$\frac{1}{2}mv_{rms}^2 = \frac{3}{2}kT.$$

Thus, one can define a scale for temperature based on the corresponding pressure and volume of the gas: the temperature in kelvins is the pressure in pascals of one mole of gas in a container of one cubic metre, divided by the gas constant. In practice, such a gas thermometer is not very convenient, but other thermometers can be calibrated to this scale.

The pressure, volume, and the number of moles of a substance are all inherently greater than or equal to zero, suggesting that temperature must also be greater than or equal to zero. As a practical matter it is not possible to use a gas thermometer to measure absolute zero temperature since the gasses tend to condense into a liquid long before the temperature reaches zero. It is possible, however, to extrapolate to absolute zero by using the ideal gas law.

Temperature of a vacuum

It is possible to use the zeroth law definition of temperature to assign a temperature to something not normally associated with temperatures, like a perfect vacuum. Because all objects emit black body radiation, a thermometer in a vacuum away from thermally radiating sources will radiate away its own thermal energy; decreasing in temperature indefinitely until it reaches the zero-point energy limit. At that point it can be said to be in equilibrium with the vacuum and by definition at the same temperature. A gas that behaved ideally all the way down to absolute zero, obeying the kinetic theory of gases, would achieve zero kinetic energy per particle, and thereby achieve absolute zero temperature. Thus, by the zeroth law a perfect, isolated vacuum is at absolute zero temperature. Note that in order to behave ideally in this context it is necessary for the atoms of the gas to have no zero point energy. It will turn out not to matter that this is not possible because the second law definition of temperature will yield the same result for any unique vacuum state.

More realistically, no such ideal vacuum exists. For instance a thermometer in a vacuum chamber which is maintained at some finite temperature (say, chamber is in the lab at room temperature) will equilibrate with the thermal radiation it receives from the chamber and with time reaches the temperature of the chamber. If a thermometer orbiting the Earth is exposed to sunlight, then it equilibrates at the temperature at which power received by the thermometer from the Sun is exactly equal to the power radiated away by thermal radiation of the thermometer. For a black body this equilibrium temperature is about 281 K (+8 °C). Since Earth has an albedo of 30%, average temperature as seen from space is lower than for a black body, 254 K, while the surface temperature is considerably higher due to the greenhouse effect.

A thermometer isolated from solar radiation (in the shade of the Earth, for example) is still exposed to thermal radiation of Earth - thus will show some equilibrium temperature at which it receives and radiates equal amount of energy. If this thermometer is close to Earth then its equilibrium temperature is about 236 K (-37 °C) provided that Earth surface is at 281 K.

A thermometer far away from the Solar system still receives cosmic microwave background radiation. Equilibrium temperature of such thermometer is about 2.725 K, which is the temperature of a photon gas constituting black body microwave background radiation at present state of expansion of Universe. This temperature is sometimes referred to as the temperature of space. This temperature is thus similar to a test charge in that it facilitates a measure of the system even though temperature is not strictly defined there.

Definition based on the second law of thermodynamics

In the previous section temperature was defined in terms of the Zeroth Law of thermodynamics. It is also possible to define temperature in terms of the second law of thermodynamics, which deals with entropy. Entropy is a measure of the disorder in a system. The second law states that any process will result in either no change or a net increase in the entropy of the universe. This can be understood in terms of probability. Consider a series of coin tosses. A perfectly ordered system would be one in which either every toss comes up heads or every toss comes up tails. This means that for a perfectly ordered set of coin tosses, there is only one set of toss outcomes possible: the set in which 100% of tosses came up the same.

On the other hand, there are multiple combinations that can result in disordered or mixed systems, where some fraction are heads and the rest tails. A disordered system can be 90% heads and 10% tails, or it could be 98% heads and 2% tails, et cetera. As the number of coin tosses increases, the number of possible combinations corresponding to imperfectly ordered systems increases. For a very large number of coin tosses, the combinations to ~50% heads and ~50% tails dominates and obtaining an outcome significantly different from 50/50 becomes extremely unlikely. Thus the system naturally progresses to a state of maximum disorder or entropy.

It has been previously stated that temperature controls the flow of heat between two systems and was just shown that the universe tends to progress so as to maximize entropy (this is expected of any natural system). Thus, it is expected that there is some relationship between temperature and entropy. To find this relationship, the relationship between heat, work and temperature is first considered. A heat engine is a device for converting heat into mechanical work and analysis of the Carnot heat engine provides the necessary relationships. The work from a heat engine corresponds to the difference between the heat put into the system at the high temperature, q_H and the heat ejected at the low temperature, q_C. The efficiency is the work divided by the heat put into the system or:

$$\text{efficiency} = \frac{w_{cy}}{q_H} = \frac{q_H - q_C}{q_H} = 1 - \frac{q_C}{q_H} \quad (2)$$

where w_{cy} is the work done per cycle. The efficiency depends only on q_C/q_H. Because q_C and q_H correspond to heat transfer at the temperatures T_C and T_H, respectively, q_C/q_H should be some function of these temperatures:

$$\frac{q_C}{q_H} = f(T_H, T_C) \quad (3)$$

Carnot's theorem states that all reversible engines operating between the same heat reservoirs are equally efficient. Thus, a heat engine operating between T_1 and T_3 must have the same efficiency as one consisting of two cycles, one between T_1 and T_2, and the second between T_2 and T_3. This can only be the case if:

$$q_{13} = \frac{q_1}{o_7}$$

Since the first function is independent of T_2, this temperature must cancel on the right side, meaning $f(T_1,T_3)$ is of the form $g(T_1)/g(T_3)$ (i.e. $f(T_1,T_3) = f(T_1,T_2)f(T_2,T_3) = g(T_1)/g(T_2)\cdot g(T_2)/g(T_3) = g(T_1)/g(T_3)$), where g is a function of a single temperature. A temperature scale can now be chosen with the property that:

$$\frac{q_C}{q_H} = \frac{T_C}{T_H} \quad (4)$$

Substituting Equation 4 back into Equation 2 gives a relationship for the efficiency in terms of temperature:

$$\text{efficiency} = 1 - \frac{q_C}{q_H} = 1 - \frac{T_C}{T_H} \quad (5)$$

Notice that for $T_C = 0$ K the efficiency is 100% and that efficiency becomes greater than 100% below 0 K. Since an efficiency greater than 100% violates the first law of thermodynamics, this implies that 0 K is the minimum possible temperature. In fact the lowest temperature ever obtained in a macroscopic system was 20 nK, which was achieved in 1995 at NIST. Subtracting the right hand side of Equation 5 from the middle portion and rearranging gives:

$$\frac{q_H}{T_H} - \frac{q_C}{T_C} = 0$$

where the negative sign indicates heat ejected from the system. This relationship suggests the existence of a state function, S, defined by:

$$dS = \frac{dq_{\mathrm{rev}}}{T} \quad (6)$$

where the subscript indicates a reversible process. The change of this state function around any cycle is zero, as is necessary for any state function. This function corresponds to the entropy of the system, which was described previously. Rearranging Equation 6 gives a new definition for temperature in terms of entropy and heat:

$$T = \frac{dq_{\mathrm{rev}}}{dS} \quad (7)$$

For a system, where entropy $S(E)$ is a function of its energy E, the temperature T is given by:

$$T^{-1} = \frac{d}{dE}S(E) \quad (8),$$

i.e. the reciprocal of the temperature is the rate of increase of entropy with respect to energy.

Definition from statistical mechanics

The argument in the previous section is how the relation between entropy and heat was arrived at historically. Modern definition of temperature is given in statistical mechanics and it is defined in terms of the fundamental degrees of freedom of a system. Eq.(8) of the previous section is then taken to be the defining relation of the temperature. Eq. (7) can be derived from the definition of entropy.

Generalized temperature from single particle statistics

It is possible to extend the definition of temperature even to systems made of few particles, like in a quantum dot. The generalized temperature is obtained by considering time ensembles instead of configuration space ensembles given in statistical mechanics in the case of thermal and particle exchange between a small system of fermions (N even less than 10) with a single/double occupancy system. The finite quantum grand partition ensemble, obtained under the hypothesis of ergodicity and orthodicity, allows to express the generalized temperature from the ratio of the average time of occupation τ_1 and τ_2 of the single/double occupancy system :

$$T = k^{-1} ln 2 \frac{\tau_2}{\tau_1}(E - E_F(1 + \frac{3}{2N}))$$

where E_F is the Fermi energy which tends to the ordinary temperature when N goes to infinity.

Negative temperature

In the macroscopic sense, a negative temperature is one below the zero-point of the measurement system used. For example, a temperature of 100K is equivalent to −173.15°C. Temperatures of macroscopic systems may have negative values in the Celsius and Fahrenheit, but not in the Kelvin or Rankine scales.

In the statistical microscopic description of some electron and nuclear spin systems it is possible to obtain a negative temperature, which is numerically less than absolute zero. This is experimentally accomplished with suitable radio frequency techniques. In these cases the temperature function shows the behavior of a singularity. As the energy increases, the temperature trends to infinity and reverses polarity (to minus infinity) before converging towards zero.

In the regime of negative temperature the system exists in a condition in which the upper states are more populated than the lower state (population inversion). Therefore, a system with a negative temperature is not colder than absolute zero, but rather it has a higher energy than at positive temperature, and is in fact *hotter* at negative temperature. When brought into contact with a system at a positive temperature, energy will be transferred from the negative temperature regime to the positive temperature system.

Examples of temperature

	Temperature		Peak emittance wavelength of black-body radiation
	Kelvin	Degrees Celsius	
Absolute zero (precisely by definition)	0 K	−273.15 °C	Infinite
Coldest measured temperature	450 pK	−273.14999999955 °C	6,400 km
One millikelvin (precisely by definition)	0.001 K	−273.149 °C	2.89777 m (radio, FM band)
Water's triple point (precisely by definition)	273.16 K	0.01 °C	10,608.3 nm (long wavelength I.R.)
Water's boiling point[A]	373.1339 K	99.9839 °C	7,766.03 nm (mid wavelength I.R.)
Incandescent lamp[B]	2500 K	≈2,200 °C	1,160 nm (near infrared)[C]
Sun's visible surface[D]	5,778 K	5,505 °C	501.5 nm (green-blue light)
Lightning bolt's channel[E]	28 kK	28,000 °C	100 nm (far ultraviolet light)
Sun's core[E]	16 MK	16 million °C	0.18 nm (X-rays)
Thermonuclear weapon (peak temperature)[E]	350 MK	350 million °C	8.3×10^{-3} nm (gamma rays)
Sandia National Labs' Z machine[E]	2 GK	2 billion °C	1.4×10^{-3} nm (gamma rays)[F]
Core of a high-mass star on its last day[E]	3 GK	3 billion °C	1×10^{-3} nm (gamma rays)
Merging binary neutron star system[E]	350 GK	350 billion °C	8×10^{-6} nm (gamma rays)
Relativistic Heavy	1 TK	1 trillion °C	3×10^{-6} nm

Ion Collider[E]			(gamma rays)
CERN's proton vs nucleus collisions[E]	10 TK	10 trillion °C	3×10^{-7} nm (gamma rays)
Universe 5.391×10^{-44} s after the Big Bang[E]	1.417×10^{32} K	1.417×10^{32} °C	1.616×10^{-26} nm (Planck frequency)

- [A] For Vienna Standard Mean Ocean Water at one standard atmosphere (101.325 kPa) when calibrated strictly per the two-point definition of thermodynamic temperature.
- [B] The 2500 K value is approximate. The 273.15 K difference between K and °C is rounded to 300 K to avoid false precision in the Celsius value.
- [C] For a true black-body (which tungsten filaments are not). Tungsten filaments' emissivity is greater at shorter wavelengths, which makes them appear whiter.
- [D] Effective photosphere temperature. The 273.15 K difference between K and °C is rounded to 273 K to avoid false precision in the Celsius value.
- [E] The 273.15 K difference between K and °C is without the precision of these values.
- [F] For a true black-body (which the plasma was not). The Z machine's dominant emission originated from 40 MK electrons (soft x–ray emissions) within the plasma.

Temperature dependencies

Effect of temperature

Temperature	Speed of sound	Density of air	Acoustic impedance
ϑ in °C	c in m·s^{-1}	ρ in kg·m^{-3}	Z in N·s·m^{-3}
+35	351.96	1.1455	403.2
+30	349.08	1.1644	406.5
+25	346.18	1.1839	409.4
+20	343.26	1.2041	413.3
+15	340.31	1.2250	416.9
+10	337.33	1.2466	420.5
+5	334.33	1.2690	424.3
±0	331.30	1.2920	428.0
-5	328.24	1.3163	432.1
-10	325.16	1.3413	436.1
-15	322.04	1.3673	440.3
-20	318.89	1.3943	444.6
-25	315.72	1.4224	449.1

Chapter- 6

Base Units

1. Metre

1 metre =	
SI units	
100 cm	1000 mm
US customary / Imperial units	
3.2808 ft	39.370 in

The **metre** (or **meter**), symbol **m**, is the base unit of length in the International System of Units (SI). Originally intended to be one ten-millionth of the distance from the Earth's equator to the North Pole (at sea level), its definition has been periodically refined to reflect growing knowledge of metrology. Since 1983, it is defined as the distance travelled by light in vacuum in $^{1}/_{299,792,458}$ of a second.

History

Name

The first recorded proposal for a decimal-based unit of length was the *universal measure* unit proposed by the English philosopher John Wilkins in 1668. In 1675 the Italian scientist Tito Livio Burattini, in his work *Misura Universale*, used the words *metro cattolico* (*metre catholic*) which was derived from the Greek μέτρον καθολικόν (*métron katholikón*), "a universal measure". This word gave rise to the French *mètre* which in 1797 was introduced into the English language.

Meridional definition

In 1668 Wilkins proposed using Christopher Wren's suggestion of a pendulum with a half-period of one second to measure a standard length that Christiaan Huygens had observed to be 38 Rhineland or 39¼ English inches (997 mm) in length.

In the eighteenth century, there were two favoured approaches to the definition of the standard unit of length. One approach followed Wilkins in defining the metre as the length of a pendulum with a half-period of one second, a 'seconds pendulum'. The other approach suggested defining the metre as one ten-millionth of the length of the Earth's meridian along a quadrant, that is the distance from the Equator to the North Pole. In 1791, the French Academy of Sciences selected the meridional definition over the pendular definition because the force of gravity varies slightly over the surface of the Earth, which affects the period of a pendulum.

In order to establish a universally accepted foundation for the definition of the metre, measurements of this meridian more accurate than those available at that time were imperative. The French Academy of Sciences commissioned an expedition led by Jean Baptiste Joseph Delambre and Pierre Méchain, lasting from 1792 to 1799, which measured the length of the meridian arc between Dunkerque and Barcelona. This portion of the meridian, which also passes near Paris, was to serve as the basis for the length of the half meridian, connecting the North Pole with the Equator. The exact shape of the Earth is not a simple mathematical shape (sphere or ellipse) at the level of precision required for defining a standard of length. The irregular and particular shape of the Earth (smoothed to sea level) is called a Geoid, which means "Earth-shaped".

However, in 1793, France adopted as its official unit of length a metre based on provisional results from the expedition. Although it was later determined that the first prototype metre bar was short by a fifth of a millimetre because of miscalculation of the flattening of the Earth, this length became the standard. The circumference of the Earth through the poles is therefore slightly more than forty million metres.

Prototype metre bar

Historical *International Prototype Metre* bar, made of an alloy of platinum and iridium, that was the standard from 1889 to 1960.

In the 1870s and in light of modern precision, a series of international conferences was held to devise new metric standards. The Metre Convention (Convention du Mètre) of 1875 mandated the establishment of a permanent International Bureau of Weights and Measures (BIPM: Bureau International des Poids et Mesures) to be located in Sèvres, France. This new organisation would preserve the new prototype metre and kilogram standards when constructed, distribute national metric prototypes, and maintain comparisons between them and non-metric measurement standards. The organisation created a new prototype bar in 1889 at the first General Conference on Weights and Measures (CGPM: Conférence Générale des Poids et Mesures), establishing the *International Prototype Metre* as the distance between two lines on a standard bar composed of an alloy of ninety percent platinum and ten percent iridium, measured at the melting point of ice.

The original international prototype of the metre is still kept at the BIPM under the conditions specified in 1889. A discussion of measurements of a standard metre bar and the errors encountered in making the measurements is found in a NIST document.

Standard wavelength of krypton-86 emission

In 1893, the standard metre was first measured with an interferometer by Albert A. Michelson, the inventor of the device and an advocate of using some particular wavelength of light as a standard of distance. By 1925, interferometry was in regular use at the BIPM. However, the International Prototype Metre remained the standard until 1960, when the eleventh CGPM defined the metre in the new SI system as equal to 1,650,763.73 wavelengths of the orange-red emission line in the electromagnetic spectrum of the krypton-86 atom in a vacuum.

Speed of light

To further reduce uncertainty, the seventeenth CGPM in 1983 replaced the definition of the metre with its current definition, thus fixing the length of the metre in terms of the second and the speed of light:

The metre is the length of the path travelled by light in vacuum during a time interval of $^{1}\!/_{299\,792\,458}$ of a second.

This definition fixed the speed of light in a vacuum at precisely 299,792,458 metres per second. Although the metre is now *defined* as the distance travelled by light in a given time, actual laboratory realisations of the metre are still *delineated* by counting the required number of wavelengths of light along the distance. Three major factors limit the accuracy attainable with laser interferometers:

- Uncertainty in vacuum wavelength of the source,
- Uncertainty in the refractive index of the medium,
- Laser count resolution of the interferometer.

Use of the interferometer to define the metre is based upon the relation:

$$\lambda = \frac{c}{nf} ,$$

where λ is the determined wavelength; c is the speed of light in ideal vacuum; n is the refractive index of the medium in which the measurement is made; and f is the frequency of the source. In this way the length is related to one of the most accurate measurements available: frequency.

An intended byproduct of the 17th CGPM's definition was that it enabled scientists to measure the wavelength of their lasers with one-fifth the uncertainty. To further facilitate reproducibility from lab to lab, the 17th CGPM also made the iodine-stabilised helium-neon laser "a recommended radiation" for realising the metre. For purposes of delineating the metre, the BIPM currently considers the HeNe laser wavelength to be as follows: λ_{HeNe} = 632.99139822 nm with an estimated relative standard uncertainty (U) of 2.5×10^{-11}. This uncertainty is currently the limiting factor in laboratory realisations of the metre as it is several orders of magnitude poorer than that of the second ($U = 5\times10^{-16}$). Consequently, a practical realisation of the metre is usually delineated (not defined) today in labs as 1,579,800.298728(39) wavelengths of helium-neon laser light in a vacuum.

Timeline of definition

- 1790 May 8 – The French National Assembly decides that the length of the new metre would be equal to the length of a pendulum with a half-period of one second.
- 1791 March 30 – The French National Assembly accepts the proposal by the French Academy of Sciences that the new definition for the metre be equal to one ten-millionth of the length of the Earth's meridian along a quadrant through Paris, that is the distance from the equator to the north pole.
- 1795 – Provisional metre bar constructed of brass.
- 1799 December 10 – The French National Assembly specifies the platinum metre bar, constructed on 23 June 1799 and deposited in the National Archives, as the final standard.
- 1889 September 28 – The first General Conference on Weights and Measures (CGPM) defines the metre as the distance between two lines on a standard bar of an alloy of platinum with ten percent iridium, measured at the melting point of ice.
- 1927 October 6 – The seventh CGPM adjusts the definition of the metre to be the distance, at 0 °C, between the axes of the two central lines marked on the prototype bar of platinum-iridium, this bar being subject to one standard atmosphere of pressure and supported on two cylinders of at least one centimetre diameter, symmetrically placed in the same horizontal plane at a distance of 571 millimetres from each other.
- 1960 October 14 – The eleventh CGPM defines the metre to be equal to 1,650,763.73 wavelengths in vacuum of the radiation corresponding to the transition between the $2p^{10}$ and $5d^{5}$ quantum levels of the krypton-86 atom.
- 1983 October 21 – The seventeenth CGPM defines the metre as equal to the distance travelled by light in vacuum during a time interval of $^{1}/_{299,792,458}$ of a second.

- 2002 – The International Committee for Weights and Measures (CIPM) considers the metre to be a unit of proper length and thus recommends this definition be restricted to "lengths ℓ which are sufficiently short for the effects predicted by general relativity to be negligible with respect to the uncertainties of realisation."

Definitions of the metre since 1795

Basis of definition	**Date**	**Absolute uncertainty**	**Relative uncertainty**
1/10000000 part of the quarter of a meridian, measurement by Delambre and Mechain	1795	0.5–0.1 mm	10^{-4}
First prototype *Metre des Archives* platinum bar standard	1799	0.05–0.01 mm	10^{-5}
Platinum-iridium bar at melting point of ice (1st CGPM)	1889	0.2–0.1 µm	10^{-7}
Platinum-iridium bar at melting point of ice, atmospheric pressure, supported by two rollers (7th CGPM)	1927	n.a.	n.a.
Hyperfine atomic transition; 1650763.73 wavelengths of light from a specified transition in Krypton 86 (11th CGPM)	1960	0.01–0.005 µm	10^{-8}
Distance traversed in vacuum by light in 1/299792458 of a second (17th CGPM)	1983	0.1 nm	10^{-10}

SI prefixed forms of metre

SI prefixes are often employed to denote decimal multiples and submultiples of the metre, as shown in the table below. As indicated in the table, some are commonly used, while others are not. Long distances are usually expressed in km, astronomical units, light-years, or parsecs, rather than in Mm, Gm, Tm, Pm, Em, Zm or Ym; "30 cm", "30 m", and "300 m" are more common than "3 dm", "3 dam", and "3 hm", respectively.

SI multiples for metre (m)

Submultiples			**Multiples**		
Value	**Symbol**	**Name**	**Value**	**Symbol**	**Name**
10^{-1} m	Dm	decimetre	10^{1} m	dam	decametre
10^{-2} m	**Cm**	**centimetre**	10^{2} m	hm	hectometre

10^{-3} m	**Mm**	**millimetre**	10^{3} m	**km**	**kilometre**
10^{-6} m	**µm**	**micrometre**	10^{6} m	Mm	megametre
10^{-9} m	**Nm**	**nanometre**	10^{9} m	Gm	gigametre
10^{-12} m	Pm	picometre	10^{12} m	Tm	terametre
10^{-15} m	Fm	femtometre	10^{15} m	Pm	petametre
10^{-18} m	Am	attometre	10^{18} m	Em	exametre
10^{-21} m	Zm	zeptometre	10^{21} m	Zm	zettametre
10^{-24} m	Ym	yoctometre	10^{24} m	Ym	yottametre

Common prefixed units are in **bold** face.

The term *micron* is often used instead of *micrometre*, but this practice is officially discouraged.

Spelling

Metre is used as the spelling of the metric unit for length in all English speaking nations except the USA.

The most recent official brochure, written in 2006, about the International System of Units (SI), *Bureau international des poids et mesures*, was written in French by the International Bureau of Weights and Measures. An English translation (using the spelling: *metre*) is included to make the SI standard "more widely accessible".

In 2008, the U.S. English translation published by the U.S. National Institute of Standards and Technology chose to use *meter* in accordance with the United States Government Printing Office Style Manual.

Measuring devices (such as parking meter, speedometer) are traditionally spelt "...meter" in all countries. The word "meter", signifying any such device, has the same derivation as the word "metre", denoting the unit of length this article is about.

Equivalents in other units

Metric unit expressed in non-SI units			**Non-SI unit expressed in metric units**		
1 metre	≈	39.37 inches	1 inch	≡	0.0254 metres
1 centimetre	≈	0.3937 inch	1 inch	≡	2.54 centimetres
1 millimetre	≈	0.03937 inch	1 inch	≡	25.4 millimetres
1 metre	≡	1×10^{10} Ångström	1 Ångström	≡	1×10^{-10} metre
1 nanometre	≡	10 Ångström	1 Ångström	≡	100 picometres

Within this table, "inch" means "international inch".

"≈" means "is approximately equal to".

"≡" means "equals by definition" or equivalently, "is exactly equal to".

A simple mnemonic aid exists to assist with conversion;

1 metre is equivalent to **3** feet, **3** and **3**/8 inches. *This gives an over-estimate of 0.125 mm*.

2. Second

The **second** (SI symbol: **s**), sometimes abbreviated **sec.**, is a unit of time, and is the International System of Units (SI) base unit of time. It may be measured using a clock.

Early definitions of the second were based on the apparent motion of the sun around the earth. The solar day was divided into 24 hours, each of which contained 60 minutes of 60 seconds each, so the second was $^{1}/_{86\,400}$ of the mean solar day. However, nineteenth- and twentieth-century astronomical observations revealed that this average time is lengthening, and thus the sun/earth motion is no longer considered a suitable basis for definition. With the advent of atomic clocks, it became feasible to define the second based on fundamental properties of nature. Since 1967, the second has been defined to be

the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

SI prefixes are frequently combined with the word *second* to denote subdivisions of the second, *e.g.*, the millisecond (one thousandth of a second), the microsecond (one millionth of a second), and the nanosecond (one billionth of a second). Though SI prefixes may also be used to form multiples of the second such as kilosecond (one thousand seconds), such units are rarely used in practice. The more common larger non-SI units of time are not formed by powers of ten; instead, the second is multiplied by 60 to form a minute, which is multiplied by 60 to form an hour, which is multiplied by 24 to form a day.

The second is also the base unit of time in the centimetre-gram-second, metre-kilogram-second, metre-tonne-second, and foot-pound-second systems of units.

International second

Under the International System of Units, the second is currently defined as

The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom.

This definition refers to a caesium atom at rest at a temperature of 0 K (absolute zero), and with appropriate corrections for gravitational time dilation. The ground state is defined at zero electric and magnetic fields. The second thus defined is consistent with the ephemeris second, which was based on astronomical measurements.

The realization of the standard second is described briefly in a special publication from the National Institute of Science and Technology, and in detail by the National Research Council of Canada.

Equivalence to other units of time

1 international second is equal to:

- 1/60 minute
- 1/3,600 hour
- 1/86,400 day (IAU system of units)
- 1/31,557,600 Julian year (IAU system of units)

History

Before mechanical clocks

The Egyptians subdivided daytime and nighttime into twelve hours each since at least 2000 BC, hence the seasonal variation of their hours. The Hellenistic astronomers Hipparchus (*c.* 150 BC) and Ptolemy (*c.* AD 150) subdivided the day sexagesimally and also used a mean hour ($\frac{1}{24}$ day), simple fractions of an hour ($\frac{1}{4}$, $\frac{2}{3}$, etc.) and time-degrees ($\frac{1}{360}$ day or four modern minutes), but not modern minutes or seconds.

The day was subdivided sexagesimally, that is by $\frac{1}{60}$, by $\frac{1}{60}$ of that, by $\frac{1}{60}$ of that, etc., to at least six places after the sexagesimal point (a precision of less than 2 microseconds) by the Babylonians after 300 BC, but they did not sexagesimally subdivide smaller units of time. For example, six fractional sexagesimal places of a day was used in their specification of the length of the year, although they were unable to measure such a small fraction of a day in real time. As another example, they specified that the mean synodic month was 29;31,50,8,20 days (four fractional sexagesimal positions), which was repeated by Hipparchus and Ptolemy sexagesimally, and is currently the mean synodic month of the Hebrew calendar, though restated as 29 days 12 hours 793 halakim (where 1 hour = 1080 halakim). The Babylonians did not use the hour, but did use a double-hour lasting 120 modern minutes, a time-degree lasting four modern minutes, and a barleycorn lasting $3\frac{1}{3}$ modern seconds (the *helek* of the modern Hebrew calendar).

In 1000, the Persian scholar al-Biruni gave the times of the new moons of specific weeks as a number of days, hours, minutes, seconds, thirds, and fourths after noon Sunday. In 1267, the medieval scientist Roger Bacon stated the times of full moons as a number of hours, minutes, seconds, thirds, and fourths (*horae*, *minuta*, *secunda*, *tertia*, and *quarta*) after noon on specified calendar dates. Although a *third* for $^1\!/_{60}$ of a second remains in some languages, for example Polish (*tercja*) and Turkish (*salise*), the modern second is subdivided decimally.

Seconds measured by mechanical clocks

The earliest clocks to display seconds appeared during the last half of the 16th century. The earliest spring-driven timepiece with a second hand which marked seconds is an unsigned clock depicting Orpheus in the Fremersdorf collection, dated between 1560 and 1570.[:417–418] During the 3rd quarter of the 16th century, Taqi al-Din built a clock with marks every five seconds. In 1579, Jost Bürgi built a clock for William of Hesse that marked seconds.[:105] In 1581, Tycho Brahe redesigned clocks that displayed minutes at his observatory so they also displayed seconds. In 1587 he complained that his four clocks disagreed by plus or minus four seconds.[:104]

The second first became accurately measurable with the development of pendulum clocks keeping *mean time* (as opposed to the *apparent time* displayed by sundials), specifically in 1670 when William Clement added a seconds pendulum to the original pendulum clock of Christian Huygens. The seconds pendulum has a period of two seconds, one second for a swing forward and one second for a swing back, enabling the longcase clock incorporating it to tick seconds. From this time, a second hand that rotated once per minute in a small subdial began to be added to the clock faces of precision clocks.

Modern measurements

In 1956 the second was defined in terms of the period of revolution of the Earth around the Sun for a particular epoch, because by then it had become recognized that the Earth's rotation on its own axis was not sufficiently uniform as a standard of time. The Earth's motion was described in Newcomb's Tables of the Sun (1895), which provide a formula estimating the motion of the Sun relative to the epoch 1900 based on astronomical observations made between 1750 and 1892. The second thus defined is

the fraction 1/31,556,925.9747 of the tropical year for 1900 January 0 at 12 hours ephemeris time.

This definition was ratified by the *Eleventh General Conference on Weights and Measures* in 1960. The *tropical year* in the definition was not measured, but calculated from a formula describing a mean tropical year that decreased linearly over time, hence the curious reference to a specific *instantaneous* tropical year. This definition of the second was in conformity with the ephemeris time scale adopted by the IAU in 1952, defined as the measure of time that brings the observed positions of the celestial bodies into accord with the Newtonian dynamical theories of their motion (those accepted for use during most of the twentieth century being Newcomb's Tables of the Sun, used from 1900 through 1983, and Brown's Tables of the Moon, used from 1923 through 1983).

With the development of the atomic clock, it was decided to use atomic clocks as the basis of the definition of the second, rather than the revolution of the Earth around the Sun.

Following several years of work, Louis Essen from the National Physical Laboratory (Teddington, England) and William Markowitz from the United States Naval Observatory (USNO) determined the relationship between the hyperfine transition frequency of the caesium atom and the ephemeris second. Using a common-view measurement method based on the received signals from radio station WWV, they determined the orbital motion of the Moon about the Earth, from which the apparent motion of the Sun could be inferred, in terms of time as measured by an atomic clock. They found that the second of ephemeris time (ET) had the duration of 9,192,631,770 ± 20 cycles of the chosen caesium frequency. As a result, in 1967 the Thirteenth General Conference on Weights and Measures defined the second of atomic time in the International System of Units as

the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.

This SI second, referred to atomic time, was later verified to be in agreement, within 1 part in 10^{10}, with the second of ephemeris time as determined from lunar observations. (Nevertheless, this SI second was already, when adopted, a little shorter than the then-current value of the second of mean solar time.)

During the 1970s it was realized that gravitational time dilation caused the second produced by each atomic clock to differ depending on its altitude. A uniform second was produced by correcting the output of each atomic clock to mean sea level (the rotating geoid), lengthening the second by about 1×10^{-10}. This correction was applied at the beginning of 1977 and formalized in 1980. In relativistic terms, the SI second is defined as the proper time on the rotating geoid.

The definition of the second was later refined at the 1997 meeting of the BIPM to include the statement

This definition refers to a caesium atom at rest at a temperature of 0 K.

The revised definition seems to imply that the ideal atomic clock contains a single caesium atom at rest emitting a single frequency. In practice, however, the definition means that high-precision realizations of the second should compensate for the effects of the ambient temperature (black-body radiation) within which atomic clocks operate, and extrapolate accordingly to the value of the second at a temperature of absolute zero.

Today, the atomic clock operating in the microwave region is challenged by atomic clocks operating in the optical region. To quote Ludlow *et al.* “In recent years, optical atomic clocks have become increasingly competitive in performance with their microwave counterparts. The overall accuracy of single trapped ion based optical standards closely approaches that of the state-of-the-art caesium fountain standards. Large ensembles of ultracold alkaline earth atoms have provided impressive clock stability for short averaging times, surpassing that of single-ion based systems. So far, interrogation of neutral atom based optical standards has been carried out primarily in free space, unavoidably including atomic motional effects that typically limit the overall system accuracy. An alternative approach is to explore the ultranarrow optical transitions of atoms held in an optical lattice. The atoms are tightly localized so that Doppler and photon-recoil related effects on the transition frequency are eliminated."

The NRC attaches a "relative uncertainty" of 2.5×10^{-11} (limited by day-to-day and device-to-device reproducibility) to their atomic clock based upon the $^{127}I_2$ molecule, and is advocating use of an ^{88}Sr ion trap instead (relative uncertainty due to linewidth of 2.2×10^{-15}).

SI multiples

SI prefixes are commonly used to measure time less than a second, but rarely for multiples of a second. Instead, the non-SI units minutes, hours, days, Julian years, Julian centuries, and Julian millennia are used.

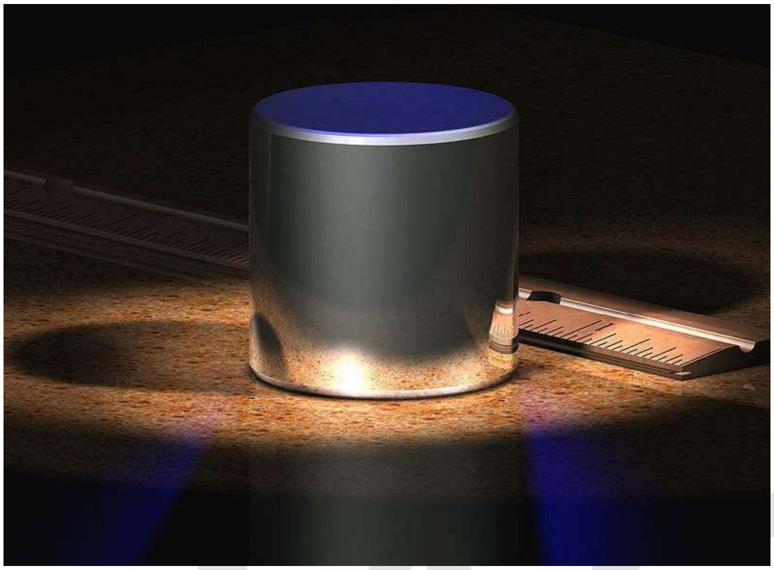
SI multiples for second (s)

Submultiples			Multiples		
Value	**Symbol**	**Name**	**Value**	**Symbol**	**Name**
10^{-1} s	ds	decisecond	10^{1} s	das	decasecond
10^{-2} s	cs	centisecond	10^{2} s	hs	hectosecond
10^{-3} s	**ms**	**millisecond**	10^{3} s	ks	kilosecond
10^{-6} s	**µs**	**microsecond**	10^{6} s	Ms	megasecond
10^{-9} s	**ns**	**nanosecond**	10^{9} s	Gs	gigasecond
10^{-12} s	ps	picosecond	10^{12} s	Ts	terasecond
10^{-15} s	fs	femtosecond	10^{15} s	Ps	petasecond
10^{-18} s	as	attosecond	10^{18} s	Es	exasecond
10^{-21} s	zs	zeptosecond	10^{21} s	Zs	zettasecond
10^{-24} s	ys	yoctosecond	10^{24} s	Ys	yottasecond

Common prefixes are in bold

3. Kilogram

Kilogram



A computer-generated image of the *International Prototype Kilogram* ("IPK"). The IPK *is* the kilogram. It sits next to an inch-based ruler for scale. The IPK is made of a platinum-iridium alloy and is stored in a vault at the BIPM in Sèvres, France. Like the other prototypes, the edges of the IPK have a four-angle chamfer to minimize wear.

Standard:	SI base unit
Quantity:	Mass
Symbol:	kg
Expressed in:	1 kg =
Natural units	$4.59467(23)\times10^{7}$ Planck masses
Energy	$1.356392733(68)\times10^{50}$ hertz
	89,875,517,873,681,764 joules (precisely)
U.S. customary	≈ 2.204622622 pounds-avoirdupois

The **kilogram** (symbol: kg) is the base unit of mass in the International System of Units (**SI**, from the French *Le **S**ystème **I**nternational d'Unités*), which is the modern standard governing the metric system. The kilogram is defined as being equal to the mass of the *International Prototype Kilogram* (**IPK**), which is almost exactly equal to the mass of one liter of water. It is the only SI base unit with an SI prefix as part of its name. It is also the only SI unit that is still defined by an artifact rather than a fundamental physical property that can be reproduced in different laboratories.

In everyday usage, the mass of an object, which is measured in kilograms, is often referred to as its weight. However, the term *weight* in strict scientific contexts refers to the gravitational force of an object. Throughout most of the world, force is measured with the SI unit newton and the non-SI unit kilogram-force. Similarly, the avoirdupois (or *international*) pound, used in both the Imperial system and U.S. customary units, is a unit of mass and its related unit of force is the pound-force. The avoirdupois pound is defined as exactly 0.45359237 kg, making one kilogram approximately equal to 2.2046 avoirdupois pounds.

Many units in the SI system are defined relative to the kilogram so its stability is important. After the International Prototype Kilogram had been found to vary in mass over time, the International Committee for Weights and Measures (known also by its French-language initials CIPM) recommended in 2005 that the kilogram be redefined in terms of a fundamental constant of nature. No final decision is expected before 2011.

Nature of mass

The chains on the swing hold all the child’s weight. If one were to stand behind her at the bottom of the arc and try to stop her, one would be acting against her inertia, which arises purely from mass, not weight.

The kilogram is a unit of mass, the measurement of which corresponds to the general, everyday notion of how “heavy” something is. However, mass is actually an *inertial* property; that is, the tendency of an object to remain at constant velocity unless acted upon by an outside force. According to Sir Isaac Newton's 323-year-old laws of motion and an important formula that sprang from his work, $F = ma$, an object with a mass, m, of one kilogram will accelerate, a, at one meter per second per second (about one-tenth the acceleration due to earth's gravity) when acted upon by a force, F, of one newton.

While the *weight* of matter is entirely dependent upon the strength of gravity, the *mass* of matter is invariant. Accordingly, for astronauts in microgravity, no effort is required to hold objects off the cabin floor; they are "weightless". However, since objects in microgravity still retain their mass and inertia, an astronaut must exert ten times as much force to accelerate a 10-kilogram object at the same rate as a 1-kilogram object.

On earth, a common swing set can demonstrate the relationship of force, mass, and acceleration without being appreciably influenced by weight (downward force). If one were to stand behind a large adult sitting stationary in a swing and give him a strong push, the adult would accelerate relatively slowly and swing only a limited distance forwards before beginning to swing backwards. Exerting that same effort while pushing on a small child would produce much greater acceleration.

History

Early definitions

On 7 April 1795, the gram was decreed in France to be equal to "the absolute weight of a volume of water equal to the cube of the hundredth part of the meter, at the temperature of melting ice." The concept of using a specified volume of water to define a unit measure of mass was first advanced by the English philosopher John Wilkins in 1668.

Since trade and commerce typically involve items significantly more massive than one gram, and since a mass standard made of water would be inconvenient and unstable, the regulation of commerce necessitated the manufacture of a *practical realization* of the water-based definition of mass. Accordingly, a provisional mass standard was made as a single-piece, metallic artifact one thousand times more massive than the gram—the kilogram.

At the same time, work was commissioned to precisely determine the mass of a cubic decimeter (one liter) of water. Although the decreed definition of the kilogram specified water at 0 °C—its highly stable *temperature* point—the French chemist, Louis Lefèvre-Gineau and the Italian naturalist, Giovanni Fabbroni after several years of research chose to redefine the standard in 1799 to water's most stable *density* point: the temperature at which water reaches maximum density, which was measured at the time as 4 °C. They concluded that one cubic decimeter of water at its maximum density was equal to 99.9265% of the target mass of the provisional kilogram standard made four years earlier. That same year, 1799, an all-platinum kilogram prototype was fabricated with the objective that it would equal, as close as was scientifically feasible for the day, the mass of one cubic decimeter of water at 4 °C. The prototype was presented to the Archives of the Republic in June and on 10 December 1799, the prototype was formally ratified as the *Kilogramme des Archives* (Kilogram of the Archives) and the kilogram was defined as being equal to its mass. This standard stood for the next ninety years.

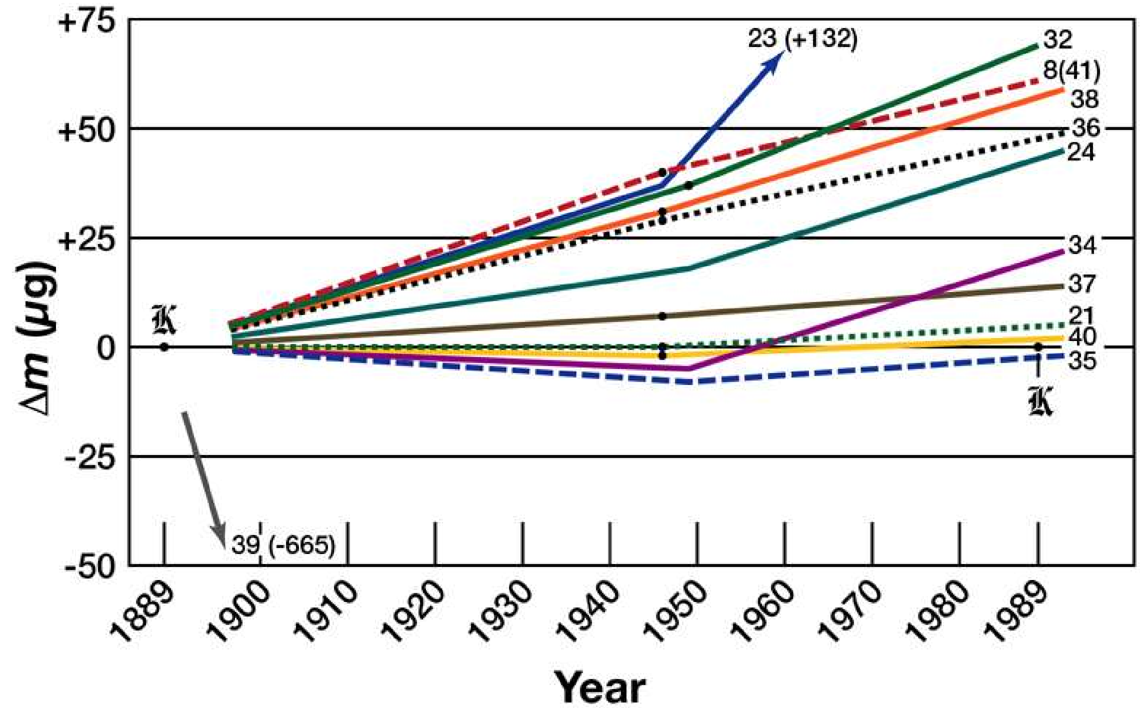
International Prototype Kilogram

Since 1889, the SI system defines the magnitude of the kilogram to be equal to the mass of the *International Prototype Kilogram*, often referred to in the professional metrology world as the "IPK". The IPK is made of a platinum alloy known as "Pt-10Ir", which is 90% platinum and 10% iridium (by mass) and is machined into a right-circular cylinder (height = diameter) of 39.17 millimeters to minimize its surface area. The addition of 10% iridium improved upon the all-platinum Kilogram of the Archives by greatly increasing hardness while still retaining platinum's many virtues: extreme resistance to oxidation, extremely high density, satisfactory electrical and thermal conductivities, and low magnetic susceptibility. The IPK and its six sister copies are stored at the International Bureau of Weights and Measures (known by its French-language initials BIPM) in an environmentally monitored safe in the lower vault located in the basement of the BIPM's House of Breteuil in Sèvres on the outskirts of Paris. Three independently controlled keys are required to open the vault. Official copies of the IPK were made available to other nations to serve as their national standards. These are compared to the IPK roughly every 50 years.

The IPK is one of three cylinders made in 1879. In 1883, it was found to be indistinguishable from the mass of the Kilogram of the Archives made eighty-four years prior, and was formally ratified as *the* kilogram by the 1st CGPM in 1889.

Modern measurements of Vienna Standard Mean Ocean Water, which is pure distilled water with an isotopic composition representative of the average of the world's oceans, show it has a density of 0.999975 ±0.000001 kg/L at its point of maximum density (3.984 °C) under one standard atmosphere (760 torr) of pressure. Thus, a cubic decimeter of water at its point of maximum density is 25 parts per million less massive than the IPK. This small difference, and the fact that the mass of the IPK was indistinguishable from the mass of the Kilogram of the Archives, speak volumes of the scientists' skills over 211 years ago when making their measurements of water's properties and in manufacturing the Kilogram of the Archives.

Stability of the International Prototype Kilogram



Mass drift over time of national prototypes K21–K40, plus two of the IPK's sister copies: K32 and K8(41). All mass changes are relative to the IPK. The initial 1889 starting-value offsets relative to the IPK have been nulled. The above are all *relative* measurements; no historical mass-measurement data is available to determine which of the prototypes has been most stable relative to an invariant of nature. There is the distinct possibility that *all* the prototypes gained mass over 100 years and that K21, K35, K40, and the IPK simply *gained less* than the others.

By definition, the error in the measured value of the IPK's mass is exactly zero; the IPK *is* the kilogram. However, any changes in the IPK's mass over time can be deduced by comparing its mass to that of its official copies stored throughout the world, a process called "periodic verification." For instance, the U.S. owns four 90% platinum / 10% iridium (Pt-10Ir) kilogram standards, two of which, K4 and K20, are from the original batch of 40 replicas delivered in 1884. The K20 prototype was designated as the primary national standard of mass for the U.S. Both of these, as well as those from other nations, are periodically returned to the BIPM for verification.

Note that none of the replicas has a mass precisely equal to that of the IPK; their masses are calibrated and documented as offset values. For instance, K20, the U.S.'s primary standard, originally had an official mass of 1 kg − 39 micrograms (µg) in 1889; that is to say, K20 was 39 µg less than the IPK. A verification performed in 1948 showed a mass of 1 kg − 19 µg. The latest verification performed in 1999 shows a mass precisely identical to its original 1889 value. Quite unlike transient variations such as this, the

U.S.'s check standard, K4, has persistently declined in mass relative to the IPK—and for an identifiable reason. Check standards are used much more often than primary standards and are prone to scratches and other wear. K4 was originally delivered with an official mass of 1 kg − 75 µg in 1889, but as of 1989 was officially calibrated at 1 kg − 106 µg and ten years later was 1 kg − 116 µg. Over a period of 110 years, K4 lost 41 µg relative to the IPK.

Beyond the simple wear that check standards can experience, the mass of even the carefully stored national prototypes can drift relative to the IPK for a variety of reasons, some known and some unknown. Since the IPK and its replicas are stored in air (albeit under two or more nested bell jars), they gain mass through adsorption of atmospheric contamination onto their surfaces. Accordingly, they are cleaned in a process the BIPM developed between 1939 and 1946 known as "the BIPM cleaning method" that comprises lightly rubbing with a chamois soaked in equal parts ether and ethanol, followed by steam cleaning with bi-distilled water, and allowing the prototypes to settle for 7–10 days before verification. Cleaning the prototypes removes between 5 and 60 µg of contamination depending largely on the time elapsed since the last cleaning. Further, a second cleaning can remove up to 10 µg more. After cleaning—even when they are stored under their bell jars—the IPK and its replicas immediately begin gaining mass again. The BIPM even developed a model of this gain and concluded that it averaged 1.11 µg per month for the first 3 months after cleaning and then decreased to an average of about 1 µg per year thereafter. Since check standards like K4 are not cleaned for routine calibrations of other mass standards—a precaution to minimize the potential for wear and handling damage—the BIPM's model of time-dependent mass gain has been used as an "after cleaning" correction factor.

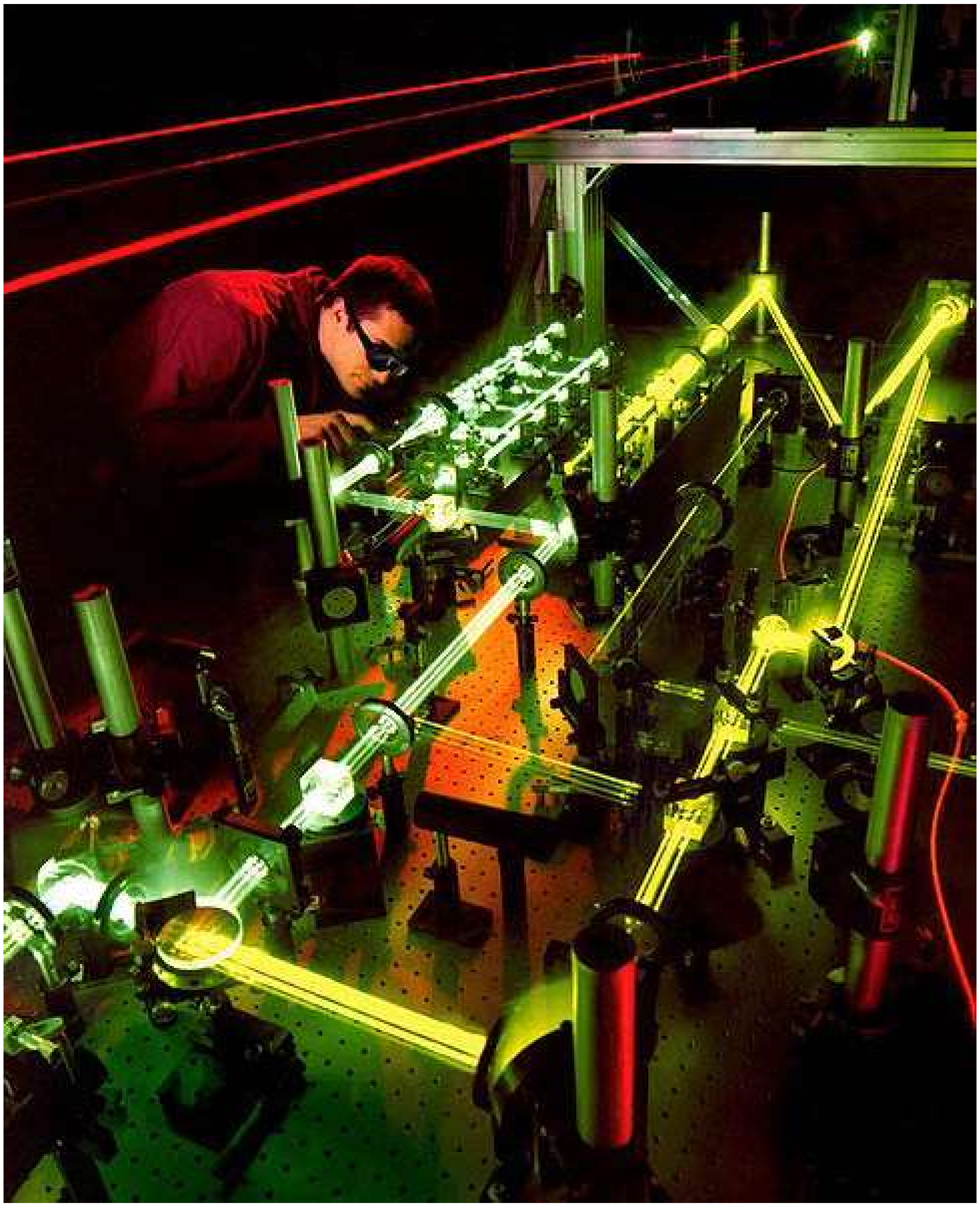
K48, above, came from the second batch of kilogram replicas to be produced. It was delivered to Denmark in 1949 with an official mass of 1 kg + 81 µg. Like all other replicas, it is stored under two nested bell jars virtually all the time. Still, its mass and that of the IPK diverged markedly in only 40 years; the mass of K48 was certified as 1 kg + 112 µg during the 1988–1992 periodic verification.

Because the first forty official copies are made of the same alloy as the IPK and are stored under similar conditions, periodic verifications using a large number of replicas—especially the national primary standards, which are rarely used—can convincingly demonstrate the stability of the IPK. What has become clear after the third periodic verification performed between 1988 and 1992 is that masses of the entire worldwide ensemble of prototypes have been slowly but inexorably diverging from each other. It is also clear that the mass of the IPK lost perhaps 50 µg over the last century, and possibly significantly more, in comparison to its official copies. The reason for this drift has eluded physicists who have dedicated their careers to the SI unit of mass. No plausible mechanism has been proposed to explain either a steady decrease in the mass of the IPK, or an increase in that of its replicas dispersed throughout the world. This *relative* nature of the changes amongst the world's kilogram prototypes is often misreported in the popular press, and even some notable scientific magazines, which often state that the IPK simply "lost 50 µg" and omit the very important caveat of *"in comparison to its official copies*." Moreover, there are no technical means available to determine whether or not the entire worldwide ensemble of prototypes suffers from even greater long-term trends upwards or downwards because their mass "relative to an invariant of nature is unknown at a level below 1000 µg over a period of 100 or even 50 years." Given the lack of data identifying which of the world's kilogram prototypes has been most stable in absolute terms, it is equally as valid to state that the first batch of replicas has, as a group, gained an average of about 25 µg over one hundred years in comparison to the IPK.

What *is* known specifically about the IPK is that it exhibits a short-term instability of about 30 µg over a period of about a month in its after-cleaned mass. The precise reason for this short-term instability is not understood but is thought to entail surface effects: microscopic differences between the prototypes' polished surfaces, possibly aggravated by hydrogen absorption due to catalysis of the volatile organic compounds that slowly deposit onto the prototypes as well as the hydrocarbon-based solvents used to clean them.

Scientists are seeing far greater variability in the prototypes than previously believed. The increasing divergence in the masses of the world's prototypes and the short-term instability in the IPK has prompted research into improved methods to obtain a smooth surface finish using diamond-turning on newly manufactured replicas and has intensified the search for a new definition of the kilogram.

Importance of the kilogram



The magnitude of many of the units comprising the SI system of measurement, including most of those used in the measurement of electricity and light, are highly dependent upon the stability of a 131-year-old, golf ball-size cylinder of metal stored in a vault in France.

The stability of the IPK is crucial because the kilogram underpins much of the SI system of measurement as it is currently defined and structured. For instance, the newton is defined as the force necessary to accelerate one kilogram at one meter per second squared. If the mass of the IPK were to change slightly, so too must the newton by a proportional degree. In turn, the pascal, the SI unit of pressure, is defined in terms of the newton. This chain of dependency follows to many other SI units of measure. For instance, the joule, the SI unit of energy, is defined as that expended when a force of one newton acts through one meter. Next to be affected is the SI unit of power, the watt, which is one joule per second. The ampere too is defined relative to the newton, and ultimately, the kilogram. With the magnitude of the primary units of electricity thus determined by the kilogram, so too follow many others; namely, the coulomb, volt, tesla, and weber. Even units used in the measure of light would be affected; the candela—following the change in the watt—would in turn affect the lumen and lux.

Because the magnitude of many of the units comprising the SI system of measurement is ultimately defined by the mass of a 131-year-old, golf ball-sized piece of metal, the quality of the IPK must be diligently protected to preserve the integrity of the SI system. Yet, in spite of the best stewardship, the average mass of the worldwide ensemble of prototypes and the mass of the IPK have likely diverged another 5 μg since the third periodic verification 21 years ago. Further, the world's national metrology laboratories must wait for the fourth periodic verification to confirm whether the historical trends persisted.

Fortunately, *definitions* of the SI units are quite different from their *practical realizations.* For instance, the meter is *defined* as the distance light travels in a vacuum during a time interval of $^1/_{299,792,458}$ of a second. However, the meter's *practical realization* typically takes the form of a helium-neon laser, and the meter's length is *delineated*—not defined—as 1,579,800.298728 wavelengths of light from this laser. Now suppose that the official measurement of the second was found to have drifted by a few parts per billion (it is actually exquisitely stable). There would be no automatic effect on the meter because the second—and thus the meter's length—is abstracted via the laser comprising the meter's practical realization. Scientists performing meter calibrations would simply continue to measure out the same number of laser wavelengths until an agreement was reached to do otherwise. The same is true with regard to the real-world dependency on the kilogram: if the mass of the IPK was found to have changed slightly, there would be no automatic effect upon the other units of measure because their practical realizations provide an insulating layer of abstraction. Any discrepancy would eventually have to be reconciled though because the virtue of the SI system is its precise mathematical and logical harmony amongst its units. If the IPK's value were definitively proven to have changed, one solution would be to simply redefine the kilogram as being equal to the mass of the IPK plus an offset value, similarly to what is currently done with its replicas; e.g., "the kilogram is equal to the mass of the IPK + 42 parts per billion" (equivalent to 42 μg).

The long-term solution to this problem, however, is to liberate the SI system's dependency on the IPK by developing a practical realization of the kilogram that can be reproduced in different laboratories by following a written specification. The units of measure in such a practical realization would have their magnitudes precisely defined and expressed in terms of fundamental physical constants. While major portions of the SI system would still be based on the kilogram, the kilogram would in turn be based on invariant, universal constants of nature. While this is a worthwhile objective and much work towards that end is ongoing, no alternative has yet achieved the uncertainty of a couple parts in 10^8 (~20 µg) required to improve upon the IPK. However, as of April 2007, the U.S.'s National Institute of Standards and Technology (NIST) had an implementation of the watt balance that was approaching this goal, with a demonstrated uncertainty of 36 µg.

Proposed future definitions

In the following section, wherever numeric equalities are shown in 'concise form'—such as 1.85487(14)×10^{43}—the two digits between the parentheses denote the uncertainty at 1σ standard deviation (68% confidence level) in the two least significant digits of the significand.

The kilogram is the only SI unit that is still defined by an artifact. Note that the meter was also once defined as an artifact (a single platinum-iridium bar with two marks on it). However, it was eventually redefined in terms of invariant, fundamental constants of nature (the wavelength of light emitted by krypton, and later the speed of light) so that the standard can be reproduced in different laboratories by following a written specification. Today, physicists are investigating various approaches to doing the same with the kilogram. Some of the approaches are fundamentally very different from each other. Some are based on equipment and procedures that enable the reproducible production of new, kilogram-mass prototypes on demand (albeit with extraordinary effort) using measurement techniques and material properties that are ultimately based on, or traceable to, fundamental constants. Others are devices that measure either the acceleration or weight of hand-tuned, kilogram test masses and which express their magnitudes in electrical terms via special components that permit traceability to fundamental constants. Measuring the weight of test masses requires the precise measurement of the strength of gravity in laboratories. All approaches would precisely fix one or more constants of nature at a defined value. These different approaches are as follows:

Atom-counting approaches

Carbon-12

Though not offering a practical realization, this definition would precisely define the magnitude of the kilogram in terms of a certain number of carbon-12 atoms. Carbon-12 (^{12}C) is an isotope of carbon. The mole is currently defined as "the quantity of entities (elementary particles like atoms or molecules) equal to the number of atoms in 12 grams of carbon-12." Thus, the current definition of the mole requires that $^{1000}\!/_{12}$ (83⅓) moles of ^{12}C has a mass of precisely one kilogram. The number of atoms in a mole, a quantity known as the Avogadro constant, is experimentally determined, and the current best estimate of its value is $6.02214179(30)\times10^{23}$ entities per mole (CODATA, 2006). This new definition of the kilogram proposes to fix the Avogadro constant at precisely 6.02214179×10^{23} with the kilogram being defined as "the mass equal to that of $^{1000}\!/_{12} \cdot 6.02214179\times10^{23}$ atoms of ^{12}C."

The accuracy of the measured value of the Avogadro constant is currently limited by the uncertainty in the value of the Planck constant—a measure relating the energy of photons to their frequency. That relative standard uncertainty has been 50 parts per billion (ppb) since 2006. By fixing the Avogadro constant, the practical effect of this proposal would be that the uncertainty in the mass of a ^{12}C atom—and the magnitude of the kilogram—could be no better than the current 50 ppb uncertainty in the Planck constant. Under this proposal, the magnitude of the kilogram would be subject to future refinement as improved measurements of the value of the Planck constant become available; electronic realizations of the kilogram would be recalibrated as required. Conversely, an electronic *definition* of the kilogram, which would precisely fix the Planck constant, would continue to allow 83⅓ moles of ^{12}C to have a mass of precisely one kilogram but the number of atoms comprising a mole (the Avogadro constant) would continue to be subject to future refinement.

A variation on a ^{12}C-based definition proposes to define the Avogadro constant as being precisely $84{,}446{,}886^3$ ($\approx 6.02214098 \times 10^{23}$) atoms. An imaginary realization of a 12-gram mass prototype would be a cube of ^{12}C atoms measuring precisely 84,446,886 atoms across on a side. With this proposal, the kilogram would be defined as "the mass equal to $84{,}446{,}886^3 \times 83⅓$ atoms of ^{12}C." The value 84,446,886 was chosen because it has a special property; its cube (the proposed new value for the Avogadro constant) is evenly divisible by twelve. Thus with this definition of the kilogram, there would be an integer number of atoms in one gram of ^{12}C: 50,184,508,190,229,061,679,538 atoms.

Avogadro project

One of the master opticians at the Australian Centre for Precision Optics (ACPO) is holding a 1 kg, single-crystal silicon sphere for the Avogadro project. These spheres are among the roundest man-made objects in the world. If the best of these spheres were scaled to the size of Earth, its high point—a continent-size area—would gently rise to a maximum elevation of only 2.4 meters above "sea level."

Another Avogadro constant-based approach, known as the *Avogadro project*, would define and delineate the kilogram as a softball-size (93.6 mm diameter) sphere of silicon atoms. Silicon was chosen because a commercial infrastructure with mature processes for creating defect-free, ultra-pure monocrystalline silicon already exists to service the semiconductor industry. To make a practical realization of the kilogram, a silicon boule (a rod-like, single-crystal ingot) would be produced. Its isotopic composition would be measured with a mass spectrometer to determine its average relative atomic mass. The boule would be cut, ground, and polished into spheres. The size of a select sphere would be measured using optical interferometry to an uncertainty of about 0.3 nm on the radius—roughly a single atomic layer. The precise lattice spacing between the atoms in its crystal structure (≈192 pm) would be measured using a scanning X-ray interferometer. This permits its atomic spacing to be determined with an uncertainty of only three parts per billion. With the size of the sphere, its average atomic mass, and its atomic spacing known, the required sphere diameter can be calculated with sufficient precision and uncertainty to enable it to be finish-polished to a target mass of one kilogram.

Experiments are being performed on the Avogadro Project's silicon spheres to determine whether their masses are most stable when stored in a vacuum, a partial vacuum, or ambient pressure. However, no technical means currently exist to prove a long-term stability any better than that of the IPK's because the most sensitive and accurate measurements of mass are made with dual-pan balances like the BIPM's FB-2 flexure-strip balance. Balances can only compare the mass of a silicon sphere to that of a reference mass. Given the latest understanding of the lack of long-term mass stability with the IPK and its replicas, there is no known, perfectly stable mass artifact to compare against. Single-pan scales, which measure weight relative to an invariant of nature, are not precise to the necessary long-term uncertainty of 10–20 parts per billion. Another issue to be overcome is that silicon oxidizes and forms a thin layer (equivalent to 5–20 silicon atoms) of silicon dioxide (quartz) and silicon monoxide. This layer slightly increases the mass of the sphere, an effect which must be accounted for when polishing the sphere to its finish dimension. Oxidation is not an issue with platinum and iridium, both of which are noble metals that are roughly as cathodic as oxygen and therefore don't oxidize unless coaxed to do so in the laboratory. The presence of the thin oxide layer on a silicon-sphere mass prototype places additional restrictions on the procedures that might be suitable to clean it to avoid changing the layer's thickness or oxide stoichiometry.

All silicon-based approaches would fix the Avogadro constant but vary in the details of the definition of the kilogram. One approach would use silicon with all three of its natural isotopes present. About 7.78% of silicon comprises the two heavier isotopes: ^{29}Si and ^{30}Si. As described in *Carbon-12* above, this method would *define* the magnitude of the kilogram in terms of a certain number of ^{12}C atoms by fixing the Avogadro constant; the silicon sphere would be the *practical realization.* This approach could accurately delineate the magnitude of the kilogram because the masses of the three silicon nuclides relative to ^{12}C are known with great precision (relative uncertainties of 1 ppb or better). An alternative method for creating a silicon sphere-based kilogram proposes to use isotopic separation techniques to enrich the silicon until it is nearly pure ^{28}Si, which has a relative atomic mass of 27.9769265325(19). With this approach, the Avogadro constant would not only be fixed, but so too would the atomic mass of ^{28}Si. As such, the definition of the kilogram would be decoupled from ^{12}C and the kilogram would instead be defined as $^{1000}/_{27.9769265325} \cdot 6.02214179 \times 10^{23}$ atoms of ^{28}Si (≈35.74374043 fixed moles of ^{28}Si atoms). Physicists could elect to define the kilogram in terms of ^{28}Si even when kilogram prototypes are made of natural silicon (all three isotopes present). Even with a kilogram definition based on theoretically pure ^{28}Si, a silicon-sphere prototype made of only nearly pure ^{28}Si would necessarily deviate slightly from the defined number of moles of silicon to compensate for various chemical and isotopic impurities as well as the effect of surface oxides.

Ion accumulation

Another Avogadro-based approach, ion accumulation, since abandoned, would have defined and delineated the kilogram by precisely creating new metal prototypes on demand. It would have done so by accumulating gold or bismuth ions (atoms stripped of an electron) and counted them by measuring the electrical current required to neutralize the ions. Gold (^{197}Au) and bismuth (^{209}Bi) were chosen because they can be safely handled and have the two highest atomic masses among the mononuclidic elements that is effectively non-radioactive (bismuth) or is perfectly stable (gold).

With a gold-based definition of the kilogram for instance, the relative atomic mass of gold could have been fixed as precisely 196.9665687, from the current value of 196.9665687(6). As with a definition based upon carbon-12, the Avogadro constant would also have been fixed. The kilogram would then have been defined as “the mass equal to that of precisely $^{1000}/_{196.9665687} \cdot 6.02214179\times10^{23}$ atoms of gold” (precisely 3,057,443,620,887,933,963,384,315 atoms of gold or about 5.07700371 fixed moles).

In 2003, German experiments with gold at a current of only 10 µA demonstrated a relative uncertainty of 1.5%. Follow-on experiments using bismuth ions and a current of 30 mA were expected to accumulate a mass of 30 g in six days and to have a relative uncertainty of better than 1 part in 10^6. Ultimately, ion-accumulation approaches proved to be unsuitable. Measurements required months and the data proved too erratic for the technique to be considered a viable future replacement to the IPK.

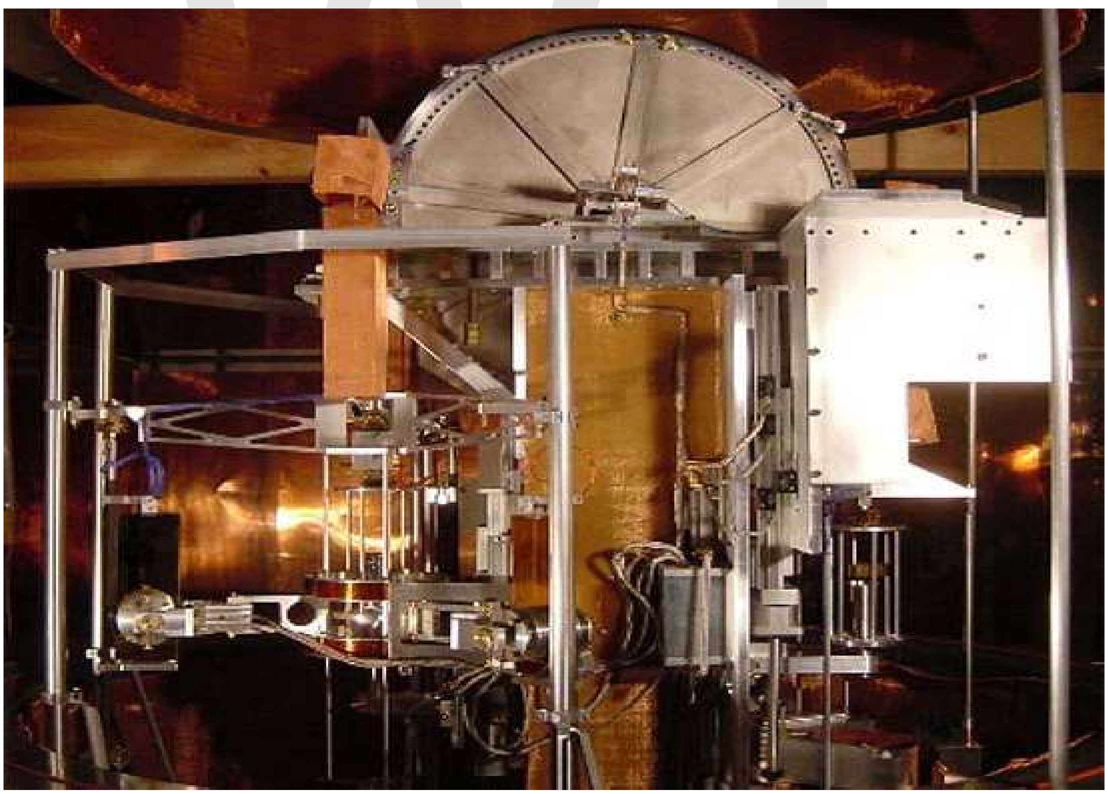
Among the many technical challenges of the ion-deposition apparatus was obtaining a sufficiently high ion current (mass deposition rate) while simultaneously decelerating the ions so they could all deposit onto a target electrode embedded in a balance pan. Experiments with gold showed the ions had to be decelerated to very low energies to avoid sputtering effects—an phenomenon whereby ions that had already been counted ricochet off the target electrode or even dislodged atoms that had already been deposited. The deposited mass fraction in the 2003 German experiments only approached very close to 100% at ion energies of less than around 1 eV (<1 km/s for gold).

If the kilogram had been defined as a precise quantity of gold or bismuth atoms deposited with an electric current, not only would the Avogadro constant and the atomic mass of gold or bismuth had to have been precisely fixed, but also the value of the elementary charge (*e*), likely to $1.602176487\times10^{-19}$ C (from the present 2006 CODATA value of $1.602176487(40)\times10^{-19}$). Doing so would have effectively defined the ampere as a flow of $^{1}/_{1.602176487\times10^{-19}}$ (6,241,509,647,120,417,390) electrons per second past a fixed point in an electric circuit. The SI unit of mass would have been fully defined by having precisely fixed the values of the Avogadro constant and elementary charge, and by exploiting the fact that the atomic masses of bismuth and gold atoms are invariant, universal constants of nature.

Beyond the slow speed of making a new mass standard and the poor reproducibility, there were other intrinsic shortcomings to the ion-accumulation approach that proved to be formidable obstacles to ion-accumulation-based techniques becoming a practical realization. The apparatus necessarily required that the deposition chamber have an integral balance system to enable the convenient calibration of a reasonable quantity of transfer standards relative to any single internal ion-deposited prototype. Furthermore, the mass prototypes produced by ion deposition techniques would have been nothing like the freestanding platinum-iridium prototypes currently in use; they would have been deposited onto—and become part of—an electrode imbedded into one pan of a special balance integrated into the device. Moreover, the ion-deposited mass wouldn't have had a hard, highly polished surface that can be vigorously cleaned like those of current prototypes. Gold, while dense and a noble metal (resistant to oxidation and the formation of other compounds), is extremely soft so an internal gold prototype would have to be kept well isolated and scrupulously clean to avoid contamination and the potential of wear from having to remove the contamination. Bismuth, which is an inexpensive metal used in low-temperature solders, slowly oxidizes when exposed to room-temperature air and forms other chemical compounds and so would not have produced stable reference masses unless it was continually maintained in a vacuum or inert atmosphere.

Electronic approaches

Watt balance



The NIST's watt balance is a project of the U.S. Government to develop an "electronic kilogram." The vacuum chamber dome, which lowers over the entire apparatus, is visible at top.

The watt balance is essentially a single-pan weighing scale that measures the electric power necessary to oppose the weight of a kilogram test mass as it is pulled by earth's gravity. It is a variation of an ampere balance in that it employs an extra calibration step that nulls the effect of geometry. The electric potential in the watt balance is delineated by a Josephson voltage standard, which allows voltage to be linked to an invariant constant of nature with extremely high precision and stability. Its circuit resistance is calibrated against a quantum Hall resistance standard.

The watt balance requires exquisitely precise measurement of gravity in a laboratory. For instance, the NIST compensates for earth's gravity gradient of 3.09 µGal per centimeter when the elevation of the center of the gravimeter differs from that of the nearby test mass in the watt balance; a change in the weight of a one-kilogram test mass that equates to about 3.16 µg/cm.

In April 2007, the NIST's implementation of the watt balance demonstrated a combined relative standard uncertainty (CRSU) of 36 µg and a short-term resolution of 10–15 µg. The UK's National Physical Laboratory's watt balance demonstrated a CRSU of 70.3 µg in 2007. That watt balance was disassembled and shipped in 2009 to Canada's Institute for National Measurement Standards (part of the National Research Council), where research and development with the device could continue.

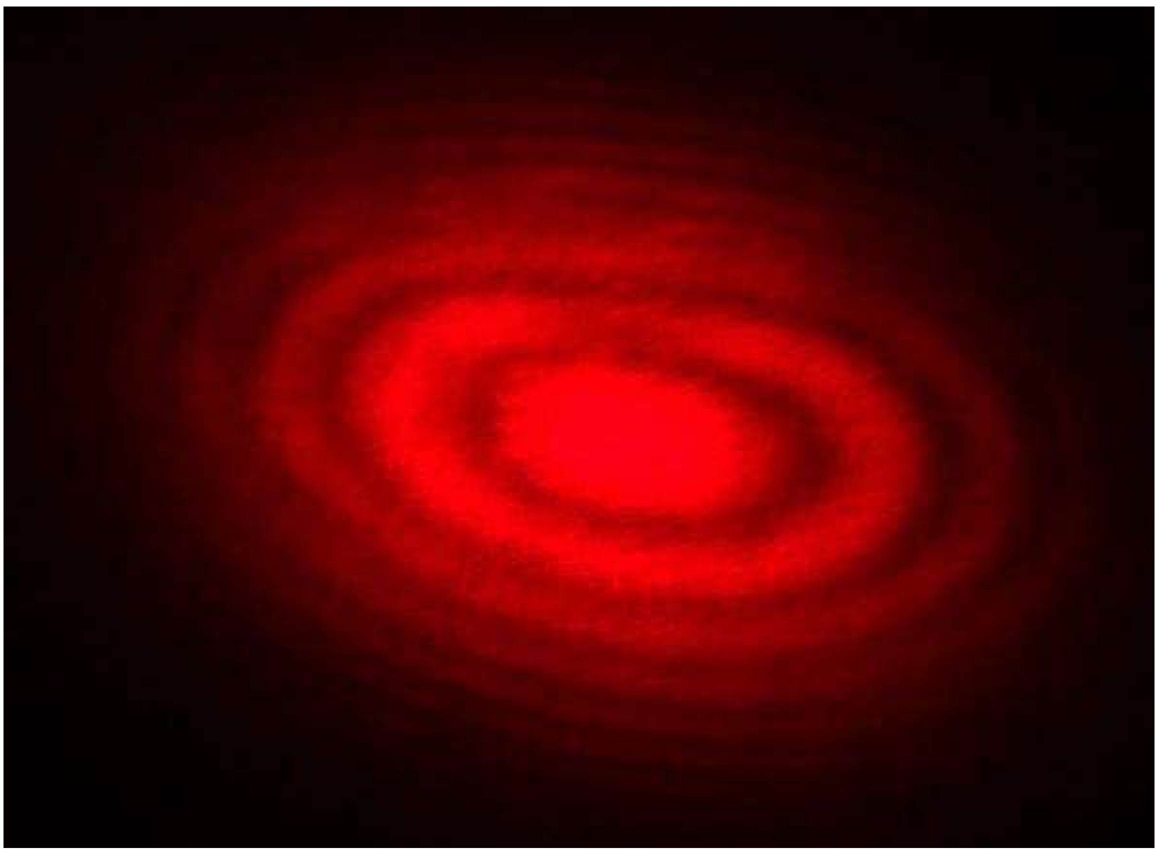
With the watt balance, the kilogram would be redefined and internationally recognized in terms of two invariants of nature: the speed of light (c) and the Planck constant (h), which is a measure that relates the energy of photons to their frequency. The Planck constant would be fixed; for example, to $h = 6.62606896\times10^{-34}$ J·s (from the 2006 CODATA value of $6.62606896(33)\times10^{-34}$ J·s). The kilogram would relate to these two fundamental constants of nature via the formula 1 kg = $c^2/h = f$, which is to say, the mass of the kilogram would be equivalent to the square of the speed of light divided by the Planck constant, where the defined amount of energy is expressed as a simple frequency. The kilogram would thus be defined as "the mass of a body at rest whose equivalent energy equals the energy of photons whose frequencies sum to ($^{299,792,458^2}/_{6.62606896\times10^{-34}}$) hertz." This quotient is approximately 1.356392733×10^{50} hertz (symbol = Hz), which is expressed here at a precision of better than one part per billion—better precision than the watt balance will be able to operate at for the foreseeable future.

When operating at its full planned output of 10 megawatts, the YAL-1 Airborne Laser will, during a single 4.5-second firing of its COIL laser, emit photons whose total energy is equivalent to one-half billionth of a kilogram.

A "sum of photon frequencies" means the photon energy that is equivalent to one kilogram could be expressed in terms of a single photon oscillating at about 1.356×10^{50} Hz, two photons that each have half that frequency (about 6.78×10^{49} Hz), 100 photons with one-hundredth the specified value (about 1.356×10^{48} Hz), or approximately 5.95×10^{35} photons from a chemical oxygen-iodine laser, which oscillate at about 2.279×10^{14} Hz. This sum of frequencies is an extraordinarily high value; a single photon oscillating at 1.356×10^{50} Hz would be oscillating over a trillion-trillion times faster than the highest-energy gamma ray photons ever observed.

By fixing the value of the Planck constant and defining the kilogram in terms of its equivalent energy, the magnitude of the kilogram is being measured in terms of Albert Einstein's famous equation $E = mc^2$, which holds that mass can be expressed in terms of energy and vice versa. Due to the way the SI system of measurement is logically structured, $E = mc^2$ means one kilogram is precisely equivalent to an energy of 89,875,517,873,681,764 joules, which is the square of the speed of light in meters per second ($299,792,458^2$), or as stated above, this energy in joules can be divided by the

Planck constant to express the energy as a sum of photon frequencies. Regardless of the units of measure employed, one kilogram has an exceedingly great deal of equivalent energy; it is comparable to that released in the explosion of a fairly powerful hydrogen bomb: a yield of approximately 21 megatons. At a slower rate, the largest electric power-producing facility in the United States, Grand Coulee Dam, would typically require over 14 months to generate nearly 90 quadrillion joules of energy. In terms of a collection of photons that have a total energy equivalent to one kilogram, the approximately 5.95×10^{35} near-infrared photons from the above-mentioned chemical oxygen-iodine laser, which is a weapon system intended for destroying intercontinental ballistic missiles, would require a beam length that could reach beyond any of the 200 stars nearest the sun; with a beam intensity of one megawatt per square centimeter and a diameter of 3.6 centimeters, it would have a length of 280 light-years.



Gravity is measured with exceptional precision with the help of a laser interferometer. The laser's pattern of interference fringes—the dark and light bands above—blooms at an ever faster rate as a free-falling corner reflector drops inside an absolute gravimeter. The pattern's frequency sweep is timed by an atomic clock.

Fortunately, the kilogram's mass/energy equivalence is established *indirectly* in the watt balance; gravity serves as a crucial translation tool that is exploited to measure the "mass vs. energy" relationship via a "force vs. electrical power" relationship. Since gravity is the weakest of nature's fundamental forces and since earth's gravity is relatively modest compared to other celestial bodies, only a relatively small amount of electrical power is required to counter the weight of one kilogram on earth. However, the force of gravity varies significantly—nearly one percent—depending upon where on earth's surface the measurement is made. Even more problematic, there are subtle seasonal variations in gravity due to changes in underground water tables, and even semimonthly and diurnal changes due to tidal distortions in the earth's shape caused by the moon. Although gravity would not be a term in the *definition* of the kilogram, gravity would be a crucial term used in the *delineation* of the kilogram when relating energy to power. Accordingly, the 'gravity' term must be measured with at least as much precision and accuracy as are the other terms. It is therefore highly desirable that gravity measurements also be traceable to fundamental constants of nature. For the most precise work in mass metrology, gravitational acceleration is measured using dropping-mass absolute gravimeters that contain an iodine-stabilized helium–neon laser interferometer. The fringe-signal, frequency-sweep output from the interferometer is measured with a rubidium atomic clock. Since this type of dropping-mass gravimeter derives its accuracy and stability from the constancy of the speed of light as well as the innate properties of helium, neon, and rubidium atoms, the 'gravity' term in the delineation of an all-electronic kilogram is also measured in terms of invariants of nature—and with very high precision. For instance, in the basement of the NIST's Gaithersburg facility in 2009, when measuring the gravity acting upon Pt-10Ir test masses (which are denser, smaller, and have a slightly lower center of gravity inside the watt balance than stainless steel masses), the measured value was typically within 8 ppb of 9.80101644 m/s^2.

The virtue of electronic realizations like the watt balance is that the definition and dissemination of the kilogram would no longer be dependent upon the stability of kilogram prototypes, which must be very carefully handled and stored. It would free physicists from the need to rely on assumptions about the stability of those prototypes, including those that would be manufactured under atom-counting schemes. Instead, hand-tuned, close-approximation mass standards would simply be weighed and documented as being equal to one kilogram plus an offset value. With the watt balance, the kilogram would not only be *delineated* in electrical and gravity terms, all of which are traceable to invariants of nature; it would be *defined* in electrical terms in a manner that is directly traceable to just two fundamental constants of nature. Mass artifacts—physical objects calibrated in a watt balance, including the IPK—would become *transfer standards*.

Scales like the watt balance also permit more flexibility in choosing materials with especially desirable properties for mass standards. For instance, Pt-10Ir could continue to be used so that the specific gravity of newly produced mass standards would be the same as existing national primary and check standards (≈21.55 g/ml). This would reduce the relative uncertainty when making mass comparisons in air. Alternately, entirely different materials and constructions could be explored with the objective of producing mass standards with greater stability. For instance, osmium-iridium alloys could be investigated if platinum's propensity to absorb hydrogen (due to catalysis of VOCs and hydrocarbon-based cleaning solvents) and atmospheric mercury proved to be sources of instability. Also, vapor-deposited, protective ceramic coatings like nitrides could be investigated for their suitability to isolate these new alloys.

The challenge with watt balances is not only in reducing their uncertainty, but also in making them truly *practical* realizations of the kilogram. Nearly every aspect of watt balances and their support equipment requires such extraordinarily precise and accurate, state-of-the-art technology that—unlike a device like an atomic clock—few countries would currently choose to fund their operation. For instance, the NIST's watt balance used four resistance standards in 2007, each of which was rotated through the watt balance every two to six weeks after being calibrated in a different part of NIST headquarters facility in Gaithersburg, Maryland. It was found that simply moving the resistance standards down the hall to the watt balance after calibration altered their values 10 ppb (equivalent to 10 µg) or more. Present-day technology is insufficient to permit stable operation of watt balances between even biannual calibrations. If the kilogram is defined in terms of the Planck constant, it is likely there will only be a few—at most—watt balances initially operating in the world.

Ampere-based force

A magnet floating above a superconductor bathed in liquid nitrogen demonstrates perfect diamagnetic levitation via the Meissner effect. Experiments with an ampere-based definition of the kilogram flipped this arrangement upside-down: an electric field accelerated a superconducting test mass supported by fixed magnets.

This approach would define the kilogram as “the mass which would be accelerated at precisely 2×10^{-7} m/s^2 when subjected to the per-meter force between two straight parallel conductors of infinite length, of negligible circular cross section, placed one meter apart in vacuum, through which flow a constant current of $^{1}/_{1.602176487\times10^{-19}}$ (≈6,241,509,647,120,417,390) elementary charges per second.”

Effectively, this would define the kilogram as a derivative of the ampere rather than present relationship, which defines the ampere as a derivative of the kilogram. This redefinition of the kilogram would specify elementary charge (e) as precisely $1.602176487\times10^{-19}$ coulomb rather than the current 2006 CODATA value of $1.602176487(40)\times10^{-19}$. Effectively, the coulomb would be the sum of 6,241,509,647,120,417,390 elementary charges. It would necessarily follow that the ampere (one coulomb per second) would also become an electrical current of this precise quantity of elementary charges per second passing a given point in an electric circuit.

The virtue of a practical realization based upon this definition is that unlike the watt balance and other scale-based methods, all of which require the careful characterization of gravity in the laboratory, this method delineates the magnitude of the kilogram directly in the very terms that define the nature of mass: acceleration due to an applied force. Unfortunately, it is extremely difficult to develop a practical realization based upon accelerating masses. Experiments over a period of years in Japan with a superconducting, 30 g mass supported by diamagnetic levitation never achieved an uncertainty better than ten parts per million. Magnetic hysteresis was one of the limiting issues. Other groups are continuing this line of research using different techniques to levitate the mass.

SI multiples

Because SI prefixes may not be concatenated (serially linked) within the name or symbol for a unit of measure, SI prefixes are used with the *gram*, not the kilogram, which already has a prefix as part of its name. For instance, one-millionth of a kilogram is 1 mg (one milligram), not 1 µkg (one microkilogram).

SI multiples for gram (g)

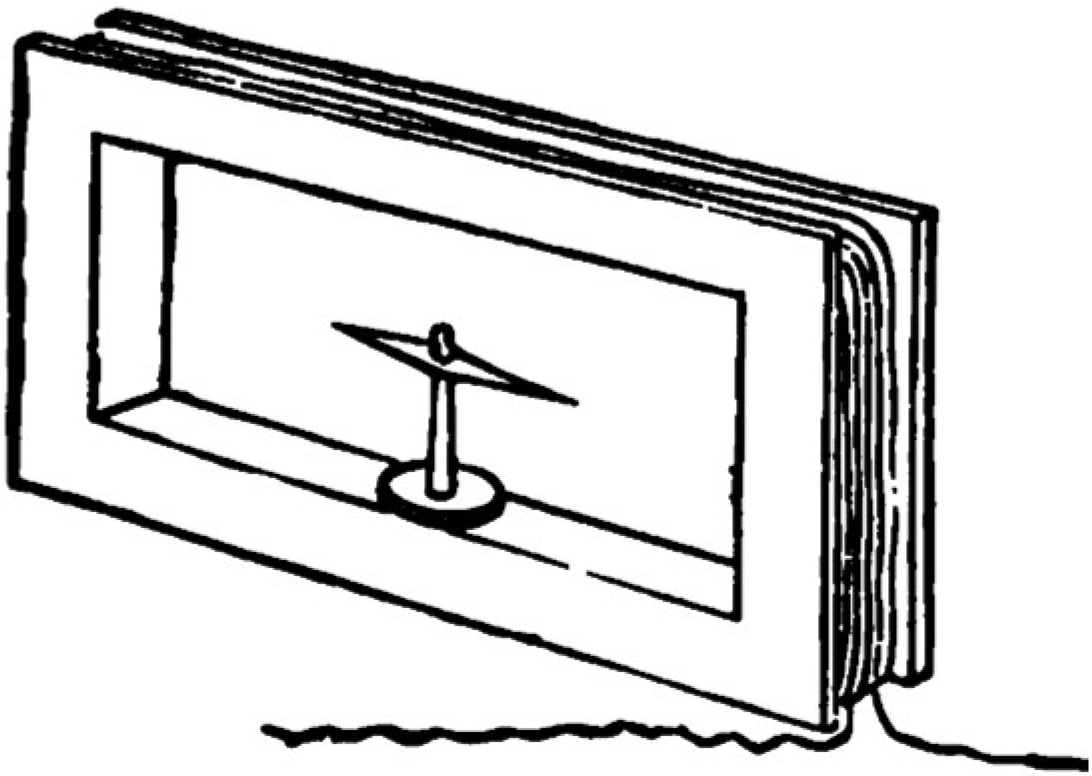
Submultiples			**Multiples**		
Value	**Symbol**	**Name**	**Value**	**Symbol**	**Name**
10^{-1} g	dg	decigram	10^{1} g	dag	decagram
10^{-2} g	cg	centigram	10^{2} g	hg	hectogram
10^{-3} g	**mg**	**milligram**	10^{3} g	**kg**	**kilogram**
10^{-6} g	**µg**	**microgram (mcg)**	10^{6} g	Mg	megagram (**tonne**)
10^{-9} g	**ng**	**nanogram**	10^{9} g	Gg	gigagram
10^{-12} g	**pg**	**picogram**	10^{12} g	Tg	teragram
10^{-15} g	fg	femtogram	10^{15} g	Pg	petagram
10^{-18} g	ag	attogram	10^{18} g	Eg	exagram

10^{-21} g	zg	zeptogram	10^{21} g	Zg	zettagram
10^{-24} g	yg	yoctogram	10^{24} g	Yg	yottagram

Common prefixes are in bold face.

- When the Greek lowercase "μ" (mu) in the symbol of microgram is typographically unavailable, it is occasionally—although not properly—replaced by Latin lowercase "u".
- The microgram is often abbreviated "mcg", particularly in pharmaceutical and nutritional supplement labeling, to avoid confusion since the "μ" prefix is not well recognized outside of technical disciplines. Note however, that the *abbreviation* "mcg", is also the *symbol* for an obsolete CGS unit of measure known as the "millicentigram", which is equal to 10 μg.
- The unit name "megagram" is rarely used, and even then, typically only in technical fields in contexts where especially rigorous consistency with the units of measure is desired. For most purposes, the unit "tonne" is instead used. The tonne and its symbol, t, were adopted by the CIPM in 1879. It is a non-SI unit accepted by the BIPM for use with the SI. According to the BIPM, "In English speaking countries this unit is usually called 'metric ton'." Note also that the unit name "megatonne" or "megaton" (Mt) is often used in general-interest literature on greenhouse gas emissions whereas the equivalent value in scientific papers on the subject is often the "teragram" (Tg).

4. Ampere



Current can be measured by a galvanometer, via the deflection of a magnetic needle in the magnetic field created by the current.

The **ampere** (symbol: A) is the SI unit of electric current (symbol: I) and is one of the seven SI base units. It is named after André-Marie Ampère (1775–1836), French mathematician and physicist, considered the father of electrodynamics. In practice, its name is often shortened to **amp**.

In practical terms, the ampere is a measure of the amount of electric charge passing a point per unit time. Around 6.241×10^{18} electrons passing a given point each second constitutes one ampere.

Definition

Ampère's force law states that there is an attractive force between two parallel wires carrying an electric current. This force is used in the formal definition of the ampere which states that it is "the constant current which will produce an attractive force of 2×10^{-7} newtons per metre of length between two straight, parallel conductors of infinite length and negligible circular cross section placed one metre apart in a vacuum".

In terms of Ampère's force law,

$$2 \times 10^{-7}\ \tfrac{\mathrm{N}}{\mathrm{m}} = k_A \frac{\mathrm{A} \cdot \mathrm{A}}{\mathrm{m}}$$

so

$$1\ \mathrm{A} = \sqrt{\frac{2 \times 10^{-7}\ \mathrm{N}}{k_A}}$$

The SI unit of charge, the coulomb, "is the quantity of electricity carried in 1 second by a current of 1 ampere." Conversely, a current of one ampere is one coulomb of charge going past a given point per second**:**

$$1\ \mathrm{A} = 1\tfrac{\mathrm{C}}{\mathrm{s}}$$

That is, in general, charge Q is determined by steady current I flowing for a time t as $Q = It$.

History

The ampere was originally defined as one tenth of the CGS system electromagnetic unit of current (now known as the abampere), the amount of current which generates a force of two dynes per centimetre of length between two wires one centimetre apart. The size of the unit was chosen so that the units derived from it in the MKSA system would be conveniently sized.

The "international ampere" was an early realisation of the ampere, defined as the current that would deposit 0.001118000 grams of silver per second from a silver nitrate solution. Later, more accurate measurements revealed that this current is 0.99985 A.

Realisation

The ampere is most accurately realised using a watt balance, but is in practice maintained via Ohm's Law from the units of electromotive force and resistance, the volt and the ohm, since the latter two can be tied to physical phenomena that are relatively easy to reproduce, the Josephson junction and the quantum Hall effect, respectively.

At present, techniques to establish the realisation of an ampere have a relative uncertainty of approximately a few parts in 10^7, and involve realisations of the watt, the ohm and the volt.

Proposed future definition

Rather than a definition in terms of the force between two current-carrying wires, it has been proposed to define the ampere in terms of the rate of flow of elementary charges. Since a coulomb is approximately equal to $6.24150948 \times 10^{18}$ elementary charges, one ampere is approximately equivalent to $6.24150948 \times 10^{18}$ elementary charges, such as electrons, moving past a boundary in one second. The proposed change would define 1 A as being the current in the direction of flow of a particular number of elementary charges per second. In 2005, the International Committee for Weights and Measures (CIPM) agreed to study the proposed change, and, depending on the outcome of experiments over the next few years, to formally propose the change at the 24th General Conference on Weights and Measures (CGPM) in 2011.

5. Kelvin

Kelvin temperature conversion formulae

	from Kelvin	**to Kelvin**
Celsius	[°C] = [K] − 273.15	[K] = [°C] + 273.15
Fahrenheit	[°F] = [K] × $\frac{9}{5}$ − 459.67	[K] = ([°F] + 459.67) × $\frac{5}{9}$
Rankine	[°R] = [K] × $\frac{9}{5}$	[K] = [°R] × $\frac{5}{9}$

For temperature *intervals* rather than specific temperatures,
1 K = 1 °C = 1.8 °F = 1.8 °R

Comparisons among various temperature scales

The **kelvin** (symbol: **K**) is a unit increment of temperature and is one of the seven SI base units. The **Kelvin scale** is a thermodynamic (absolute) temperature scale referenced to absolute zero, the absence of all thermal energy. So by definition, the temperature of a substance at absolute zero is zero kelvin (0 K). The secondary reference point on the Kelvin scale is the triple point of water (0.01 degrees Celsius). The Kelvin scale is the difference between these two reference points, with the kelvin defined as one 273.16th of this scale. The Kelvin scale and the kelvin are named after the Belfast-born physicist and engineer William Thomson, 1st Baron Kelvin (1824–1907), who wrote of the need for an "absolute thermometric scale". Unlike the degree Fahrenheit and degree Celsius, the kelvin is not referred to as a "degree", nor is it typeset with a degree symbol; that is, it is written K and not °K. The kelvin and the degree Celsius are often used together, as they have the same interval, and 0 kelvin is −273.15 degrees Celsius.

History

1848

Lord Kelvin (William Thomson), wrote in his paper, *On an Absolute Thermometric Scale,* of the need for a scale whereby "infinite cold" (absolute zero) was the scale's null point, and which used the degree Celsius for its unit increment. Thomson calculated that absolute zero was equivalent to −273 °C on the air thermometers of the time. This absolute scale is known today as the Kelvin thermodynamic temperature scale. Thomson's value of "−273" was the reciprocal of 0.00366 - the accepted expansion coefficient of gas per degree Celsius relative to the ice point, giving a remarkable consistency to the currently accepted value.

1954

Resolution 3 of the 10th CGPM gave the Kelvin scale its modern definition by designating the triple point of water as its second defining point and assigned its temperature to exactly 273.16 kelvin.

1967/1968

Resolution 3 of the 13th CGPM renamed the unit increment of thermodynamic temperature "kelvin", symbol K, replacing "degree absolute", symbol °K. Furthermore, feeling it useful to more explicitly define the magnitude of the unit increment, the 13th CGPM also held in Resolution 4 that "The kelvin, unit of thermodynamic temperature, is equal to the fraction 1/273.16 of the thermodynamic temperature of the triple point of water."

2005

The Comité International des Poids et Mesures (CIPM), a committee of the CGPM, affirmed that for the purposes of delineating the temperature of the triple point of water, the definition of the Kelvin thermodynamic temperature scale would refer to water having an isotopic composition specified as *VSMOW*.

Usage conventions

When reference is made to the unit kelvin (either a specific temperature or a temperature interval), kelvin is always spelled with a lowercase k unless it is the first word in a sentence. When reference is made to the "Kelvin *scale*", the word "kelvin"—which is normally a noun—functions adjectivally to modify the noun "scale" and is capitalized.

Until the 13th General Conference on Weights and Measures (CGPM) in 1967–1968, the unit kelvin was called a "degree", the same as with the other temperature scales at the time. It was distinguished from the other scales with either the adjective suffix "Kelvin" ("degree Kelvin") or with "absolute" ("degree absolute") and its symbol was °K. The latter (degree absolute), which was the unit's official name from 1948 until 1954, was rather ambiguous since it could also be interpreted as referring to the Rankine scale. Before the 13th CGPM, the plural form was "degrees absolute". The 13th CGPM changed the name to simply "kelvin" (symbol K). The omission of "degree" indicates that it is not relative to an arbitrary reference point like the Celsius and Fahrenheit scales, but rather an absolute unit of measure which can be manipulated algebraically (e.g., multiplied by two to indicate twice the amount of "mean energy" available among elementary degrees of freedom of the system).

This SI unit is named after William Thomson, 1st Baron Kelvin. As with every SI unit whose name is derived from the proper name of a person, the first letter of its symbol is uppercase (**K**). When an SI unit is spelled out in English, it should always begin with a lowercase letter (**kelvin**), except where *any* word would be capitalized, such as at the beginning of a sentence or in capitalized material such as a title. Note that "degree Celsius" conforms to this rule because the "d" is lowercase.
—Based on *The International System of Units*, section 5.2.

The kelvin symbol is always a roman, non-italic capital K. In the SI naming convention, all symbols named after a person are capitalized; in the case of the kelvin, capitalizing also distinguishes the symbol from the SI prefix "kilo", which has the lowercase k as its symbol. The admonition against italicizing the symbol K applies to all SI unit symbols; only symbols for variables and constants (e.g., P = pressure, and c = 299,792,458 m/s) are italicized in scientific and engineering papers. As with most other SI unit symbols (angle symbols, e.g. 45° 3′ 4″, are the exception) there is a space between the numeric value and the kelvin symbol (e.g. "99.987 K").

Unicode provides a compatibility character for the kelvin at U+212A (decimal 8490), for compatibility with CJK encodings that provide such a character (as such, in most fonts the width is the same as for fullwidth characters).

Use in conjunction with Celsius

In science and in engineering, the Celsius scale and the kelvin are often used simultaneously in the same article (e.g., "...its measured value was 0.01028 °C with an uncertainty of 60 µK..."). This practice is permissible because the degree Celsius is a special name for the kelvin for use in expressing Celsius temperatures and the magnitude of the degree Celsius is exactly equal to that of the kelvin. Notwithstanding that the official endorsement provided by Resolution 3 of the 13th CGPM states, "a temperature interval may also be expressed in degrees Celsius," the practice of simultaneously using both "°C" and "K" remains widespread throughout the scientific world as the use of SI prefixed forms of the degree Celsius (such as "µ°C" or "microdegrees Celsius") to express a temperature interval has not been widely adopted. A helpful way to think of the kelvin system is thinking that nothing can be colder than 0 kelvin (−273.15 degrees Celsius).

Proposed redefinition

In 2005 the CIPM embarked on a program to redefine, amongst others, the Kelvin using a more rigorous basis than was in use. The current (2010) definition is unsatisfactory for temperatures below 20 K and above 1300 K. It is anticipated that the program will be completed in time for its adoption by the CGPM at its 2011 meeting. The committee proposes defining the kelvin as the temperature scale for which Boltzmann's Constant is $1.3806505 \times 10^{-23}$ J/K exactly.

From a scientific point of view, this will link temperature to the rest of SI and result in a stable definition that is independent of any particular substance. From a practical point of view the redefinition will pass unnoticed; water will still freeze at 0 °C (273.15 K).

Practical uses

Color temperature

The kelvin is often used in the measure of the color temperature of light sources. Color temperature is based upon the principle that a black body radiator emits light whose color depends on the temperature of the radiator. Black bodies with temperatures below about 4000 K appear reddish whereas those above about 7500 K appear bluish. Color temperature is important in the fields of image projection and photography where a color temperature of approximately 5600 K is required to match "daylight" film emulsions. In astronomy, the stellar classification of stars and their place on the Hertzsprung-Russell diagram are based, in part, upon their surface temperature, known as effective temperature. The photosphere of the Sun, for instance, has an effective temperature of 5778 K.

Kelvin as a measure of noise

In electronics, the kelvin is used as an indicator of how noisy a circuit is in relation to an ultimate noise floor, i.e. the noise temperature. The so-called Johnson–Nyquist noise of discrete resistors and capacitors is a type of thermal noise derived from the Boltzmann constant and can be used to determine the noise temperature of a circuit using the Friis formulas for noise.

6. Mole (unit)

The **mole** is a unit of measurement for the amount of substance or *chemical amount*. It is one of the base units in the International System of Units, and has the unit symbol mol.

The name *mole* is an 1897 translation of the German unit *Mol*, coined by the chemist Wilhelm Ostwald in 1893, although the related concept of equivalent mass had been in use at least a century earlier. The name is assumed to be derived from the German word *Molekül* (molecule).

The mole is defined as the amount of substance that contains as many elementary entities (e.g., atoms, molecules, ions, electrons) as there are atoms in 0.012 kg of the isotope carbon-12 (^{12}C). Thus, by definition, one mole of pure ^{12}C has a mass of *exactly* 12 g. The amount of substance n (in mol) and the number of atoms or molecules contained in it, N, are proportional, and the proportionality constant N/n is known as Avogadro constant. By definition it has the dimension of the inverse of amount of substance (unit mol^{-1}) and its experimentally determined value is $6.02214179(30)\times10^{23}$ mol^{-1}.

By this definition, a mole of any pure substance has a mass in grams exactly equal to that substance's molecular or atomic mass; e.g., 1mol of calcium-40 is approximately equal to 40g, because the Ca-40 isotope has a mass of 39.9625906 amu on the C-12 scale. In other words, the numerical value of a substance's molecular or atomic mass in atomic mass units is the same as that of its molar mass—the mass of one mole of that substance—in grams.

The most common method of determining the amount, expressed in moles, of pure substance the value of whose molar mass is known, is to measure its mass in grams and then to divide by its molar mass (expressed in g/mol). Molar masses may be easily calculated from tabulated values of atomic weights and the molar mass constant (which has a convenient defined value of 1 g/mol). Other methods include the use of the molar volume or the measurement of electric charge.

The current definition of the mole was approved during the 1960s. Earlier definitions had been based on the atomic mass of hydrogen (about one gram of hydrogen-1 gas, excluding its heavy isotopes), the atomic weight of oxygen, and the relative atomic mass of oxygen-16; the four different definitions were equivalent to within 1%.

The names **gram-atom** (abbreviated gat.) and **gram-molecule** have also been used in the same sense as "mole". However, modern conventions define the gram-atom and the mole differently. While the elementary entity defining a mole will vary depending on the substance, the elementary entity for the gram-atom is always the atom. For example, 1 mole of He is equivalent to 1 gram-atom of He, but 1 mole of MgB_2 is equivalent to 3 gram-atoms of MgB_2.

History

The first table of atomic weights was published by John Dalton (1766–1844) in 1805, based on a system in which the atomic weight of hydrogen was defined as 1. These atomic weights were based on the stoichiometric proportions of chemical reactions and compounds, a fact which greatly aided their acceptance: it was not necessary for a chemist to subscribe to atomic theory (an unproven hypothesis at the time) to make practical use of the tables. This would lead to some confusion between atomic weights (promoted by proponents of atomic theory) and equivalent weights (promoted by its opponents and which sometimes differed from atomic weights by an integer factor), which would last throughout much of the nineteenth century.

Jöns Jacob Berzelius (1779–1848) was instrumental in the determination of atomic weights to ever increasing accuracy. He was also the first chemist to use oxygen as the standard to which other weights were referred. Oxygen is a useful standard, as, unlike hydrogen, it forms compounds with most other elements, especially metals. However he chose to fix the atomic weight of oxygen as 100, an innovation which did not catch on.

Charles Frédéric Gerhardt (1816–56), Henri Victor Regnault (1810–78) and Stanislao Cannizzaro (1826–1910) expanded on Berzelius' work, resolving many of the problems of unknown stoichiometry of compounds, and the use of atomic weights attracted a large consensus by the time of the Karlsruhe Congress (1860). The convention had reverted to defining the atomic weight of hydrogen as 1, although at the level of precision of measurements at that time—relative uncertainties of around 1%—this was numerically equivalent to the later standard of oxygen = 16. However the chemical convenience of having oxygen as the primary atomic weight standard became ever more evident with advances in analytical chemistry and the need for ever more accurate atomic weight determinations.

Scale basis	Scale basis relative to $^{12}C = 12$	Relative deviation from the $^{12}C = 12$ scale
Atomic weight of hydrogen = 1	1.007 94(7)	−0.788%
Atomic weight of oxygen = 16	15.9994(3)	+37.5 ppm
Relative atomic mass of $^{16}O = 16$	15.994 914 6221(15)	+318 ppm

The mole as a unit

Since its adoption into the International System of Units, there have been a number of criticisms of the concept of the mole being a unit like the metre or the second. These criticisms may be briefly summarized as:

- amount of substance is not a true physical quantity (or dimension), and is redundant to mass, so should not have its own base unit;
- the mole is simply a shorthand way of referring to a large number.

In chemistry, it has been known since Proust's law of definite proportions (1794) that knowledge of the mass of each of the components in a chemical system is not sufficient to define the system. Amount of substance can be described as mass divided by Proust's "definite proportions", and contains information which is missing from the measurement of mass alone. As demonstrated by Dalton's law of partial pressures (1803), a measurement of mass is not even necessary to measure the amount of substance (although in practice it is usual). There are many physical relationships between amount of substance and other physical quantities, most notably the ideal gas law (where the relationship was first demonstrated in 1857). The term "mole" was first used in a textbook describing these colligative properties.

Other units called "mole"

Chemical engineers use the concept extensively, but the unit is rather small for industrial use. For convenience in avoiding conversions, some American engineers adopted the **pound-mole** (noted **lb-mol** or **lbmol**), which is defined as the number of entities in 12 lb of ^{12}C. One lb-mol is equal to 453.59237 mol. In the metric system, chemical engineers once used the **kilogram-mole** (noted **kg-mol**), which is defined as the number of entities in 12 kg of ^{12}C, and often referred to the mole as the **gram-mole** (noted **g-mol**), when dealing with laboratory data. However modern chemical engineering practice is to use the **kilomole** (kmol), which is identical to the kilogram-mole, but whose name and symbol adopt the SI convention for standard multiples of metric units.

Proposed future definition

Kilogram

As with other SI base units, there have been proposals to redefine the kilogram in such a way as to define some currently measured physical constants to fixed values. One proposed definition of the kilogram is:

> *The kilogram is the mass of exactly* ($^{6.0221415\times1023}/_{0.012}$) *unbound carbon-12 atoms at rest and in their ground state.*

This would have the effect of defining the Avogadro constant to be precisely 6.0221415×10^{23} elementary entities per mole.

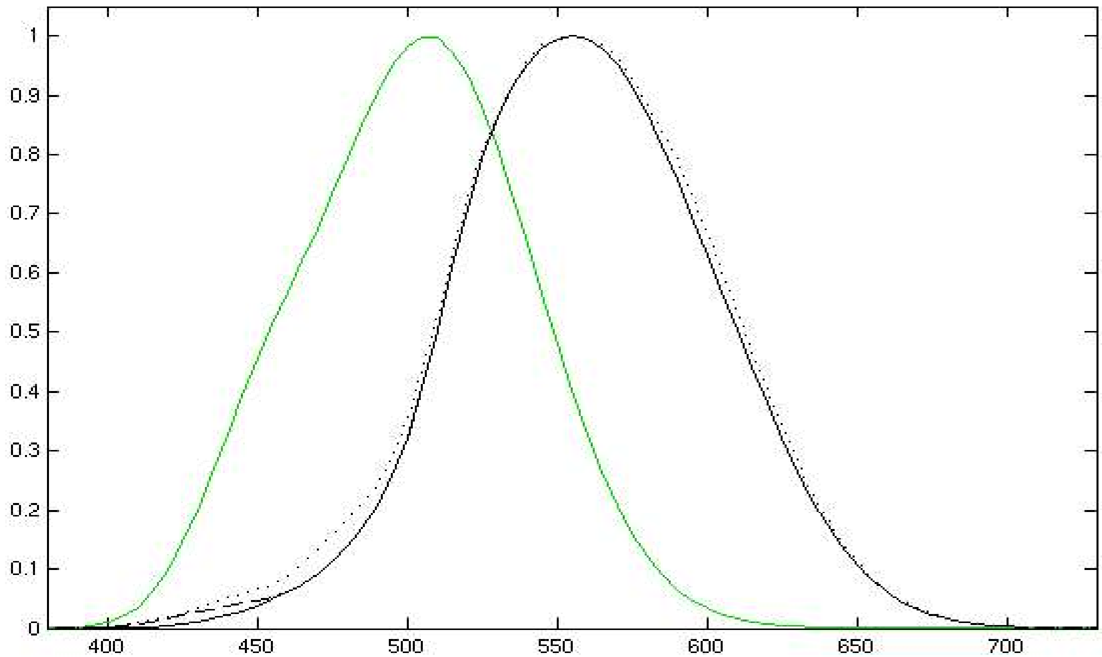
Related units

The SI units for molar concentration are mol/m^3. However, most chemical literature traditionally uses mol/dm^3, or mol dm^{-3}, which is the same as mol/L. These traditional units are often denoted by a capital letter M (pronounced "molar"), sometimes preceded by an SI prefix, for example millimoles per litre (mmol/L) or millimolar (mM), micromoles/litre (µmol/L) or micromolar (µM), or nanomoles/L (nmol/L) or nanomolar (nM).

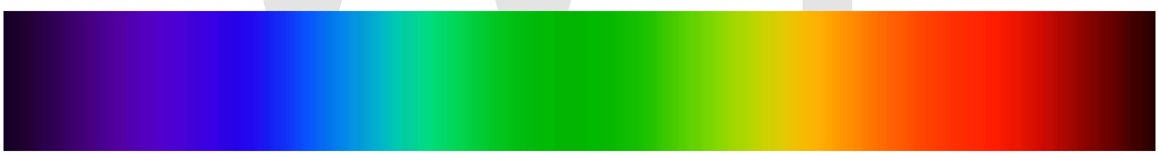
The unit's holiday

October 23 is called Mole Day. It is an informal holiday in honor of the unit among chemists in North America. The date is derived from Avogadro's number, which is approximately 6.02×10^{23}. It officially starts at 6:02 A.M. and ends at 6:02 P.M.

7. Candela



Photopic (black) and scotopic (green) luminosity functions. The photopic includes the CIE 1931 standard (solid), the Judd-Vos 1978 modified data (dashed), and the Sharpe, Stockman, Jagla & Jägle 2005 data (dotted). The horizontal axis is wavelength in nm.



The **candela** is the SI base unit of luminous intensity; that is, power emitted by a light source in a particular direction, weighted by the luminosity function (a standardized model of the sensitivity of the human eye to different wavelengths, also known as the *luminous efficiency function*). A common candle emits light with a luminous intensity of roughly one candela. If emission in some directions is blocked by an opaque barrier, the emission would still be approximately one candela in the directions that are not obscured.

The word *candela* means *candle* in Latin, Spanish and Italian.

Definition

Like other SI base units, the candela has an operational definition—it is defined by a description of a physical process that will produce one candela of luminous intensity. Since the 16th General Conference on Weights and Measures (CGPM) in 1979, the candela has been defined as:

The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 × 10^{12} hertz and that has a radiant intensity in that direction of $^1\!/_{683}$ watt per steradian.

The definition describes how to produce a light source that (by definition) emits one candela. Such a source could then be used to calibrate instruments designed to measure luminous intensity.

The candela is sometimes still called by the old name *candle*, such as in *foot-candle* and the modern definition of *candlepower*.

Explanation

The frequency chosen is in the visible spectrum near green, corresponding to a wavelength of about 555 nanometers. The human eye is most sensitive to this frequency, when adapted for bright conditions. At other frequencies, more radiant intensity is required to achieve the same luminous intensity, according to the frequency response of the human eye. The luminous intensity for light of a particular wavelength λ is given by

$$I_v(\lambda) = 683.002\,\overline{y}(\lambda) I(\lambda)$$

where $I_v(\lambda)$ is the luminous intensity in candelas, $I(\lambda)$ is the radiant intensity in W/sr and $\overline{y}(\lambda)$ is the standard luminosity function. If more than one wavelength is present (as is usually the case), one must sum or integrate over the spectrum of wavelengths present to get the total luminous intensity.

A common candle emits roughly 1 cd. A 100 W incandescent lightbulb emits about 120 cd.

Origin

Prior to 1948, various standards for luminous intensity were in use in a number of countries. These were typically based on the brightness of the flame from a "standard candle" of defined composition, or the brightness of an incandescent filament of specific design. One of the best-known of these was the English standard of candlepower. One candlepower was the light produced by a pure spermaceti candle weighing one sixth of a pound and burning at a rate of 120 grains per hour. Germany, Austria and Scandinavia used the hefnerkerze, a unit based on the output of a Hefner lamp.

It became clear that a better-defined unit was needed. The Commission Internationale de l'Éclairage (International Commission on Illumination) and the CIPM proposed a “new candle” based on the luminance of a Planck radiator (a black body) at the temperature of freezing platinum. The value of the new unit was chosen to make it similar to the earlier unit candlepower. The decision was promulgated by the CIPM in 1946:

The value of the **new candle** is such that the brightness of the full radiator at the temperature of solidification of platinum is 60 new candles per square centimetre.

It was then ratified in 1948 by the 9th CGPM which adopted a new name for this unit, the *candela*. In 1967 the 13th CGPM removed the term "new candle" and gave an amended version of the candela definition, specifying the atmospheric pressure applied to the freezing platinum:

The candela is the luminous intensity, in the perpendicular direction, of a surface of 1/600 000 square metre of a black body at the temperature of freezing platinum under a pressure of 101 325 newtons per square metre.

In 1979, because of the difficulties in realizing a Planck radiator at high temperatures and the new possibilities offered by radiometry, the 16th CGPM adopted the modern definition of the candela. The arbitrary (1/683) term was chosen so that the new definition would exactly match the old definition. Although the candela is now defined in terms of the second (an SI base unit) and the watt (a derived SI unit), the candela remains a base unit of the SI system, by definition.

SI photometric light units

SI photometry units

Quantity	Symbol	SI unit	Abbr.	Notes
Luminous energy	Q_v	lumen second	lm·s	units are sometimes called talbots
Luminous flux	F	lumen (= cd·sr)	lm	also called *luminous power*
Luminous intensity	I_v	**candela** (= lm/sr)	cd	an SI base unit
Luminance	L_v	candela per square metre	cd/m^2	units are sometimes called "nits"
Illuminance	E_v	lux (= lm/m^2)	lx	Used for light incident on a surface

Luminous emittance	M_v	lux (= lm/m^2)	lx	Used for light emitted from a surface
Luminous efficacy		lumen per watt	lm/W	ratio of luminous flux to radiant flux

Relationship between luminous intensity and luminous flux

If a source emits a known luminous intensity I_v (in candelas) in a well-defined cone, the total luminous flux F in lumens is given by

$$F = I_v \times 2\pi \times (1 - \cos(A/2)),$$

where A is the *radiation angle* of the lamp—the full vertex angle of the emission cone. For example, a lamp that emits 590 cd with a radiation angle of 40° emits about 223 lumens.

If the source emits light uniformly in all directions, the flux can be found by multiplying the intensity by 4π: a uniform 1 candela source emits 12.6 lumens.

Chapter- 7

Dimensionless Quantity

In dimensional analysis, a **dimensionless quantity** is a quantity without an associated physical dimension. It is thus a "pure" number, and as such always has a dimension of 1. Dimensionless quantities are widely used in mathematics, physics, engineering, economics, and in everyday life (such as in counting). Numerous well-known quantities, such as π, e, and φ, are dimensionless.

Dimensionless quantities are often defined as products or ratios of quantities that are not dimensionless, but whose dimensions cancel out when their powers are multiplied. This is the case, for instance, with the engineering strain, a measure of deformation. It is defined as change in length over initial length but, since these quantities both have dimensions *L* (length), the result is a dimensionless quantity.

Properties

- Even though a dimensionless quantity has no physical dimension associated with it, it can still have dimensionless units. It is sometimes helpful to use the same units in both the numerator and denominator, such as kg/kg, to show the quantity being measured (for example, to distinguish a mass ratio from a volume ratio). The quantity may also be given as a ratio of two different units that have the same dimension (for instance, light years over meters). This may be the case when calculating slopes in graphs, or when making unit conversions. Such notation does not indicate the presence of physical dimensions, and is purely a notational convention. Other common dimensionless units are % (= 0.01), ‰ (= 0.001), ppm (= 10^{-6}), ppb (= 10^{-9}), ppt (= 10^{-12}) and angle units (radians, grad, degrees). Units of amount such as the dozen and the gross are also dimensionless.
- The – dimensionless – ratio of two quantities with the same dimensions has the same value regardless of the units used to calculate them. For instance, if body **A** exerts a force of magnitude *F* on body **B**, and **B** exerts a force of magnitude *f* on **A**, then the ratio *F*/*f* will always be equal to 1, regardless of the actual units used to measure *F* and *f*. This is a fundamental property of dimensionless proportions and follows from the assumption that the laws of physics are independent of the system of units used in their expression. In this case, if the ratio *F*/*f* was not always equal to 1, but changed if we switched from SI to CGS, for instance, that would mean that Newton's Third Law's truth or falsity would depend on the system of units used, which would contradict this fundamental hypothesis. The

assumption that the laws of physics are not contingent upon a specific unit system is also closely related to the Buckingham π theorem. A formulation of this theorem is that any physical law can be expressed as an identity (always true equation) involving only dimensionless combinations (ratios or products) of the variables linked by the law (e. g., pressure and volume are linked by Boyle's Law – they are inversely proportional). If the dimensionless combinations' values changed with the systems of units, then the equation would not be an identity, and Buckingham's theorem would not hold.

Buckingham π theorem

Another consequence of the Buckingham π theorem of dimensional analysis is that the functional dependence between a certain number (say, n) of variables can be reduced by the number (say, k) of independent dimensions occurring in those variables to give a set of $p = n - k$ independent, dimensionless quantities. For the purposes of the experimenter, different systems which share the same description by dimensionless quantity are equivalent.

Example

The power consumption of a stirrer with a given shape is a function of the density and the viscosity of the fluid to be stirred, the size of the stirrer given by its diameter, and the speed of the stirrer. Therefore, we have $n = 5$ variables representing our example.

Those $n = 5$ variables are built up from $k = 3$ dimensions which are:

- Length: L (m)
- Time: T (s)
- Mass: M (kg).

According to the π-theorem, the $n = 5$ variables can be reduced by the $k = 3$ dimensions to form $p = n - k = 5 - 3 = 2$ independent dimensionless numbers which are, in case of the stirrer:

- Reynolds number (a dimensionless number describing the fluid flow regime)
- Power number (describing the stirrer and also involves the density of the fluid)

Standards efforts

The CIPM Consultative Committee for Units contemplated defining the unit of 1 as the 'uno', but the idea was dropped.

Examples

Consider this example: Sarah says, *"Out of every 10 apples I gather, 1 is rotten."*. The rotten-to-gathered ratio is (1 apple) / (10 apples) = 0.1 = 10%, which is a dimensionless quantity. Another more typical example in physics and engineering is the measure of plane angles. An angle is measured as the ratio of the length of a circle's arc subtended by an angle whose vertex is the centre of the circle to some other length. The ratio, length divided by length, is dimensionless. When using radians as the unit, the length that is compared is the length of the radius of the circle. When using degree as the units, the arc's length is compared to 1/360 of the circumference of the circle.

List of dimensionless quantities

All numbers are dimensionless quantities. Certain dimensionless quantities of some importance are given below:

Name	Standard symbol	Definition	Field of application
Abbe number	V		optics (dispersion in optical materials)
Activity coefficient	γ		chemistry (Proportion of "active" molecules or atoms)
Albedo	α		climatology, astronomy (reflectivity of surfaces or bodies)
Archimedes number	Ar		motion of fluids due to density differences
Arrhenius number	α		Ratio of activation energy to thermal energy
Atomic weight	M		chemistry
Bagnold number	Ba		flow of bulk solids such as grain and sand.
Blowdown circulation number	BC		deviation from isothermal flow in blowdown (rapid depressurization) of a pressure vessel
Bejan number (thermodynamics)	Be		the ratio of heat transfer irreversibility to total irreversibility due to heat transfer and fluid friction
Bejan number (fluid mechanics)	Be		dimensionless pressure drop along a channel
Bingham number	Bm	$Bm = \frac{\tau_y L}{\mu V}$	Ratio of yield stress to viscous stress

Biot number	*Bi*		surface vs. volume conductivity of solids
Blake number	*Bl* or *B*		relative importance of inertia compared to viscous forces in fluid flow through porous media
Bodenstein number			residence-time distribution
Bond number	*Bo*		capillary action driven by buoyancy
Brinkman number	*Br*		heat transfer by conduction from the wall to a viscous fluid
Brownell-Katz number			combination of capillary number and Bond number
Capillary number	*Ca*		fluid flow influenced by surface tension
Coefficient of static friction	μ_s		friction of solid bodies at rest
Coefficient of kinetic friction	μ_k		friction of solid bodies in translational motion
Colburn j factor			dimensionless heat transfer coefficient
Courant-Friedrich-Levy number	ν		numerical solutions of hyperbolic PDEs
Damkohler number	*Da*		reaction time scales vs. transport phenomena
Damping ratio	ζ	$\zeta = \frac{c}{2\sqrt{km}}$	the level of damping in a system
Darcy friction factor	C_f or f		fluid flow
Dean number	*D*		vortices in curved ducts
Deborah number	*De*		rheology of viscoelastic fluids
Decibel	*dB*		ratio of two intensities, often sound
Drag coefficient	C_d		flow resistance
Dukhin number	*Du*		ratio of electric surface conductivity to the electric bulk conductivity in heterogeneous systems
Euler's number	*e*		mathematics
Eckert number	*Ec*		convective heat transfer

Ekman number	*Ek*		geophysics (frictional (viscous) forces)
Elasticity (economics)	*E*		widely used to measure how demand or supply responds to price changes
Eötvös number	*Eo*		determination of bubble/drop shape
Ericksen number	*Er*		liquid crystal flow behavior
Euler number	*Eu*		hydrodynamics (pressure forces vs. inertia forces)
Fanning friction factor	*f*		fluid flow in pipes
Feigenbaum constants	α,δ		chaos theory (period doubling)
Fine structure constant	α	$\alpha = \frac{e^2}{2\varepsilon_0 hc}$	quantum electrodynamics (QED)
f-number	*f*		optics, photography
Foppl–von Karman number			thin-shell buckling
Fourier number	*Fo*		heat transfer
Fresnel number	*F*		slit diffraction
Froude number	*Fr*		wave and surface behaviour
Gain			electronics (signal output to signal input)
Galilei number	*Ga*		gravity-driven viscous flow
Golden ratio	φ		mathematics and aesthetics
Graetz number	*Gz*		heat flow
Grashof number	*Gr*		free convection
Gravitational coupling constant	α_G	$\alpha_G = \frac{Gm_e^2}{\hbar c}$	Gravitation
Hatta number	*Ha*		adsorption enhancement due to chemical reaction
Hagen number	*Hg*		forced convection
Hydraulic gradient	*i*		groundwater flow
Karlovitz number			turbulent combustion turbulent combustion
Keulegan–Carpenter number	K_C		ratio of drag force to inertia for a bluff object in oscillatory fluid flow

Knudsen number	Kn	ratio of the molecular mean free path length to a representative physical length scale
Kt/V		medicine
Kutateladze number	K	counter-current two-phase flow
Laplace number	La	free convection within immiscible fluids
Lewis number	Le	ratio of mass diffusivity and thermal diffusivity
Lift coefficient	C_L	lift available from an airfoil at a given angle of attack
Lockhart-Martinelli parameter	χ	flow of wet gases
Lundquist number	S	ratio of a resistive time to an Alfvén wave crossing time in a plasma
Mach number	M	gas dynamics
Magnetic Reynolds number	R_m	magnetohydrodynamics
Manning roughness coefficient	n	open channel flow (flow driven by gravity)
Marangoni number	Mg	Marangoni flow due to thermal surface tension deviations
Morton number	Mo	determination of bubble/drop shape
Mpemba number	K_M	thermal conduction and diffusion in freezing of a solution
Nusselt number	Nu	heat transfer with forced convection
Ohnesorge number	Oh	atomization of liquids, Marangoni flow
Péclet number	Pe	advection–diffusion problems
Peel number		adhesion of microstructures with substrate
Pi	π	mathematics (ratio of a circle's circumference to its diameter)
Poisson's ratio	ν	elasticity (load in transverse and longitudinal direction)

Porosity	φ		geology
Power factor			electronics (real power to apparent power)
Power number	N_p		power consumption by agitators
Prandtl number	Pr	$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$	convection heat transfer (thickness of thermal and momentum boundary layers)
Pressure coefficient	C_P		pressure experienced at a point on an airfoil
Q factor	Q		describes how under-damped an oscillator or resonator is
Radian	rad		measurement of angles
Rayleigh number	Ra		buoyancy and viscous forces in free convection
Refractive index	n		electromagnetism, optics
Reynolds number	Re	$Re = \frac{vL\rho}{\mu}$	Ratio of fluid inertial and viscous forces
Relative density	RD		hydrometers, material comparisons
Richardson number	Ri		effect of buoyancy on flow stability
Rockwell scale			mechanical hardness
Rolling resistance coefficient	C_{rr}	$C_{rr} = \frac{N_f}{F}$	Vehicle dynamics
Rossby number	R_o		inertial forces in geophysics
Rouse number	Z or P		sediment transport
Schmidt number	Sc		fluid dynamics (mass transfer and diffusion)
Shape factor	H		ratio of displacement thickness to momentum thickness in boundary layer flow
Sherwood number	Sh		mass transfer with forced convection
Shields parameter	τ_* or θ		threshold of sediment movement due to fluid motion
Sommerfeld number			boundary lubrication
Stanton number	St		heat transfer in forced convection
Stefan number	Ste		heat transfer during phase change
Stokes number	Stk		particle dynamics
Strain	ε		materials science, elasticity

Strouhal number	*St* or *Sr*		nondimensional frequency, continuous and pulsating flow
Taylor number	*Ta*		rotating fluid flows
Ursell number	*U*		nonlinearity of surface gravity waves on a shallow fluid layer
Vadasz number	*Va*	$Va = \frac{\phi Pr}{Da}$	governs the effects of porosity φ, the Prandtl number and the Darcy number on flow in a porous medium
van 't Hoff factor	*i*		quantitative analysis (K_f and K_b)
Wallis parameter	J^*		nondimensional superficial velocity in multiphase flows
Weaver flame speed number			laminar burning velocity relative to hydrogen gas
Weber number	*We*		multiphase flow with strongly curved surfaces
Weissenberg number	*Wi*		viscoelastic flows
Womersley number	α		continuous and pulsating flows

Dimensionless physical constants

Certain fundamental physical constants, such as the speed of light in a vacuum, the universal gravitational constant, and the constants of Planck and Boltzmann, are normalized to 1 if the units for time, length, mass, charge, and temperature are chosen appropriately. The resulting system of units is known as natural. However, not all physical constants cannot be eliminated in **any** system of units; the values of the remaining ones must be determined experimentally. Resulting constants include:

- α, the fine structure constant, the coupling constant for the electromagnetic interaction;
- μ or β, the proton-to-electron mass ratio, the rest mass of the proton divided by that of the electron. More generally, the rest masses of all elementary particles relative to that of the electron;
- α_s, the coupling constant for the strong force;
- α_G, the gravitational coupling constant.

Chapter- 8

Natural Units

In physics, **natural units** are physical units of measurement based only on universal physical constants. For example the elementary charge *e* is a natural unit of electric charge, or the speed of light *c* is a natural unit of speed. A purely natural system of units is defined in such a way that some set of selected universal physical constants are normalized to unity; that is, their numerical values in terms of these units become exactly 1.

Introduction

Natural units are intended to elegantly simplify particular algebraic expressions appearing in physical law or to normalize some chosen physical quantities that are properties of universal elementary particles and that may be reasonably believed to be constant. However, what may be believed and forced to be constant in one system of natural units can very well be allowed or even assumed to vary in another natural unit system.

Natural units are natural because the origin of their definition comes only from properties of nature and not from any human construct. Planck units are often, without qualification, called "natural units", when in fact they constitute only one of several systems of natural units, albeit the best known such system. Planck units might be considered unique in that the set of units are not based on properties of any prototype, object, or particle but are solely derived from the properties of free space.

As with other systems of units, the base units of a set of natural units will include definitions and values for length, mass, time, temperature, and electric charge (in lieu of electric current). Some physicists do not recognize temperature as a fundamental physical quantity, since it simply expresses the energy per degree of freedom of a particle, which can be expressed in terms of energy (or mass, length, and time). Virtually every system of natural units normalizes Boltzmann's constant k_B to 1, which can be thought of as simply a way of defining the unit temperature.

In the SI unit system, electric charge is a separate fundamental dimension of physical quantity, but in natural unit systems charge is expressed in terms of the mechanical units of mass, length, and time, similarly to cgs. There are two common ways to relate charge to mass, length, and time: In "rationalized" (or "Heaviside-Lorentz") natural unit systems, Coulomb's law is $F=q_1q_2/(4\pi r^2)$, and in "non-rationalized" (or "Gaussian"), Coulomb's law is $F=q_1q_2/r^2$. Both possibilities are incorporated into different natural unit systems.

Notation and use

Natural units are most commonly used by *setting the units to one*. For example, many natural unit systems include the equation $c = 1$ in the unit-system definition, where c is the speed of light. If a velocity v is half the speed of light, then from the equations $v = \frac{1}{2}c$ and $c = 1$, the consequence is $v = \frac{1}{2}$. The equation $v = \frac{1}{2}$ means "the velocity v has the value one-half when measured in Planck units", or "the velocity v is one-half the Planck unit of velocity".

The equation $c = 1$ can be plugged in anywhere else. For example, Einstein's equation $E = mc^2$ can be rewritten in Planck units as $E = m$. This equation means "The rest-energy of a particle, measured in Planck units of energy, equals the rest-mass of a particle, measured in Planck units of mass."

Advantages and disadvantages

Compared to SI or other unit systems, natural units have both advantages and disadvantages:

- **Simplified equations:** By setting constants to 1, equations containing those constants appear more compact and in some cases may be simpler to understand. For example, the special relativity equation $E^2 = p^2c^2 + m^2c^4$ appears somewhat complicated, but the natural units version, $E^2 = p^2 + m^2$, appears simpler.

- **Physical interpretation:** Natural unit systems automatically incorporate dimensional analysis. For example, in Planck units, the units are defined by properties of quantum mechanics and gravity. Not coincidentally, the Planck unit of length is approximately the length where quantum gravity effects become important. Likewise, atomic units are based on the mass and charge of an electron, and not coincidentally the atomic unit of length is the Bohr radius describing the orbit of the electron in a hydrogen atom.

- **No prototypes:** A prototype is a physical object that defines a unit, such as the International Prototype Kilogram, a certain cylinder whose mass is by definition exactly one kilogram. A prototype definition always has imperfect reproducibility between different places and between different times, and it is an advantage of natural unit systems that they use no prototypes. (They share this advantage with other *non-natural* unit systems, such as conventional electrical units.)

- **Less precise measurements:** SI units are designed to be used in precision measurements. For example, the second is defined by an atomic transition frequency in cesium atoms, because this transition frequency can be precisely reproduced with atomic clock technology. Natural unit systems are *not* generally based on quantities that can be precisely reproduced in a lab. Therefore, a quantity measured in natural units can have fewer digits of precision than the same quantity measured in SI. For example, Planck units use the gravitational constant G, which is measurable in a laboratory only to four significant digits.

- **Greater potential for ambiguity:** Consider the equation $a = 10^{10}$ in Planck units. If a represents a length, then the equation means $a = 1.6\times10{-}25$ m. If a represents a mass, then the equation means $a = 220$ kg. Therefore, if the variable a was not clearly defined, then the equation $a = 10^{10}$ might be misinterpreted. By contrast, in SI units, the equation would be $a = 220$ kg, and it would be automatically clear that a represents a mass, not a length or anything else. In fact, natural units are especially useful when this ambiguity is *deliberate*: For example, in special relativity space and time are so closely related that it can be useful to not specify whether a variable represents a distance or a time.

Candidate physical constants used in natural unit systems

The candidate physical constants to be normalized are chosen from those in the following table. Note that only a smaller subset of the following can be normalized in any one system of units without contradiction in definition (*e.g.*, m_e and m_p cannot both be defined as the unit mass in a single system).

Constant	Symbol	Dimension
Speed of light in vacuum	$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$	L T^{-1}
Magnetic constant	$\mu_0 = \frac{1}{\epsilon_0 c^2}$	Q^{-2} M L
Electric constant	$\epsilon_0 = \frac{1}{\mu_0 c^2}$	Q^{2} M^{-1} L^{-3} T^{2}
Coulomb constant	$k_e = \frac{1}{4\pi\epsilon_0} = \frac{\mu_0 c^2}{4\pi}$	Q^{-2} M L^{3} T^{-2}

Characteristic impedance of free space	$Z_0 = \mu_0 c = \frac{1}{\epsilon_0 c}$	Q^{-2}	M	L^2	T^{-1}	
Gravitational constant	G		M^{-1}	L^3	T^{-2}	
Planck constant (reduced)	$\hbar = \frac{h}{2\pi}$		M	L^2	T^{-1}	
Boltzmann constant	k_B		M	L^2	T^{-2}	Θ^{-1}
Elementary charge	e	Q				
Electron mass	m_e		M			
Proton mass	m_p		M			

Judiciously choosing units can only normalize physical constants that have dimension. Dimensionless physical constants cannot take on a different numerical value no matter what system of units is used. One such constant important to physics is the fine-structure constant, α:

$$\alpha \equiv \frac{e^2}{\hbar c(4\pi\epsilon_0)} = \frac{1}{137.035999679} = 7.2973525376 \cdot 10^{-3}$$

Since α is a fixed dimensionless number not equal to 1, it is not possible to define a system of natural units that will normalize *all* of the physical constants that comprise α. Any three of the four constants: c, $\hbar$, e, or $4\pi\varepsilon_0$, can be normalized (leaving the remaining physical constant to take on a value that is a simple function of α, attesting to the fundamental nature of the fine-structure constant) but not all four.

Systems of natural units

Planck units

Quantity	Expression	Metric value
Length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	1.616252×10^{-35} m
Mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}}$	$2.17644(11) \times 10^{-8}$ kg
Time (T)	$t_P = \sqrt{\frac{\hbar G}{c^5}}$	5.39124×10^{-44} s
Electric charge (Q)	$q_P = \sqrt{\hbar c (4\pi\epsilon_0)}$	$1.87554573 \times 10^{-18}$ C
Temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G {k_B}^2}}$	1.416785×10^{32} K

$$c = G = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$e = \sqrt{\alpha} \approx 0.08542$$

Planck units are unique among systems of natural units, because they are not defined in terms of properties of any prototype, physical object, or even elementary particle. They only refer to the basic structure of the laws of physics: c and G are part of the structure of spacetime in general relativity, and $\hbar$ captures the relationship between energy and frequency which is at the foundation of quantum mechanics. This makes Planck units particularly useful and common in theories of quantum gravity, including string theory.

Some may consider Planck units to be "more natural" even than other natural unit systems discussed below. For example, some other systems use the mass of an electron as a parameter to be normalized. But the electron is just one of 15 known massive elementary particles, all with different masses, and there is no compelling reason, within fundamental physics, to emphasize the electron mass over some other elementary particle's mass.

"Natural units" (particle physics)

Unit	Metric value	Derivation
1 eV^{-1} of length	1.97×10^{-7} m	$= (1\text{eV}^{-1})\hbar c$
1 eV of mass	1.78×10^{-36} kg	$= (1\text{eV}) / c^2$
1 eV^{-1} of time	6.58×10^{-16} s	$= (1\text{eV}^{-1})\hbar$
1 unit of electric charge (for rationalized)	5.29×10^{-19} C	$= \sqrt{\hbar c \epsilon_0}$

1 eV of temperature 1.16×10^4 K $= 1\text{eV} / k_B$

In particle physics, the phrase "natural units" generally means:

$$\hbar = c = k_B = 1.$$

This is yet not enough information to define a unit system. Next, this equation is almost always supplemented with a definition of charge, from one of two possibilities:

- **Rationalized (Heaviside-Lorentz units)**:

  $\varepsilon_0 = \mu_0 = Z_0 = 1$

- **Not rationalized (Gaussian units)**:

$$\frac{1}{4\pi\epsilon_0} = \mu_0 = 1$$

In Heaviside-Lorentz units, there are factors of 4π in Coulomb's law and the Biot-Savart law but not in Maxwell's equations; In Gaussian units, it is the reverse. Both systems are used, although Heaviside-Lorentz is more common. The value of the elementary charge in the two systems is:

$e = \sqrt{\alpha} \approx 0.08542$ (Gaussian).
$e = \sqrt{4\pi\alpha} \approx 0.3028$ (Lorentz-Heaviside).

Finally, one more unit is needed. Most commonly, electron-volt (eV) is used, despite the fact that this is not a "natural" unit in the sense discussed above. (The SI prefixed multiples of eV are used as well: keV, MeV, GeV, etc.)

With the addition of eV (or any other auxiliary unit), any quantity can be expressed. For example, a distance of 1 cm can be expressed in terms of eV, in natural units, as:

$$1\,\text{cm} = \frac{1\,\text{cm}}{\hbar c} \approx 51000\,\text{eV}^{-1}$$

Stoney units

Quantity	Expression	Metric Value
Length (L)	$l_S = \sqrt{\frac{Ge^2}{c^4(4\pi\epsilon_0)}}$	1.38068×10^{-36} m

Mass (M)	$m_S = \sqrt{\frac{e^2}{G(4\pi\epsilon_0)}}$	1.85921×10^{-9} kg
Time (T)	$t_S = \sqrt{\frac{Ge^2}{c^6(4\pi\epsilon_0)}}$	4.60544×10^{-45} s
Electric charge (Q)	$q_S = e$	1.60218×10^{-19} C
Temperature (Θ)	$T_S = \sqrt{\frac{c^4 e^2}{G(4\pi\epsilon_0){k_B}^2}}$	1.21028×10^{31} K

Stoney units are defined by:

$$c = G = e = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$\hbar = \frac{1}{\alpha}$$

where α is the fine-structure constant.

George Johnstone Stoney was the first physicist to introduce the concept of natural units. He presented the idea in a lecture entitled "On the Physical Units of Nature" delivered to the British Association in 1874. Stoney units differ from Planck units by fixing the elementary charge at 1, instead of Planck's constant (only discovered after Stoney's proposal).

Stoney units are rarely used in modern physics for calculations, but they are of historical interest.

Atomic units

Quantity	Expression (Hartree atomic units)	Metric Value (Hartree atomic units)
Length (L)	$l_A = \frac{\hbar^2(4\pi\epsilon_0)}{m_e e^2}$	5.29177×10^{-11} m
Mass (M)	$m_A = m_e$	9.10938×10^{-31} kg
Time (T)	$t_A = \frac{\hbar^3(4\pi\epsilon_0)^2}{m_e e^4}$	2.41889×10^{-17} s
Electric charge (Q)	$q_A = e$	1.60218×10^{-19} C

Temperature (Θ) $T_A = \frac{m_e e^4}{\hbar^2 (4\pi\epsilon_0)^2 k_B}$ 3.15774×10^5 K

There are two types of atomic units, closely related: **Hartree atomic units**:

$$e = m_e = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$c = \frac{1}{\alpha}$$

Rydberg atomic units:

$$\frac{e}{\sqrt{2}} = 2m_e = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$c = \frac{2}{\alpha}$$

These units are designed to simplify atomic and molecular physics and chemistry, especially the hydrogen atom, and are widely used in these fields. The Hartree units were first proposed by Douglas Hartree, and are more common than the Rydberg units.

The units are designed especially to characterize the behavior of an electron in the ground state of a hydrogen atom. For example, using the Hartree convention, in the Bohr model of the hydrogen atom, an electron in the ground state has orbital velocity = 1, orbital radius = 1, angular momentum = 1, ionization energy = ½, etc.

The unit of energy is called the Hartree energy in the Hartree system and the Rydberg energy in the Rydberg system. They differ by a factor of 2. The speed of light is relatively large in atomic units (137 in Hartree or 274 in Rydberg), which comes from the fact that an electron in hydrogen tends to move much slower than the speed of light. The gravitational constant is extremely small in atomic units (around 10^{-45}), which comes from the fact that the gravitational force between two electrons is far weaker than the Coulomb force. The unit length, m_A, is the Bohr radius, a_0.

Quantum chromodynamics (QCD) system of units

Quantity	Expression
Length (L)	$l_{\mathrm{QCD}} = \frac{\hbar}{m_p c}$
Mass (M)	$m_{\mathrm{QCD}} = m_p$
Time (T)	$t_{\mathrm{QCD}} = \frac{\hbar}{m_p c^2}$

Electric charge (Q) $q_{\text{QCD}} = e$

Temperature (Θ) $T_{\text{QCD}} = \frac{m_p c^2}{k_B}$

$$c = e = m_p = \hbar = k_B = 1$$
$$\frac{1}{4\pi\epsilon_0} = \alpha$$

The electron mass is replaced with that of the proton and the permittivity of free space is not fixed by definition. *Strong units* are convenient for work in QCD and nuclear physics, where quantum mechanics and relativity are omnipresent and the proton is an object of central interest.

Geometrized units

$$c = G = 1$$

The geometrized unit system is not a completely defined or unique system. In this system, the base physical units are chosen so that the speed of light and the gravitational constant are set equal to unity, leaving latitude to also set some other constant such as the Boltzmann constant and Coulomb force constant equal to unity:

$$k_B = 1$$
$$\frac{1}{4\pi\epsilon_0} = 1$$

If the reduced Planck constant is also set equal to unity,

$$\hbar = 1$$

then geometrized units are identical to Planck units.

Summary table

Quantity / Symbol	Planck	Stoney	Atomic (Hartree)	"Natural Units" (Rationalized)	QCD
Speed of light in vacuum c	1	1	$\frac{1}{\alpha}$	1	1

Electric constant (vacuum permittivity) ϵ_0	$\frac{1}{4\pi}$	$\frac{1}{4\pi}$	$\frac{1}{4\pi}$	1	$\frac{1}{4\pi\alpha}$
Magnetic constant (vacuum permeability) $\mu_0 = \frac{1}{\epsilon_0 c^2}$	4π	4π	$4\pi\alpha^2$	1	$4\pi\alpha$
Characteristic impedance of vacuum $Z_0 = \frac{1}{\epsilon_0 c}$	4π	4π	$4\pi\alpha$	1	$4\pi\alpha$
Planck's constant (reduced) $\hbar = \frac{h}{2\pi}$	1	$\frac{1}{\alpha}$	1	1	1
Elementary charge e	$\sqrt{\alpha}$	1	1	$\sqrt{4\pi\alpha}$	1
Josephson constant $K_J = \frac{e}{\pi\hbar}$	$\frac{\sqrt{\alpha}}{\pi}$	$\frac{\alpha}{\pi}$	$\frac{1}{\pi}$	$\sqrt{\frac{4\alpha}{\pi}}$	$\frac{1}{\pi}$
von Klitzing constant $R_K = \frac{2\pi\hbar}{e^2}$	$\frac{2\pi}{\alpha}$	$\frac{2\pi}{\alpha}$	2π	$\frac{1}{2\alpha}$	2π

Gravitational constant G	1	1	$\alpha_G = \left(\frac{m_e}{m_P}\right)^2$	$G c^{-5} \hbar^{-1} \approx 10^{-56}\,\mathrm{eV}^{-2}$	$\mu^2 \alpha_G$
Electron mass m_e	$\sqrt{\alpha_G}$	$\sqrt{\frac{\alpha_G}{\alpha}}$	1	$m_e c^2 \approx 511\,\mathrm{keV}$	$\frac{1}{\mu}$

where:

- α is the fine-structure constant, $7.2973525376 \times 10^{-3} = (137.035999679)^{-1}$,
- α_G is the gravitational coupling constant, $(m_e/m_{Planck})^2 \approx 1.7518 \times 10^{-45}$,
- μ is the proton-to-electron mass ratio, about 1836.15267247.

Chapter- 9

Buckingham π Theorem

The **Buckingham π theorem** is a key theorem in dimensional analysis. It is a formalization of Rayleigh's method of dimensional analysis. The theorem loosely states that if we have a physically meaningful equation involving a certain number, n, of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of $p = n - k$ dimensionless parameters constructed from the original variables: it is a scheme for nondimensionalization. This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Statement

More formally, the number of dimensionless terms that can be formed, p, is equal to the nullity of the dimensional matrix, and k is the rank. For the purposes of the experimenter, different systems which share the same description in terms of these dimensionless numbers are equivalent.

In mathematical terms, if we have a physically meaningful equation such as

$$f(q_1, q_2, \ldots, q_n) = 0$$

where the q_i are the n physical variables, and they are expressed in terms of k independent physical units, then the above equation can be restated as

$$F(\pi_1, \pi_2, \ldots, \pi_p) = 0$$

where the π_i are dimensionless parameters constructed from the q_i by $p = n - k$ equations of the form

$$\pi_i = q_1^{a_1}\, q_2^{a_2} \cdots q_n^{a_n}$$

where the exponents a_i are rational numbers (they can always be taken to be integers: just raise it to a power to clear denominators).

The use of the π_i as the dimensionless parameters was introduced by Edgar Buckingham in his original 1914 paper on the subject from which the theorem draws its name.

Significance

The Vaschy-Buckingham π theorem provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Two systems for which these parameters coincide are called *similar* (as with similar triangles, they differ only in scale); they are equivalent for the purposes of the equation, and the experimentalist who wants to determine the form of the equation can choose the most convenient one.

Proof

Outline

It will be assumed that the space of fundamental and derived physical units forms a vector space over the rational numbers, with the fundamental units as basis vectors, and with multiplication of physical units as the "vector addition" operation, and raising to powers as the "scalar multiplication" operation: represent a dimensional variable as the set of exponents needed for the fundamental units (with a power of zero if the particular fundamental unit is not present). For instance, the gravitational constant g has units of $\ell/t^2 = \ell^1 t^{-2}$ (distance over time squared), so it is represented as the vector $(1, -2)$ with respect to the basis of fundamental units (distance,time).

Making the physical units match across sets of physical equations can then be regarded as imposing linear constraints in the physical unit vector space.

Formal proof

Given a system of n dimensional variables (physical variables), in k (physical) dimensions, write the *dimensional matrix M*, whose rows are the dimensions and whose columns are the variables: the *(i,j)*th entry is the power of the *i*th unit in the *j*th variable. The matrix can be interpreted as taking in a combination of the dimensional quantities and giving out the dimensions of this product. So

$$M \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

is the units of

$$q_1^{a_1} \, q_2^{a_2} \cdots q_n^{a_n}.$$

A dimensionless variable is a combination whose units are all zero (hence, dimensionless), which is equivalent to the kernel of this matrix; a dimensionless variable is a linear relation between units of dimensional variables.

By the rank-nullity theorem, a system of *n* vectors in *k* dimensions (where all dimensions are necessary) satisfies a *(p=n-k)*-dimensional space of relations. Any choice of basis will have *p* elements, which are the dimensionless variables.

The dimensionless variables can always be taken to be integer combinations of the dimensional variables (by clearing denominators). There is mathematically no natural choice of dimensionless variables; some choices of dimensionless variables are more physically meaningful, and these are what are ideally used.

Examples

Speed

This example is elementary, but demonstrates the general procedure: Suppose a car is driving at 100 km/hour; how long does it take it to go 200 km?

This question has 2 fundamental physical units: time *t* and length ℓ, and 3 dimensional variables: distance *D*, time taken *T*, and velocity *V*. Thus there is 3-2=1 dimensionless quantity. The units of the dimensional quantities are:

$$D \sim \ell, \; T \sim t, \; V \sim \ell/t.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The rows correspond to the dimensions ℓ, and *t*, and the columns to the dimensional variables *D*, *T*, *V*. For instance, the 3rd column, (1, −1), states that the *V* (velocity) variable has units of $\ell^1 t^{-1} = \ell/t$.

For a dimensionless constant $\pi = D^{a_1}T^{a_2}V^{a_3}$ we are looking for a vector a = [a_1,a_2,a_3] such that the matrix product of M on a yields the zero vector [0,0]. In linear algebra, this vector is known as the kernel of the dimensional matrix, and it spans the nullspace of the dimensional matrix, which in this particular case is one dimensional. The dimensional matrix as written above is in reduced row echelon form, so one can read off that a kernel vector may be written (to within a multiplicative constant) by:

$$a = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

If the dimensional matrix were not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix in order to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\begin{aligned} \pi &= D^{-1}T^1V^1 \\ &= TV/D \end{aligned}$$

or, in dimensional terms:

$$\pi \sim (\ell)^{-1}(t)^1(\ell/t)^1 \sim 1$$

Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant.

Dimensional analysis has thus provided a general equation relating the three physical variables:

$$f(\pi) = 0$$

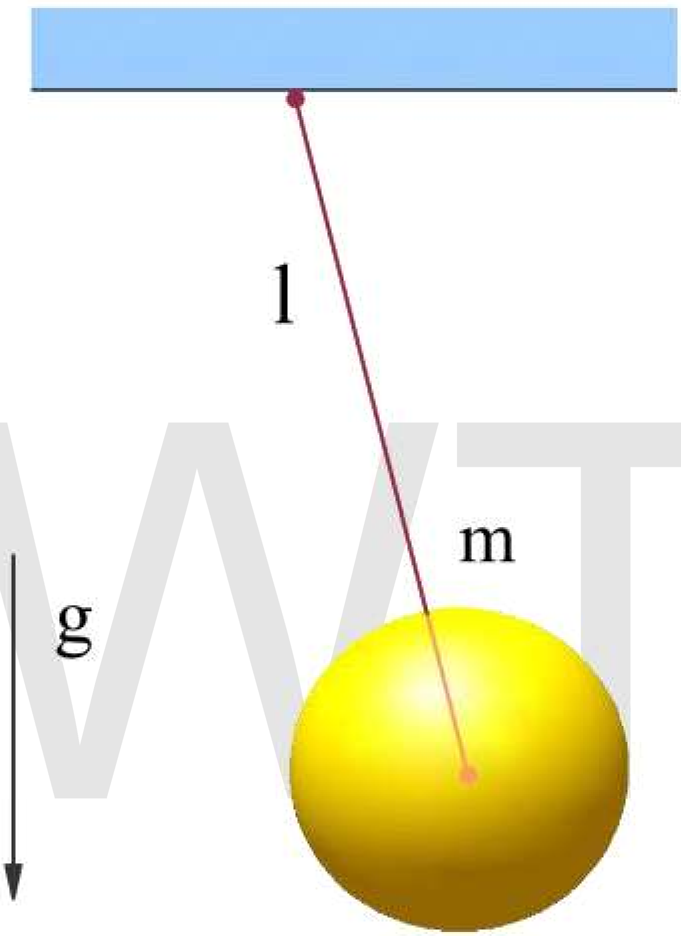
which may be written:

$$T = \frac{CD}{V}$$

where C is one of a set of constants, such that $C = f^{-1}(0)$. The actual relationship between the three variables is simply $D = VT$ so that the actual dimensionless equation ($f(\pi) = 0$) is written:

$$f(\pi) = \pi - 1 = VT / D - 1 = 0$$

In other words, there is only one value of C and it is unity. The fact that there is only a single value of C and that it is equal to unity is a level of detail not provided by the technique of dimensional analysis.



The simple pendulum

We wish to determine the period T of small oscillations in a simple pendulum. It will be assumed that it is a function of the length L, the mass M, and the acceleration due to gravity on the surface of the Earth g, which has units of length divided by time squared. The model is of the form

$$f(T, M, L, g) = 0.$$

(Note that it is written as a relation, not as a function: *T* isn't here written as a function of *M*, *L*, and *g*.)

There are 3 fundamental physical units in this equation: time *t*, mass *m*, and length *l*, and 4 dimensional variables, *T*, *M*, *L*, and *g*. Thus we need only 4−3=1 dimensionless parameter, denoted π, and the model can be re-expressed as

$$f(\pi) = 0$$

where π is given by

$$\pi = T^{a1} M^{a2} L^{a3} g^{a4}$$

for some values of $a_1, ..., a_4$.

The units of the dimensional quantities are:

$$T = t, M = m, L = \ell, g = \ell/t^2.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(The rows correspond to the dimensions *t*, *m*, and *l*, and the columns to the dimensional variables *T*, *M*, *L* and *g*. For instance, the 4th column, (−2, 0, 1), states that the *g* variable has units of $t^{-2} m^0 \ell^1$.)

We are looking for a kernel vector a = [a1,a2,a3,a4] such that the matrix product of *M* on *a* yields the zero vector [0,0,0]. The dimensional matrix as written above is in reduced row echelon form, so one can read off that a kernel vector may be written (to within a multiplicative constant) by:

$$a = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Were it not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix in order to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\begin{aligned}\pi &= T^2 M^0 L^{-1} g^1 \\ &= gT^2/L\end{aligned}$$

In dimensional terms:

$$\pi = (t)^2(m)^0(\ell)^{-1}(\ell/t^2)^1 = 1$$

which is dimensionless. Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant.

This example is easy because 3 of the dimensional quantities are fundamental units, so the last (*g*) is a combination of the previous. Note that if a_2 were non-zero there would be no way to cancel the *M* value—therefore a_2 *must* be zero. Dimensional analysis has allowed us to conclude that the period of the pendulum is not a function of its mass. (In the 3D space of powers of mass, time, and distance, we can say that the vector for mass is linearly independent from the vectors for the three other variables. Up to a scaling factor, $\vec{g} - 2\vec{T} - \vec{L}$ is the only nontrivial way to construct a vector of a dimensionless parameter.)

The model can now be expressed as:

$$f(gT^2/L) = 0.$$

Assuming the zeroes of f are discrete, we can say $gT^2/L = C_n$ where C_n is the *n*th zero. If there is only one zero, then $gT^2/L = C$. It requires more physical insight or an experiment to show that there is indeed only one zero and that the constant is in fact given by $C = 4\pi^2$.

For large oscillations of a pendulum, the analysis is complicated by an additional dimensionless parameter, the maximum swing angle. The above analysis is a good approximation as the angle approaches zero.

The atomic bomb

Sir Geoffrey I. Taylor used dimensional analysis to estimate the energy released in an atomic bomb explosion (Taylor, 1950a,b). The first atomic bomb was detonated near Alamogordo, New Mexico on July 16, 1945. In 1947, movies of the explosion were declassified, allowing Sir Geoffrey to complete the analysis and estimate the energy released in the explosion, even though the energy release was still classified. The actual energy released (about 20ktons of TNT) was later declassified and its value was remarkably close to Taylor's estimate (16.8ktons).

Taylor supposed that the process was adequately described by five physical quantities: the time t since the detonation, the energy E which is released at a single point in space at detonation, the radius R of the shock wave at time T, the atmospheric pressure p and the ambient density ρ. There are only three fundamental physical units in this equation: mass, time, and length. Thus we need only 5 − 3 = 2 dimensionless parameters. The dimensional matrix for rows corresponding to L, t and m and columns corresponding to R, T, p, ρ and E is

$$M = \begin{bmatrix} 1 & 0 & -1 & -3 & 2 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The nullspace is two dimensional. Two kernel vectors which span this space are:

$$A_0 = [5, -2, 0, 1, -1] \qquad \text{and} \qquad A_1 = [0, 6, 5, -3, -2]$$

which yield the dimensionless constants:

$$\pi_0 = R\left(\frac{\rho}{Et^2}\right)^{1/5} \qquad \text{and} \qquad \pi_1 = p\left(\frac{t^6}{E^2\rho^3}\right)^{1/5}$$

Note that there are an infinite number of dimensionless constant pairs that could be defined by taking linear combinations of the above two particular kernel vectors. The process can now be described by an equation of the form

$$f(\pi_0, \pi_1) = 0,$$

Assuming that inverting the equation yields a single possible value for *R*, this may be written:

$$R = \left(\frac{Et^2}{\rho}\right)^{1/5} g(\pi_1),$$

where $g(\pi_1)$ is some function of π_1. The energy in the explosion is expected to be huge, so that for times of the order of a second after the explosion, we can estimate π_1 to be approximately zero, and experiments using light explosives can be conducted to determine that $g(0)$ is on the order of unity so that

$$R \approx \left(\frac{Et^2}{\rho}\right)^{1/5}.$$

This is Taylor's equation which, once he knew the radius of the explosion as a function of the time, allowed him to calculate the energy of the explosion. (Wan, 1989)

Chapter- 10

Homogeneity (Physics) & Fermi Problem

Homogeneity (Physics)

In general, **homogeneity** is defined as the quality or state of being *homogeneous* (of the same or similar nature, from Greek ὁμός meaning 'same'. It also means having a uniform structure throughout). For instance, a uniform electric field (which has the same strength and the same direction at each point) would be compatible with homogeneity (all points experience the same physics). A material constructed with different constituents can be described as effectively homogeneous in the electromagnetic domain, when interacting with a directed radiation field (light, microwave frequencies, etc.) In physics, homogeneous usually means describing a material or system that has the same properties at every point of the space; in other words, uniform without irregularities. In physics, it also describes a substance or an object whose properties do not vary with position. For example, an object of uniform density is sometimes described as homogeneous.

Another related definition is simply a substance that is uniform in composition.

Mathematically, homogeneity has the connotation of invariance, as all components of the equation have the same degree of value whether or not each of these components are scaled to different values, for example, by multiplication or addition. Cumulative distribution fits this description. "The state of having identical cumulative distribution function or values".

Context

The definition of homogeneous strongly depends on the context used. For example, a composite material is made up of different individual materials, known as "*constituents*" of the material, but may be defined as a homogeneous material when assigned a function. For example, asphalt paves our roads, but is a composite material consisting of asphalt binder and mineral aggregate, and then laid down in layers and compacted.

In another context, any material is actually made up of atoms and molecules. At this level, then, any material is not homogeneous. However, in our normal everyday world, a block of wood, a pane of glass, or a sheet of metal is described as wood, glass, or stainless steel. In other words, these are each described as a homogeneous material.

A few other instances of context are: *Dimensional homogeneity* is the quality of an equation having quantities of same units on both sides; *Homogeneity (in space)* implies conservation of momentum; and *homogeneity in time* implies conservation of energy.

Homogeneous alloy

In the context of composite metals is an alloy. A blend of a metal with one or more metallic or nonmetallic materials is an alloy. The components of an alloy do not combine chemically but, rather, are very finely mixed. An alloy might be homogeneous or might contain small particles of components that can be viewed with a microscope. Brass is an example of an alloy, being a homogeneous mixture of copper and zinc. Another example is steel, which is an alloy of iron with carbon and possibly other metals. The purpose of alloying is to produce desired properties in a metal that naturally lacks them. Brass, for example, is harder than copper and has a more gold-like color. Steel is harder than iron and can even be made rust proof (stainless steel).

Homogeneous cosmology

Homogeneity, in another context plays a role in cosmology. From the perspective of 19th-century cosmology (and before), the universe was infinite, unchanging, homogeneous, and therefore filled with stars. In 1826, this being true according to the German astronomer Heinrich Olbers, then the entire night sky should be filled with light and as bright as daytime, but it is dark. He presented a technical paper in 1826 that attempted to answer this conundrum. The faulty premise, unknown in Olbers' time, was that the universe is not infinite, static, and homogeneous. The Big Bang cosmology replaced this model (expanding, finite, and inhomogeneous universe). However, modern astronomers supply reasonable explanations to answer this question. One of at least several explanations is that distant stars and galaxies are red shifted, which weakens their apparent light and makes the night sky dark.

Translation invariance

By translation invariance, one means independence of (absolute) position, especially when referring to a law of physics, or to the evolution of a physical system.

Fundamental laws of physics should not (explicitly) depend on position in space. That would make them quite useless. In some sense, this is also linked to the requirement that experiments should be reproducible. This principle is true for all laws of mechanics (Newton's laws, etc.), electrodynamics, quantum mechanics, etc.

In practice, this principle is usually violated, since one studies only a small subsystem of the universe, which of course "feels" the influence of rest of the universe. This situation gives rise to "external fields" (electric, magnetic, gravitational, etc.) which make the description of the evolution of the system depending on the position (potential wells, etc.). This only stems from the fact that the objects creating these external fields are not considered as (a "dynamical") part of the system.

Translational invariance as described above is equivalent to shift invariance in system analysis, although here it is most commonly used in linear systems, whereas in physics the distinction is not usually made.

The notion of isotropy, for properties independent of direction, is not a consequence of homogeneity. For example, a uniform electric field (i.e., which has the same strength and the same direction at each point) would be compatible with homogeneity (at each point physics will be the same), but not with isotropy, since the field singles out one "preferred" direction.

Consequences

In Lagrangian formalism, homogeneity (in space) implies conservation of momentum, and homogeneity in time implies conservation of energy. This is shown, using variational calculus, in standard textbooks like the classical reference [Landau & Lifshitz] cited below. This is a particular application of Noether's theorem.

Dimensional homogeneity

As said in the introduction, *dimensional homogeneity* is the quality of an equation having quantities of same units on both sides. A valid equation in physics must be homogeneous, since equality cannot apply between quantities of different nature. This can be used to spot errors in formula or calculations. For example, if one is calculating a speed, units must always combine to [length]/[time]; if one is calculating an energy, units must always combine to [mass]•[length]^2/[time]^2, etc. For example, the following formulae could be valid expressions for some energy:

$$E_k = \frac{1}{2}mv^2; \quad E = mc^2; \quad E = pv; \quad E = hc/\lambda$$

if m is a mass, v and c are velocities, p is a momentum, h is Planck's constant, λ a length. On the other hand, if the units of the right hand side do not combine to [mass]•[length]^2/[time]^2, it cannot be a valid expression for some energy.

Being homogeneous does not necessarily mean the equation will be true, since it does not take into account numerical factors. For example, $E = m \bullet v^2$ could be or could not be the correct formula for the energy of a particle of mass m traveling at speed v, and one cannot know if $h \bullet c/\lambda$ should be divided or multiplied by 2π.

Nevertheless, this is a very powerful tool in finding characteristic units of a given problem.

Theoretical physicists tend to express everything in natural units given by constants of nature, for example by taking $c = \hbar = k = 1$; once this is done, one partly loses the possibility of the above checking. The atomic number was also known as a proton number.

Fermi Problem

In science, particularly in physics or engineering education, a **Fermi problem**, **Fermi question**, or **Fermi estimate** is an estimation problem designed to teach dimensional analysis, approximation, and the importance of clearly identifying one's assumptions. Named after physicist Enrico Fermi, such problems typically involve making justified guesses about quantities that seem impossible to compute given limited available information.

Fermi was known for his ability to make good approximate calculations with little or no actual data, hence the name. One example is his estimate of the strength of the atomic bomb detonated at the Trinity test, based on the distance travelled by pieces of paper dropped from his hand during the blast. Fermi's estimate of 10 kilotons of TNT was remarkably close to the now-accepted value of around 20 kilotons, a difference of less than one order of magnitude.

Examples of Fermi problems

The classic Fermi problem, generally attributed to Fermi, is "*How many piano tuners are there in Chicago?*" A typical solution to this problem would involve multiplying together a series of estimates that would yield the correct answer if the estimates were correct. For example, we might make the following assumptions:

1. There are approximately 5,000,000 people living in Chicago.
2. On average, there are two persons in each household in Chicago.
3. Roughly one household in twenty has a piano that is tuned regularly.
4. Pianos that are tuned regularly are tuned on average about once per year.
5. It takes a piano tuner about two hours to tune a piano, including travel time.
6. Each piano tuner works eight hours in a day, five days in a week, and 50 weeks in a year.

From these assumptions we can compute that the number of piano tunings in a single year in Chicago is

> (5,000,000 persons in Chicago) / (2 persons/household) × (1 piano/20 households) × (1 piano tuning per piano per year) = 125,000 piano tunings per year in Chicago.

We can similarly calculate that the average piano tuner performs

> (50 weeks/year)×(5 days/week)×(8 hours/day)/(1 piano tuning per 2 hours per piano tuner) = 1000 piano tunings per year per piano tuner.

Dividing gives

(125,000 piano tunings per year in Chicago) / (1000 piano tunings per year per piano tuner) = 125 piano tuners in Chicago.

A famous example of a Fermi-problem-like estimate is the Drake equation, which seeks to estimate the number of intelligent civilizations in the galaxy. The basic question of why, if there is a significant number of such civilizations, ours has never encountered any others is called the Fermi paradox.

Advantages and scope

Scientists often look for Fermi estimates of the answer to a problem before turning to more sophisticated methods to calculate a precise answer. This provides a useful check on the results: where the complexity of a precise calculation might obscure a large error, the simplicity of Fermi calculations makes them far less susceptible to such mistakes. (Performing the Fermi calculation first is preferable because the intermediate estimates might otherwise be biased by knowledge of the calculated answer.)

Fermi estimates are also useful in approaching problems where the optimal choice of calculation method depends on the expected size of the answer. For instance, a Fermi estimate might indicate whether the internal stresses of a structure are low enough that it can be accurately described by linear elasticity; or if the estimate already bears significant relationship in scale relative to some other value, for example, if a structure will be over-engineered to withstand loads several times greater than the estimate.

Although Fermi calculations are often not accurate, as there may be many problems with their assumptions, this sort of analysis does tell us what to look for to get a better answer. For the above example, we might try to find a better estimate of the number of pianos tuned by a piano tuner in a typical day, or look up an accurate number for the population of Chicago. It also gives us a rough estimate that may be good enough for some purposes: if we want to start a store in Chicago that sells piano tuning equipment, and we calculate that we need 10,000 potential customers to stay in business, we can reasonably assume that the above estimate is far enough below 10,000 that we should consider a different business plan (and, with a little more work, we could compute a rough upper bound on the number of piano tuners by considering the most extreme *reasonable* values that could appear in each of our assumptions).

A Fermi calculation that involves the multiplication of several estimated factors (e.g. the number of piano tuners in Chicago) will *probably* be more accurate than might be first supposed (assuming that there is no consistent bias in the estimated factors). This is because if there is no consistent bias, then there will probably (with a binomial distribution) be some factors that are estimated too high and other factors that are estimated too low, and such errors will partially cancel each other out.

Chapter- 11

Nondimensionalization

Nondimensionalization is the partial or full removal of units from an equation involving physical quantities by a suitable substitution of variables. This technique can simplify and parameterize problems where measured units are involved. It is closely related to dimensional analysis. In some physical systems, the term **scaling** is used interchangeably with *nondimensionalization*, in order to suggest that certain quantities are better measured relative to some appropriate unit. These units refer to quantities intrinsic to the system, rather than units such as SI units. Nondimensionalization is not the same as converting extensive quantities in an equation to intensive quantities, since the latter procedure results in variables that still carry units.

Nondimensionalization can also recover characteristic properties of a system. For example, if a system has an intrinsic resonance frequency, length, or time constant, nondimensionalization can recover these values. The technique is especially useful for systems that can be described by differential equations. One important use is in the analysis of control systems. One of the simplest characteristic units is the doubling time of a system experiencing exponential growth, or conversely the half-life of a system experiencing exponential decay; a more natural pair of characteristic units is mean age/mean lifetime, which correspond to base *e* rather than base 2.

Many illustrative examples of nondimensionalization originate from simplifying differential equations. This is because a large body of physical problems can be formulated in terms of differential equations. Consider the following:

- List of dynamical systems and differential equations topics
- List of partial differential equation topics
- Differential equations of mathematical physics

Although nondimensionalization is well adapted for these problems, it is not restricted to them. An example of a non-differential-equation application is dimensional analysis, while another is normalization in statistics.

Measuring devices are practical examples of nondimensionalization occurring in everyday life. Measuring devices are calibrated relative to some known unit. Subsequent measurements are made relative to this standard. Then, the absolute value of the measurement is recovered by scaling with respect to the standard.

Rationale

Suppose a pendulum is swinging with a particular period *T*. For such a system, it is advantageous to perform calculations relating to the swinging relative to *T*. In some sense, this is normalizing the measurement with respect to the period.

Measurements made relative to an intrinsic property of a system will apply to other systems which also have the same intrinsic property. It also allows one to compare a common property of different implementations of the same system. Nondimensionalization determines in a systematic manner the **characteristic units** of a system to use, without relying heavily on prior knowledge of the system's intrinsic properties (one should not confuse characteristic units of a *system* with natural units of *nature*). In fact, nondimensionalization can suggest the parameters which should be used for analyzing a system. However, it is necessary to start with an equation that describes the system appropriately.

Nondimensionalization steps

To nondimensionalize a system of equations, one must do the following:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order polynomial or derivative term;
4. Choose judiciously the definition of the characteristic unit for each variable so that the coefficients of as many terms as possible become 1;
5. Rewrite the system of equations in terms of their new dimensionless quantities.

The last three steps are usually specific to the problem where nondimensionalization is applied. However, almost all systems require the first two steps to be performed.

As an illustrative example, consider a first order differential equation with constant coefficients:

$$a\frac{dx}{dt} + bx = Af(t).$$

1. In this equation the independent variable here is *t*, and the dependent variable is *x*.
2. Set $x = \chi x_c,\ t = \tau t_c$. This results in the equation

$$a\frac{x_c}{t_c}\frac{d\chi}{d\tau} + bx_c\chi = Af(\tau t_c) \stackrel{\text{def}}{=} AF(\tau).$$

3. The coefficient of the highest ordered term is in front of the first derivative term. Dividing by this gives

$$\frac{d\chi}{d\tau} + \frac{bt_c}{a}\chi = \frac{At_c}{ax_c}F(\tau).$$

4. The coefficient in front of χ only contains one characteristic variable t_c, hence it is easiest to choose to set this to unity first:

$$\frac{bt_c}{a} = 1 \Rightarrow t_c = \frac{a}{b}.$$ Subsequently, $$\frac{At_c}{ax_c} = \frac{A}{bx_c} = 1 \Rightarrow x_c = \frac{A}{b}.$$

5. The final dimensionless equation in this case becomes completely independent of any parameters with units:

$$\frac{d\chi}{d\tau} + \chi = F(\tau).$$

Substitutions

Suppose for simplicity that a certain system is characterized by two variables - a dependent variable x and an independent variable t, where x is a function of t. Both x and t represent quantities with units. To scale these two variables, assume there are two intrinsic units of measurement x_c and t_c with the same units as x and t respectively, such that these conditions hold:

$$\kappa = \frac{t}{t_c} \Rightarrow t = \kappa t_c$$
$$\chi = \frac{x}{x_c} \Rightarrow x = \chi x_c.$$

These equations are used to replace x and t when nondimensionalizing. If differential operators are needed to describe the original system, their scaled counterparts become dimensionless differential operators.

Conventions

There are no restrictions on the variable names used to replace "*x*" and "*t*". However, they are generally chosen so that it is convenient and intuitive to use for the problem at hand. For example, if "*x*" represented mass, the letter "*m*" might be an appropriate symbol to represent the dimensionless mass quantity.

Here, the following conventions have been used:

- t - represents the independent variable - usually a time quantity. Its nondimensionalized counterpart is τ.
- x - represents the dependent variable - can be mass, voltage, or any measurable quantity. Its nondimensionalized counterpart is χ.

A subscripted c added to a quantity's variable-name is used to denote the characteristic unit used to scale that quantity. For example, if x is a quantity, then x_c is the characteristic unit used to scale it.

Differential operators

Consider the relationship

$$t = \tau t_c \Rightarrow dt = t_c d\tau \Rightarrow \frac{d\tau}{dt} = \frac{1}{t_c}.$$

The dimensionless differential operators with respect to the independent variable becomes

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{t_c}\frac{d}{d\tau} \Rightarrow \frac{d^n}{dt^n} = \left(\frac{d}{dt}\right)^n = \left(\frac{1}{t_c}\frac{d}{d\tau}\right)^n = \frac{1}{t_c^n}\frac{d^n}{d\tau^n}.$$

Forcing function

If a system has a forcing function $f(t)$, then

$$f(t) = f(\tau t_c) = f(t(\tau)) = F(\tau).$$

Hence, the new forcing function F is made to be dependent on the dimensionless quantity τ.

Linear differential equations with constant coefficients

First order system

Let us consider the differential equation for a first order system:

$$a\frac{dx}{dt} + bx = Af(t).$$

The derivation of the characteristic units for this system gives

$$t_c = \frac{a}{b},\ x_c = \frac{A}{b}.$$

Second order system

A second order system has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Af(t).$$

Substitution step

Replace the variables *x* and *t* with their scaled quantities. The equation becomes

$$a\frac{x_c}{t_c^2}\frac{d^2\chi}{d\tau^2} + b\frac{x_c}{t_c}\frac{d\chi}{d\tau} + cx_c\chi = Af(\tau t_c) = AF(\tau).$$

This new equation is not dimensionless, although all the variables with units are isolated in the coefficients. Dividing by the coefficient of the highest ordered term, the equation becomes

$$\frac{d^2\chi}{d\tau^2} + t_c\frac{b}{a}\frac{d\chi}{d\tau} + t_c^2\frac{c}{a}\chi = \frac{At_c^2}{ax_c}F(\tau).$$

Now it is necessary to determine the quantities of x_c and t_c so that the coefficients become normalized. Since there are two free parameters, at most only two coefficients can be made to equal unity.

Determination of characteristic units

Consider the variable t_c:

1. If $t_c = \frac{a}{b}$ the first order term is normalized.
2. If $t_c = \sqrt{\frac{a}{c}}$ the zeroth order term is normalized.

Both substitutions are valid. However, for pedagogical reasons, the latter substitution is used for second order systems. Choosing this substitution allows x_c to be determined by normalizing the coefficient of the forcing function:

$$1 = \frac{At_c^2}{ax_c} = \frac{A}{cx_c} \Rightarrow x_c = \frac{A}{c}.$$

The differential equation becomes

$$\frac{d^2\chi}{d\tau^2} + \frac{b}{\sqrt{ac}}\frac{d\chi}{d\tau} + \chi = F(\tau).$$

The coefficient of the first order term is unitless. Define

$$2\zeta \stackrel{\text{def}}{=} \frac{b}{\sqrt{ac}}.$$

The factor 2 is present so that the solutions can be parameterized in terms of ζ. In the context of mechanical or electrical systems, ζ is known as the damping ratio, and is an important parameter required in the analysis of control systems. 2ζ is also known as the linewidth of the system. The result of the definition is the universal oscillator equation.

$$\frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = F(\tau).$$

Higher order systems

The general n-th order linear differential equation with constant coefficients has the form:

$$a_n\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \ldots + a_1\frac{dx(t)}{dt} + a_0 x(t) = \sum_{k=0}^{n} a_k \frac{d^k x(t)}{dt^k} = Af(t).$$

The function $f(t)$ is known as the forcing function.

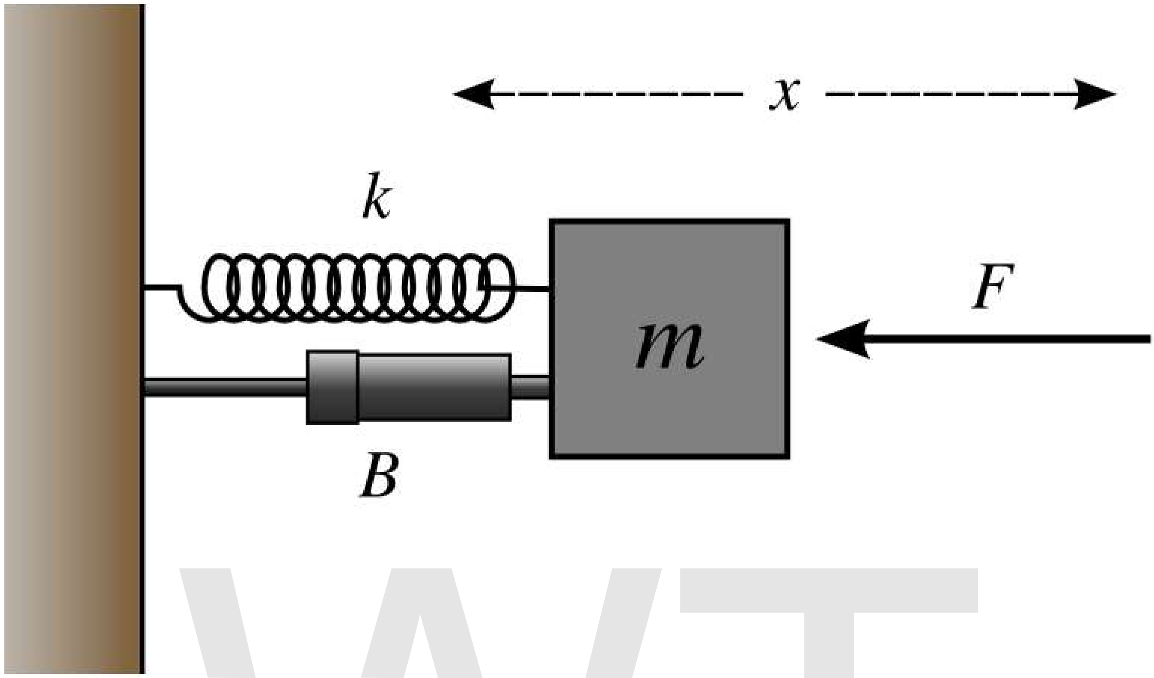
If the differential equation only contains real (not complex) coefficients, then the properties of such a system behaves as a mixture of first and second order systems only. This is because the roots of its characteristic polynomial are either real, or complex conjugate pairs. Therefore, understanding how nondimensionalization applies to first and second ordered systems allows the properties of higher order systems to be determined through superposition.

The number of free parameters in a nondimensionalized form of a system increases with its order. For this reason, nondimensionalization is rarely used for higher order differential equations. The need for this procedure has also been reduced with the advent of symbolic computation.

Examples of recovering characteristic units

A variety of systems can be approximated as either first or second order systems. These include mechanical, electrical, fluidic, caloric, and torsional systems. This is because the fundamental physical quantities involved within each of these examples are related through first and second order derivatives.

Mechanical oscillations



A mass attached to a spring and a dashpot.

Suppose we have a mass attached to a spring and a damper, which in turn are attached to a wall, and a force acting on the mass along the same line.

Define

x = displacement from equilibrium [m]
t = time [s]
f = external force or "disturbance" applied to system [kg m s^{-2}]
m = mass of the block [kg]
B = damping constant of dashpot [kg s^{-1}]
k = force constant of spring [kg s^{-2}]

Suppose the applied force is a sinusoid $F = F_0 \cos(\omega t)$, the differential equation that describes the motion of the block is

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = F_0\cos(\omega t)$$

Nondimensionalizing this equation the same way as described under second order system yields several characteristics of the system.

The intrinsic unit x_c corresponds to the distance the block moves per unit force

$$x_c = \frac{F_0}{k}.$$

The characteristic variable t_c is equal to the period of the oscillations

$$t_c = \sqrt{\frac{m}{k}}$$

and the dimensionless variable 2ζ corresponds to the linewidth of the system. ζ itself is the damping ratio.

$$2\zeta = \frac{B}{\sqrt{mk}}$$

Electrical oscillations

First-order series RC circuit

For a series RC attached to a voltage source

$$R\frac{dQ}{dt} + \frac{Q}{C} = V(t) \Rightarrow \frac{d\chi}{d\tau} + \chi = F(\tau)$$

with substitutions

$$Q = \chi x_c,\ t = \tau t_c,\ x_c = CV_0,\ t_c = RC,\ F = V.$$

The first characteristic unit corresponds to the total charge in the circuit. The second characteristic unit corresponds to the time constant for the system.

Second-order series RLC circuit

For a series configuration of R,C,L components where Q is the charge in the system

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0\cos(\omega t) \Rightarrow \frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = \cos(\Omega\tau)$$

with the substitutions

$$Q = \chi x_c,\ t = \tau t_c,\ \ x_c = CV_0,\ t_c = \sqrt{LC},\ 2\zeta = R\sqrt{\frac{C}{L}},\ \Omega = t_c\omega.$$

The first variable corresponds to the maximum charge stored in the circuit. The resonance frequency is given by the reciprocal of the characteristic time. The last expression is the linewidth of the system. The Ω can be considered as a normalized forcing function frequency.

Nonlinear differential equation example

Since there are no general methods of solving nonlinear differential equations, each case has to be considered on an individual basis when nondimensionalizing.

Quantum harmonic oscillator

The Schrödinger equation for the one dimensional time independent quantum harmonic oscillator is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2\right)\psi(x) = E\psi(x).$$

The wavefunction ψ itself represents probability, which is in a sense already dimensionless and normalized. Therefore, there is no need to nondimensionalize the wavefunction. However, it should be rewritten as a function of a dimensionless variable. Furthermore, the variable x has units of length. Hence substitute

$$\chi = \frac{x}{x_c},\ \psi(x) = \psi(x(\chi)) = \psi(\chi).$$

The differential equation becomes

$$\left(-\frac{\hbar^2}{2mx_c^2}\frac{d^2}{d\chi^2} + \frac{m\omega^2 x_c^2}{2}\chi^2\right)\psi(\chi) = E\psi(\chi) \Rightarrow \left(-\frac{d^2}{d\chi^2} + \frac{m^2\omega^2 x_c^4}{\hbar^2}\chi^2\right)\psi(\chi) = \frac{2Emx_c^2}{\hbar^2}\psi(\chi).$$

To make the term in front of χ^2 unitless, set

$$\frac{m^2\omega^2 x_c^4}{\hbar^2} = 1 \Rightarrow x_c = \sqrt{\frac{\hbar}{m\omega}}.$$

Hence, the fully nondimensionalized equation is

$$\left(-\frac{d^2}{d\chi^2} + \chi^2\right)\psi(\chi) = \frac{2E}{\hbar\omega}\psi(\chi) \stackrel{\text{def}}{=} \mathcal{E}\psi(\chi).$$

The nondimensionalization factor for the energy is the same as the ground state of the harmonic oscillator. Usually, the energy term is not made dimensionless because a primary emphasis of quantum mechanics is determining the energies of the states of a system. Rearranging the first equation, the familiar equation for the harmonic oscillator is

$$\frac{\hbar\omega}{2}\left(-\frac{d^2}{d\chi^2}+\chi^2\right)\psi(\chi)=E\psi(\chi).$$

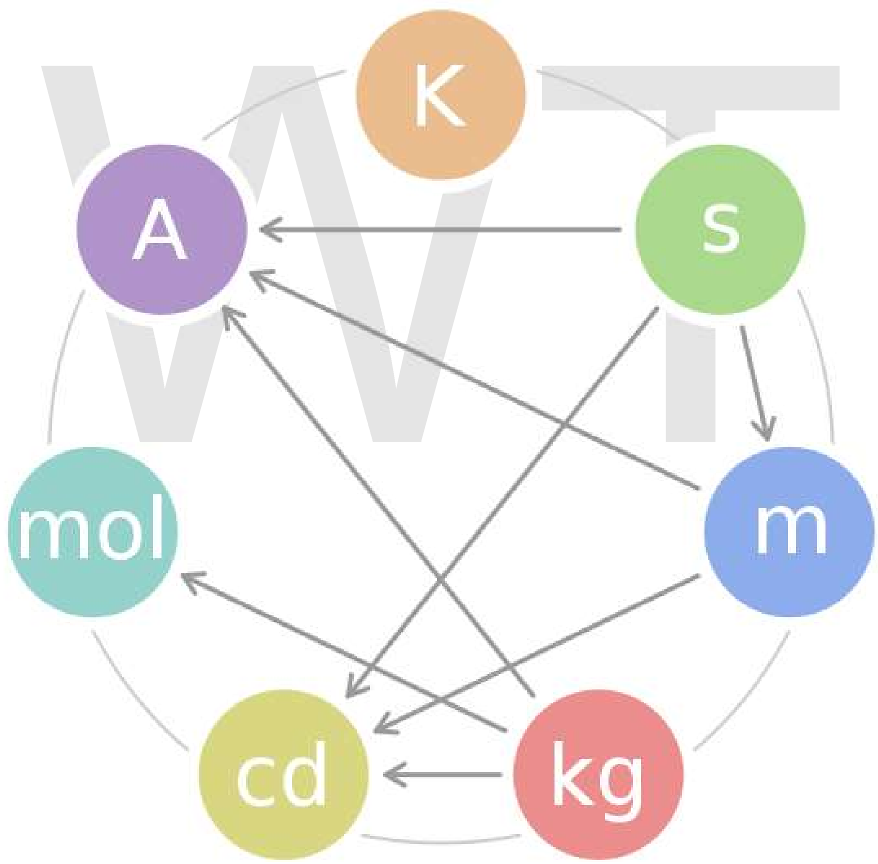
Statistical analogs

In statistics, the analogous process is usually dividing a difference (a distance) by a scale factor (a measure of statistical dispersion), which yields a dimensionless number, which is called *normalization.* Most often, this is dividing errors or residuals by the standard deviation or sample standard deviation, respectively, yielding standard scores and studentized residuals.

Chapter- 12

SI Base Unit & Similitude (Model)

SI Base Unit



The seven SI base units and the interdependency of their definitions

The International System of Units (**SI**) defines seven units of measure as a basic set from which all other SI units are derived. These **SI base units** and their physical quantities are:

- metre for length
- kilogram for mass
- second for time
- ampere for electric current
- kelvin for temperature
- candela for luminous intensity
- mole for the amount of substance.

The SI base quantities form a set of mutually independent dimensions as required by dimensional analysis commonly employed in science and technology. However, in a given realization of these units they may well be interdependent, i.e. defined in terms of each other.

The names of all SI units are written in lowercase characters (e.g., the *metre* has the symbol m), except that the symbols of units named after persons are written with an initial capital letter (e.g., the *ampere* has the uppercase symbol A).

Many other units, such as the litre, are formally not part of the SI, but are accepted for use with SI.

SI base units

Name	Symbol	Measure	Definition	Historical Origin / Justification
metre	**m**	length	"The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second." *17th CGPM (1983, Resolution 1, CR, 97)*	$^{1}\!/_{10,000,000}$ of the distance from the Earth's equator to the North Pole measured on the circumference through Paris.
kilogram	**kg**	mass	"The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram." *3rd CGPM (1901, CR, 70)*	The mass of one litre of water. A litre is one thousandth of a cubic metre.
second	**s**	time	"The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition	The day is divided in 24 hours, each hour divided in 60 minutes, each minute divided in 60 seconds.

			between the two hyperfine levels of the ground state of the caesium 133 atom." *13th CGPM (1967/68, Resolution 1; CR, 103)* "This definition refers to a caesium atom at rest at a temperature of 0 K." *(Added by CIPM in 1997)*	A second is $^{1}/_{(24 \times 60 \times 60)}$ of the day
ampere	**A**	electric current	"The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length." *9th CGPM (1948)*	The original "International Ampere" was defined electrochemically as the current required to deposit 1.118 milligrams of silver per second from a solution of silver nitrate. Compared to the SI ampere, the difference is 0.015%.
kelvin	**K**	thermodynamic temperature	"The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." *13th CGPM (1967/68, Resolution 4; CR, 104)* "This definition refers to water having the isotopic composition defined exactly by the following amount of substance ratios: 0.000 155 76 mole of ^{2}H per mole of ^{1}H, 0.000 379 9 mole of ^{17}O per mole of ^{16}O, and 0.002 005 2	The Celsius scale: the Kelvin scale uses the degree Celsius for its unit increment, but is a thermodynamic scale (0 K is absolute zero).

			mole of ^{18}O per mole of ^{16}O." *(Added by CIPM in 2005)*	
mole	**mol**	amount of substance	"1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is “mol.” 2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles." *14th CGPM (1971, Resolution 3; CR, 78)* "In this definition, it is understood that unbound atoms of carbon 12, at rest and in their ground state, are referred to." *(Added by CIPM in 1980)*	Atomic weight or molecular weight divided by the molar mass constant, 1 g/mol.

candela	**cd**	luminous intensity	"The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." *16th CGPM (1979, Resolution 3; CR, 100)*	The candlepower, which is based on the light emitted from a burning candle of standard properties.

Future redefinitions

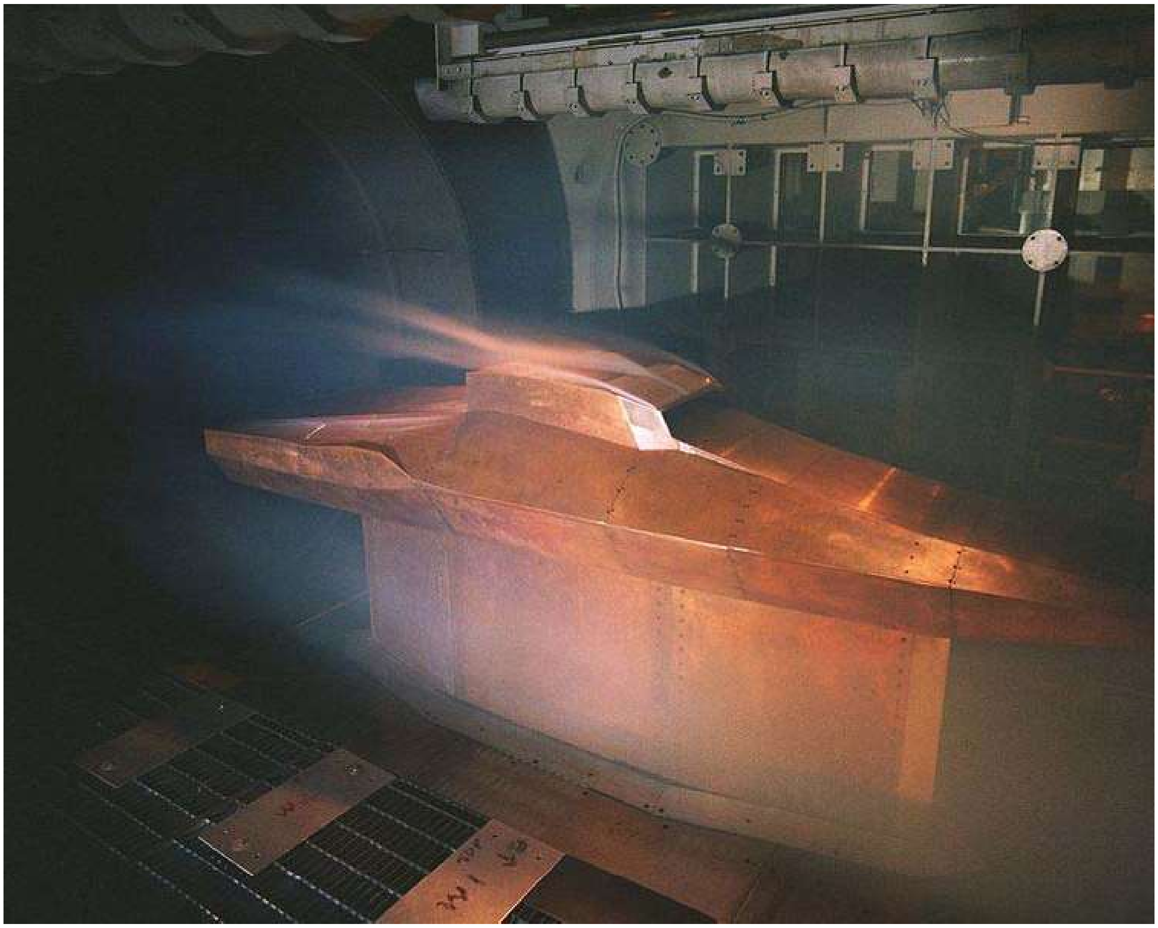
There have been several modifications to the definitions of the base units, and additions of base units, since the Metre Convention in 1875. Since the redefinition of the metre in 1960, the kilogram is the only unit which is directly defined in terms of a physical artifact rather than a property of nature. However, the mole, the ampere and the candela are also linked through their definitions to the mass of this platinum–iridium cylinder stored in a vault near Paris. It has long been an objective of metrology to find a way to define the kilogram in terms of a fundamental constant, in the same way that the metre is now defined in terms of the speed of light.

The 21st General Conference on Weights and Measures (CGPM, 1999) placed these efforts on an official footing, and recommended "that national laboratories continue their efforts to refine experiments that link the unit of mass to fundamental or atomic constants with a view to a future redefinition of the kilogram." Two main possibilities have attracted attention: the Planck constant and the Avogadro constant.

In 2005, the International Committee for Weights and Measures (CIPM) approved the preparation of new definitions for the kilogram, the ampere, and the kelvin and it noted the possibility of a new definition for the mole based on the Avogadro constant. The 23rd CGPM (2007) decided to postpone any legal change until the next General Conference in 2011.

In a note to the CIPM in October 2009, Ian Mills, the President of the CIPM *Consultative Committee - Units (CCU)* cataloged the uncertainties of the fundamental constants of physics according to the current definitions and their values under the proposed new definition. He urged the CIPM to accept the proposed changes in the definition of the *kilogram*, *ampere*, *kelvin* and *mole* so that they are referenced to the values of the fundamental constants, namely Planck's constant (h), the electron charge (e), Boltzmann's constant (k), and Avogadro's constant (N_A).

Similitude (Model)



A full scale X-43 Wind tunnel test. The test is designed to have **dynamic similitude** with the real application to ensure valid results.

Similitude is a concept used in the testing of engineering models. A model is said to have similitude with the real application if the two share geometric similarity, kinematic similarity and dynamic similarity. *Similarity* and *similitude* are interchangeable in this context.

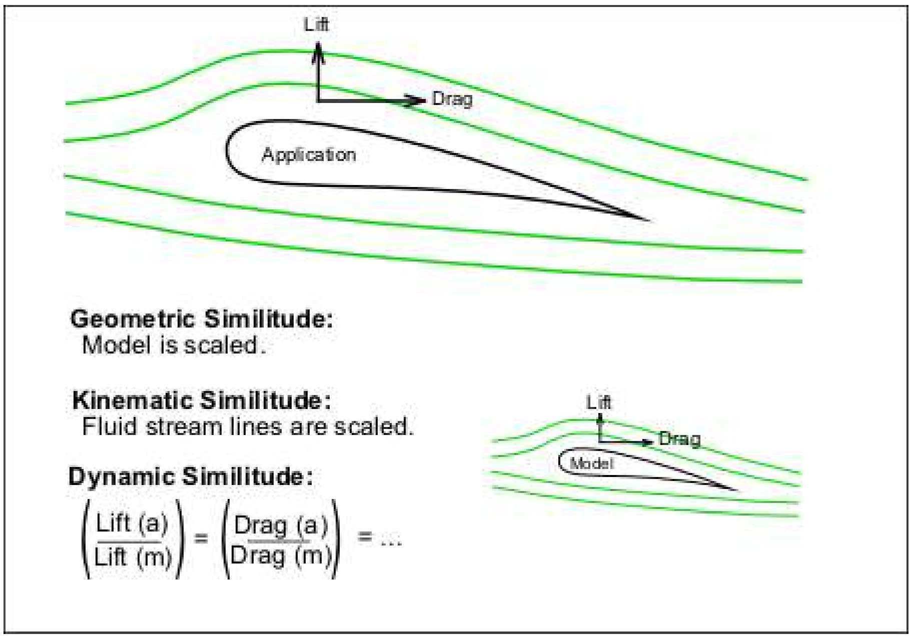
The term **dynamic similitude** is often used as a catch-all because it implies that geometric and kinematic similitude have already been met.

Similitude's main application is in hydraulic and aerospace engineering to test fluid flow conditions with scaled models. It is also the primary theory behind many textbook formulas in fluid mechanics.

Overview

Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations aren't reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process.

Construction of a scale model, however, must be accompanied by an analysis to determine what conditions it is tested under. While the geometry may be simply scaled, other parameters, such as pressure, temperature or the velocity and type of fluid may need to be altered. Similitude is achieved when testing conditions are created such that the test results are applicable to the real design.



The three conditions required for a model to have similitude with an application.

The following criteria are required to achieve similitude;

- **Geometric similarity** - The model is the same shape as the application, usually scaled.
- **Kinematic similarity** - Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)
- **Dynamic similarity** - Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

To satisfy the above conditions the application is analyzed;

1. All parameters required to describe the system are identified using principles from continuum mechanics.

2. Dimensional analysis is used to express the system with as few independent variables and as many dimensionless parameters as possible.
3. The values of the dimensionless parameters are held to be the same for both the scale model and application. This can be done because they are *dimensionless* and will ensure dynamic similitude between the model and the application. The resulting equations are used to derive *scaling laws* which dictate model testing conditions.

It is often impossible to achieve strict similitude during a model test. The greater the departure from the application's operating conditions, the more difficult achieving similitude is. In these cases some aspects of similitude may be neglected, focusing on only the most important parameters.

The design of marine vessels remains more of an art than a science in large part because dynamic similitude is especially difficult to attain for a vessel that is partially submerged: a ship is affected by wind forces in the air above it, by hydrodynamic forces within the water under it, and especially by wave motions at the interface between the water and the air. The scaling requirements for each of these phenomena differ, so models cannot replicate what happens to a full sized vessel nearly so well as can be done for an aircraft or submarine—each of which operates entirely within one medium.

Similitude is a term used widely in fracture mechanics relating to the strain life approach. Under given loading conditions the fatigue damage in an un-notched specimen is comparable to that of a notched specimen. Similitude suggests that the component fatigue life of the two objects will also be similar.

An example

Consider a submarine modeled at 1/40th scale. The application operates in sea water at 0.5 °C, moving at 5 m/s. The model will be tested in fresh water at 20 °C. Find the power required for the submarine to operate at the stated speed.

A free body diagram is constructed and the relevant relationships of force and velocity are formulated using techniques from continuum mechanics. The variables which describe the system are:

Variable	Application	Scaled mode	Units
L (diameter of submarine)	1	1/40	(m)
V (speed)	5	*calculate*	(m/s)
ρ (density)	1028	998	(kg/m^3)
μ (dynamic viscosity)	$1.88\text{x}10^{-3}$	$1.00\text{x}10^{-3}$	Pa·s (N s/m^2)
F (force)	*calculate*	*to be measured*	N (kg m/s^2)

This example has five independent variables and three fundamental units. The fundamental units are: metre, kilogram, second.

Invoking the Buckingham π theorem shows that the system can be described with two dimensionless numbers and one independent variable.

Dimensional analysis is used to re-arrange the units to form the Reynolds number (R_e) and pressure coefficient (C_p). These dimensionless numbers account for all the variables listed above except F, which will be the test measurement. Since the dimensionless parameters will stay constant for both the test and the real application, they will be used to formulate scaling laws for the test.

Scaling laws:

$$R_e = \left(\frac{\rho V L}{\mu}\right) \longrightarrow V_{model} = V_{application} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{L_a}{L_m}\right) \times \left(\frac{\mu_m}{\mu_a}\right)$$

$$C_p = \left(\frac{2\Delta P}{\rho V^2}\right), F = \Delta P L^2 \longrightarrow F_{application} = F_{model} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{V_a}{V_m}\right)^2 \times \left(\frac{L_a}{L_m}\right)^2$$

This gives a required test velocity of:

$$V_{model} = V_{application} \times 21.9.$$

The force measured from the model at that velocity is then scaled to find the force that can be expected for the real application:

$$F_{application} = F_{model} \times 3.44$$

The power P in watts required by the submarine is then:

$$P[\mathrm{W}] = F_{application} \times V_{application} = F_{model}[\mathrm{N}] \times 17.2\ \mathrm{m/s}$$

Note that even though the model is scaled smaller, the water velocity needs to be increased for testing. This remarkable result shows how similitude in nature is often counterintuitive.

Typical applications

Similitude has been well documented for a large number of engineering problems and is the basis of many textbook formulas and dimensionless quantities. These formulas and quantities are easy to use without having to repeat the laborious task of dimensional analysis and formula derivation. Simplification of the formulas (by neglecting some aspects of similitude) is common, and needs to be reviewed by the engineer for each application.

Similitude can be used to predict the performance of a new design based on data from an existing, similar design. In this case, the model is the existing design. Another use of similitude and models is in validation of computer simulations with the ultimate goal of eliminating the need for physical models altogether.

Another application of similitude is to replace the operating fluid with a different test fluid. Wind tunnels, for example, have trouble with air liquefying in certain conditions so helium is sometimes used. Other applications may operate in dangerous or expensive fluids so the testing is carried out in a more convenient substitute.

Some common applications of similitude and associated dimensionless numbers;

Incompressible flow -	Reynolds number, Pressure coefficient, (Froude number and Weber number for open channel hydraulics)
Compressible flows -	Reynolds number, Mach number, Prandtl number, Specific heat ratio
Flow excited vibration	Strouhal number
Centrifugal compressors -	Reynolds number, Mach number, Pressure coefficient, Velocity ratio

Chapter- 13

Units Conversion by Factor-Label

Many, if not most, parameters and measurements in the physical sciences and engineering are expressed as a numerical quantity and a corresponding dimensional unit; for example: 1000 kg/m³, 100 kPa/bar, 50 miles per hour, 1000 Btu/lb. Converting from one dimensional unit to another is often somewhat complex and being able to perform such conversions is an important skill to acquire. The **factor-label method**, also known as the **unit-factor method** or **dimensional analysis**, is a widely used approach for performing such conversions. It is also used for determining whether the two sides of a mathematical equation involving dimensions have the same dimensional units.

The factor-label method for converting units

The factor-label method is the sequential application of conversion factors expressed as fractions and arranged so that any dimensional unit appearing in both the numerator and denominator of any of the fractions can be cancelled out until only the desired set of dimensional units is obtained. For example, 10 miles per hour can be converted to meters per second by using a sequence of conversion factors as shown below:

$$\frac{10\ \cancel{\text{mile}}}{1\ \cancel{\text{hour}}} \times \frac{1609\ \text{meter}}{1\ \cancel{\text{mile}}} \times \frac{1\ \cancel{\text{hour}}}{3600\ \text{second}} = 4.47\ \frac{\text{meter}}{\text{second}}$$

It can be seen that each conversion factor is equivalent to the value of one. For example, starting with 1 mile = 1609 meters and dividing both sides of the equation by 1 mile yields 1 mile / 1 mile = 1609 meters / 1 mile, which when simplified yields 1 = 1609 meters / 1 mile.

So, when the units *mile* and *hour* are cancelled out and the arithmetic is done, 10 miles per hour converts to 4.47 meters per second.

As a more complex example, the concentration of nitrogen oxides (i.e., NOx) in the flue gas from an industrial furnace can be converted to a mass flow rate expressed in grams per hour (i.e., g/h) of NOx by using the following information as shown below:

NOx concentration

= 10 parts per million by volume = 10 ppmv = 10 volumes/10^6 volumes

NOx molar mass

= 46 kg/kgmol (sometimes also expressed as 46 kg/kmol)

Flow rate of flue gas

= 20 cubic meters per minute = 20 m^3/min

The flue gas exits the furnace at 0 °C temperature and 101.325 kPa absolute pressure.

The molar volume of a gas at 0 °C temperature and 101.325 kPa is 22.414 m^3/kgmol.

$$\frac{10\ \cancel{m^3\ NOx}}{10^6\ \cancel{m^3\ gas}} \times \frac{20\ \cancel{m^3\ gas}}{1\ \cancel{minute}} \times \frac{60\ \cancel{minute}}{1\ hour} \times \frac{1\ \cancel{kgmol\ NOx}}{22.414\ \cancel{m^3\ NOx}} \times \frac{46\ \cancel{kg}\ NOx}{1\ \cancel{kgmol\ NOx}} \times \frac{1000\ g}{1\ \cancel{kg}} = 24.63\ \frac{g\ NOx}{hour}$$

After cancelling out any dimensional units that appear both in the numerators and denominators of the fractions in the above equation, the NOx concentration of 10 ppm_v converts to mass flow rate of 24.63 grams per hour.

Checking equations that involve dimensions

The factor-label method can also be used on any mathematical equation to check whether or not the dimensional units on the left hand side of the equation are the same as the dimensional units on the right hand side of the equation. Having the same units on both sides of an equation does not guarantee that the equation is correct, but having different units on the two sides of an equation does guarantee that the equation is wrong.

For example, check the Universal Gas Law equation of $P \cdot V = n \cdot R \cdot T$, when:

- the pressure P is in pascals (Pa)
- the volume V is in cubic meters (m^3)
- the amount of substance n is in moles (mol)
- the universal gas law constant R is 8.3145 Pa·m^3/(mol·K)
- the temperature T is in kelvins (K)

$$(Pa)(m^3) = \frac{\cancel{mol}}{1} \times \frac{(Pa)(m^3)}{(\cancel{mol})(\cancel{K})} \times \frac{\cancel{K}}{1}$$

As can be seen, when the dimensional units appearing in the numerator and denominator of the equation's right hand side are cancelled out, both sides of the equation have the same dimensional units.

Limitations

The factor-label method can convert only unit quantities for which the units are in a linear relationship intersecting at 0. Most units fit this paradigm. An example for which it cannot be used is the conversion between degrees Celsius and kelvins (or Fahrenheit). Between degrees Celsius and kelvins, there is a constant difference rather than a constant ratio, while between Celsius and Fahrenheit there is both a constant difference and a constant ratio. Instead of multiplying the given quantity by a single conversion factor to obtain the converted quantity, it is more logical to think of the original quantity being divided by its unit, being added or subtracted by the constant difference, and the entire operation being multiplied by the new unit. Mathematically, this is an affine transform ($ax + b$), not a linear transform (ax). Formally, one starts with a displacement (in some units) from one point, and ends with a displacement (in some other units) from some other point.

For instance, the freezing point of water is 0° in Celsius and 32° in Fahrenheit, and a 5° change in Celsius correspond to a 9° change in Fahrenheit. Thus to convert from Fahrenheit to Celsius one subtracts 32° (displacement from one point), multiplies by 5 and divides by 9 (scales by the ratio of units), and adds 0 (displacement from new point). Reversing this yields the formula for Celsius; one could have started with the equivalence between 100° Celsius and 212° Fahrenheit, though this would yield the same formula at the end.

[°F = 1.8(°C) + 32°]

To convert Celsius to Fahrenheit, simply plug in the known numbers in the above formula.

[°C = (°F-32°) ÷ 1.8]

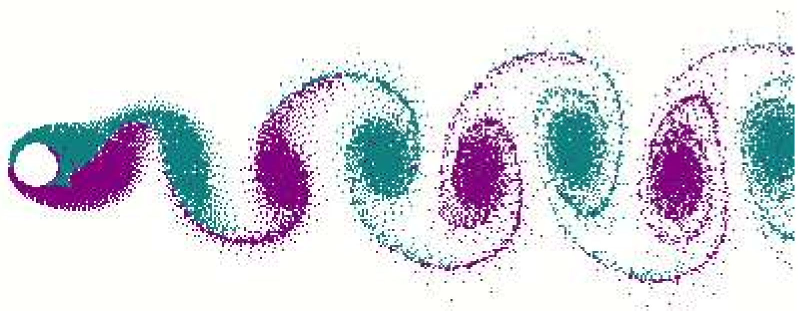
To convert Fahrenheit to Celsius (Centigrade), plug the known temperature into the above formula.

EX. °F = 1.8(-40°C) + 32° = -40°F (Identical temperature point in °C and °F)

EX. °C = (98.6°F-32°) ÷ 1.8 = 37°C (Known standard body temperature in °C and °F)

Chapter- 14

Reynolds Number



A vortex street around a cylinder. This occurs around cylinders, independently of the fluid, the cylinder size and the fluid speed, provided that there is a Reynolds number of between 250 and 200,000. Picture courtesy, Cesareo de La Rosa Siqueira.

In fluid mechanics, the **Reynolds number** Re is a dimensionless number that gives a measure of the ratio of inertial forces $\rho v^2/L$ to viscous forces $\mu v/L^2$ and consequently quantifies the relative importance of these two types of forces for given flow conditions. The concept was introduced by George Gabriel Stokes in 1851, but the Reynolds number is named after Osborne Reynolds (1842–1912), who popularized its use in 1883.

Reynolds numbers frequently arise when performing dimensional analysis of fluid dynamics problems, and as such can be used to determine dynamic similitude between different experimental cases. They are also used to characterize different flow regimes, such as laminar or turbulent flow: laminar flow occurs at low Reynolds numbers, where viscous forces are dominant, and is characterized by smooth, constant fluid motion, while turbulent flow occurs at high Reynolds numbers and is dominated by inertial forces, which tend to produce chaotic eddies, vortices and other flow instabilities.

Definition

Reynolds number can be defined for a number of different situations where a fluid is in relative motion to a surface (the definition of the Reynolds number is not to be confused with the Reynolds Equation or lubrication equation). These definitions generally include the fluid properties of density and viscosity, plus a velocity and a *characteristic length* or *characteristic dimension*. This dimension is a matter of convention - for example a radius or diameter are equally valid for spheres or circles, but one is chosen by convention. For aircraft or ships, the length or width can be used. For flow in a pipe or a sphere moving in a fluid the internal diameter is generally used today. Other shapes (such as rectangular pipes or non-spherical objects) have an *equivalent diameter* defined. For fluids of variable density (e.g. compressible gases) or variable viscosity (non-Newtonian fluids) special rules apply. The velocity may also be a matter of convention in some circumstances, notably stirred vessels.

$$\mathrm{Re} = \frac{\rho \mathrm{V} L}{\mu} = \frac{\mathrm{V} L}{\nu}$$

where:

- V is the mean velocity of the object relative to the fluid (SI units: m/s)
- L is a characteristic linear dimension, (travelled length of the fluid; hydraulic diameter when dealing with river systems) (m)
- μ is the dynamic viscosity of the fluid (Pa·s or N·s/m² or kg/(m·s))
- ν is the kinematic viscosity ($\nu = \mu / \rho$) (m²/s)
- ρ is the density of the fluid (kg/m³)

Note that this is equal to the ratio between $\frac{\rho \mathrm{V}^2}{L}$, which is the drag (up to a numerical factor, half the drag coefficient), and $\frac{\mu \mathrm{V}}{L^2}$, which is the force due to viscosity (up to a numerical factor depending on the form of the flow).

Significance

$$\mathrm{Re} = \frac{\text{total momentum transfer}}{\text{molecular momentum transfer}}$$

Flow in Pipe

For flow in a pipe or tube, the Reynolds number is generally defined as:

$$\mathrm{Re} = \frac{\rho \mathrm{V} D_H}{\mu} = \frac{\mathrm{V} D_H}{\nu} = \frac{\mathrm{Q} D_H}{\nu A}$$

where:

- D_H is the hydraulic diameter of the pipe (m).
- Q is the volumetric flow rate (m^3/s)
- A is the pipe *cross-sectional* area (m²).

Flow in a non-circular duct (annulus)

For shapes such as squares, rectangular or annular ducts (where the height and width are comparable) the characteristic dimension for internal flow situations is taken to be the *hydraulic diameter*, D_H, defined as 4 times the cross-sectional area (of the fluid), divided by the **wetted perimeter**. The wetted perimeter for a channel is the total perimeter of all channel walls that are in contact with the flow. This means the length of the water exposed to air is NOT included in the wetted perimeter

$$D_H = \frac{4A}{P}.$$

For a circular pipe, the hydraulic diameter is exactly equal to the inside pipe diameter, as can be shown mathematically.

For an annular duct, such as the outer channel in a tube-in-tube heat exchanger, the hydraulic diameter can be shown algebraically to reduce to

$$D_{H,\text{annulus}} = D_o - D_i$$

where

D_o is the inside diameter of the outside pipe, and
D_i is the outside diameter of the inside pipe.

For calculations involving flow in non-circular ducts, the hydraulic diameter can be substituted for the diameter of a circular duct, with reasonable accuracy.

Flow in a Wide Duct

For a fluid moving between two plane parallel surfaces (where the width is much greater than the space between the plates) then the characteristic dimension is twice the distance between the plates.

Flow in an Open Channel

For flow of liquid with a free surface, the *hydraulic radius* must be determined. This is the cross-sectional area of the channel divided by the wetted perimeter. For a semi-circular channel, it is half the radius. For a rectangular channel, the hydraulic radius is the cross-sectional area divided by the wetted perimeter. Some texts then use a characteristic dimension that is 4 times the hydraulic radius (chosen because it gives the same value of Re for the onset of turbulence as in pipe flow), while others the hydraulic radius as the characteristic length-scale with consequently different values of Re for transition and turbulent flow.

Object in a fluid

The Reynolds number for an object in a fluid, called the particle Reynolds number and often denoted **Re_p**, is important when considering the nature of flow around that grain, whether or not vortex shedding will occur, and its fall velocity.

Sphere in a fluid

For a sphere in a fluid, the characteristic length-scale is the diameter of the sphere and the characteristic velocity is that of the sphere relative to the fluid some distance away from the sphere (such that the motion of the sphere does not disturb that reference parcel of fluid). The density and viscosity are those belonging to the fluid. Note that purely laminar flow only exists up to Re = 0.1 under this definition.

Under the condition of low Re, the relationship between force and speed of motion is given by Stokes' law.

Oblong object in a fluid

The equation for an oblong object is identical to that of a sphere, with the object being approximated as an ellipsoid and the axis of length being chosen as the characteristic length scale. Such considerations are important in natural streams, for example, where there are few perfectly spherical grains. For grains in which measurement of each axis is impractical (e.g., because they are too small), sieve diameters are used instead as the characteristic particle length-scale. Both approximations alter the values of the critical Reynolds number.

Fall velocity

The particle Reynolds number is important in determining the fall velocity of a particle. When the particle Reynolds number indicates laminar flow, Stokes' law can be used to calculate its fall velocity. When the particle Reynolds number indicates turbulent flow, a turbulent drag law must be constructed to model the appropriate settling velocity.

Packed Bed

For flow of fluid through a bed of approximately spherical particles of diameter ***D*** in contact, if the voidage (fraction of the bed not filled with particles) is **ε** and the superficial velocity **V** (i.e. the velocity through the bed as if the particles were not there - the actual velocity will be higher) then a Reynolds number can be defined as:

$$\mathrm{Re} = \frac{\rho V D}{\mu(1-\epsilon)}.$$

Laminar conditions apply up to Re = 10, fully turbulent from 2000.

Stirred Vessel

In a cylindrical vessel stirred by a central rotating paddle, turbine or propellor, the characteristic dimension is the diameter of the agitator *D*. The velocity is *ND* where *N* is the rotational speed (revolutions per second). Then the Reynolds number is:

$$\mathrm{Re} = \frac{\rho N D^2}{\mu}.$$

The system is fully turbulent for values of Re above 10 000.

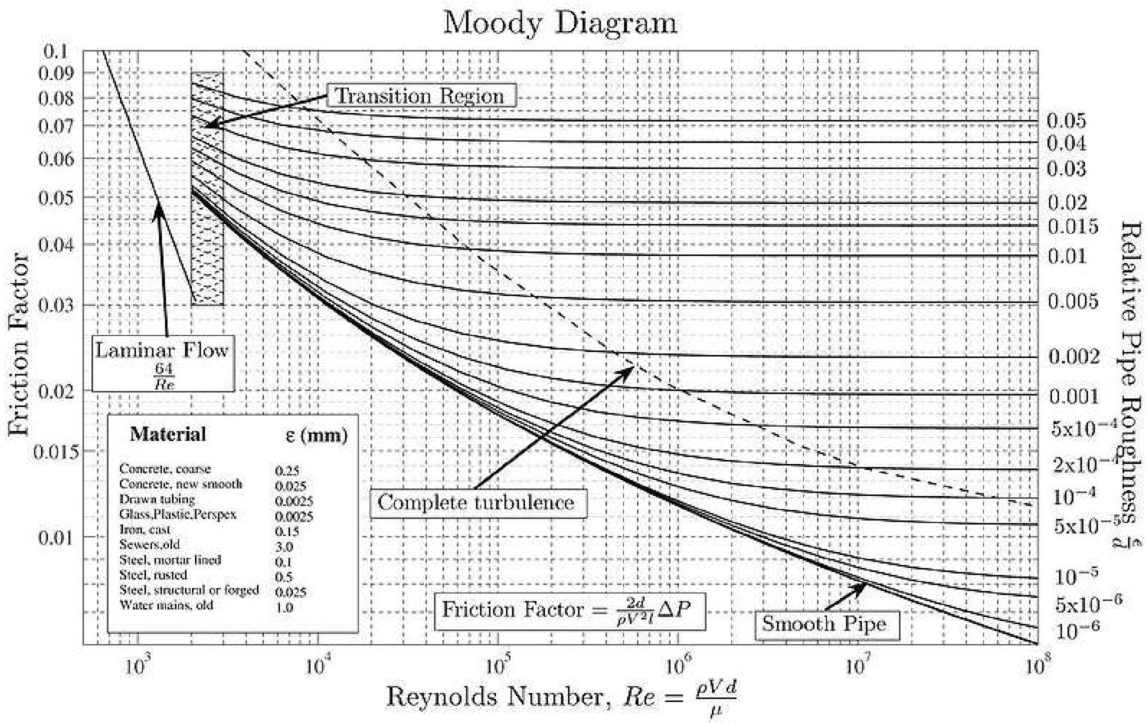
Transition Reynolds number

In boundary layer flow over a flat plate, experiments can confirm that, after a certain length of flow, a laminar boundary layer will become unstable and become turbulent. This instability occurs across different scales and with different fluids, usually when $\mathrm{Re}_x \approx 5 \times 10^5$, where *x* is the distance from the leading edge of the flat plate, and the flow velocity is the freestream velocity of the fluid outside the boundary layer.

For flow in a pipe of diameter *D*, experimental observations show that for 'fully developed' flow (Note:), laminar flow occurs when $\mathrm{Re}_D < 2300$ and turbulent flow occurs when $\mathrm{Re}_D > 4000$. In the interval between 2300 and 4000, laminar and turbulent flows are possible ('transition' flows), depending on other factors, such as pipe roughness and flow uniformity). This result is generalised to non-circular channels using the hydraulic diameter, allowing a transition Reynolds number to be calculated for other shapes of channel.

These transition Reynolds numbers are also called *critical Reynolds numbers*, and were studied by Osborne Reynolds around 1895.

Reynolds number in pipe friction



Pressure drops seen for fully-developed flow of fluids through pipes can be predicted using the Moody diagram which plots the Darcy–Weisbach friction factor f against Reynolds number Re and relative roughness ε / D. The diagram clearly shows the laminar, transition, and turbulent flow regimes as Reynolds number increases. The nature of pipe flow is strongly dependent on whether the flow is laminar or turbulent

The similarity of flows

In order for two flows to be similar they must have the same geometry, and have equal Reynolds numbers and Euler numbers. When comparing fluid behaviour at corresponding points in a model and a full-scale flow, the following holds:

$$\mathrm{Re}_m = \mathrm{Re}$$
$$\mathrm{Eu}_m = \mathrm{Eu} \qquad \text{i.e.} \quad \frac{p_m}{\varrho_m {v_m}^2} = \frac{p}{\varrho v^2} \,,$$

quantities marked with 'm' concern the flow around the model and the others the actual flow. This allows engineers to perform experiments with reduced models in water channels or wind tunnels, and correlate the data to the actual flows, saving on costs during experimentation and on lab time. Note that true dynamic similitude may require matching other dimensionless numbers as well, such as the Mach number used in compressible flows, or the Froude number that governs open-channel flows. Some flows involve more dimensionless parameters than can be practically satisfied with the available apparatus and fluids (for example air or water), so one is forced to decide which parameters are most important. For experimental flow modeling to be useful, it requires a fair amount of experience and judgement of the engineer.

Typical values of Reynolds number

- Ciliate ~ 1 x 10^{-1}
- Smallest Fish ~ 1

- Blood flow in brain ~ 1 × 10^2
- Blood flow in aorta ~ 1 × 10^3

Onset of turbulent flow ~ 2.3 × 10^3 to 5.0 × 10^4 for pipe flow to 10^6 for boundary layers

- Typical pitch in Major League Baseball ~ 2 × 10^5
- Person swimming ~ 4 × 10^6
- Fastest Fish ~ 1 x 10^6
- Blue Whale ~ 3 × 10^8
- A large ship (RMS Queen Elizabeth 2) ~ 5 × 10^9

Reynolds number sets the smallest scales of turbulent motion

In a turbulent flow, there is a range of scales of the time-varying fluid motion. The size of the largest scales of fluid motion (sometimes called eddies) are set by the overall geometry of the flow. For instance, in an industrial smoke stack, the largest scales of fluid motion are as big as the diameter of the stack itself. The size of the smallest scales is set by the Reynolds number. As the Reynolds number increases, smaller and smaller scales of the flow are visible. In a smoke stack, the smoke may appear to have many very small velocity perturbations or eddies, in addition to large bulky eddies. In this sense, the Reynolds number is an indicator of the range of scales in the flow. The higher the Reynolds number, the greater the range of scales. The largest eddies will always be the same size; the smallest eddies are determined by the Reynolds number.

What is the explanation for this phenomenon? A large Reynolds number indicates that viscous forces are not important at large scales of the flow. With a strong predominance of inertial forces over viscous forces, the largest scales of fluid motion are undamped—there is not enough viscosity to dissipate their motions. The kinetic energy must "cascade" from these large scales to progressively smaller scales until a level is reached for which the scale is small enough for viscosity to become important (that is, viscous forces become of the order of inertial ones). It is at these small scales where the dissipation of energy by viscous action finally takes place. The Reynolds number indicates at what scale this viscous dissipation occurs. Therefore, since the largest eddies are dictated by the flow geometry and the smallest scales are dictated by the viscosity, the Reynolds number can be understood as the ratio of the largest scales of the turbulent motion to the smallest scales.

Example of the importance of the Reynolds number

If an airplane wing needs testing, one can make a scaled down model of the wing and test it in a wind tunnel using the same Reynolds number that the actual airplane is subjected to. If for example the scale model has linear dimensions one quarter of full size, the flow velocity of the model would have to be multiplied by a factor of 4 to obtain similar flow behavior.

Alternatively, tests could be conducted in a water tank instead of in air (provided the compressibility effects of air are not significant). As the kinematic viscosity of water is around 13 times less than that of air at 15 °C, in this case the scale model would need to be about one thirteenth the size in all dimensions to maintain the same Reynolds number, assuming the full-scale flow velocity was used.

The results of the laboratory model will be similar to those of the actual plane wing results. Thus there is no need to bring a full scale plane into the lab and actually test it. This is an example of "dynamic similarity".

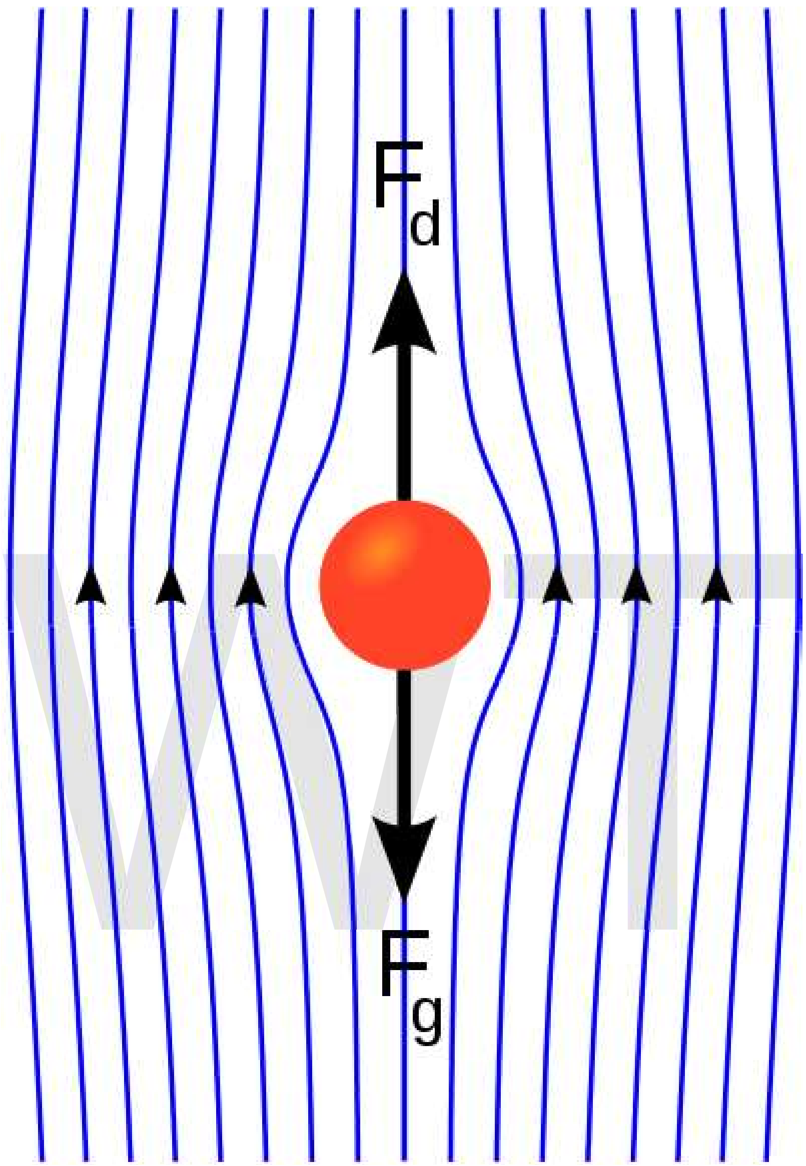
Reynolds number is important in the calculation of a body's drag characteristics. A notable example is that of the flow around a cylinder. Above roughly 3×10^6 Re the drag coefficient drops considerably. This is important when calculating the optimal cruise speeds for low drag (and therefore long range) profiles for airplanes.

Reynolds number in physiology

Poiseuille's law on blood circulation in the body is dependent on laminar flow. In turbulent flow the flow rate is proportional to the square root of the pressure gradient, as opposed to its direct proportionality to pressure gradient in laminar flow.

Using the definition of the Reynolds number we can see that a large diameter with rapid flow, where the density of the blood is high, tends towards turbulence. Rapid changes in vessel diameter may lead to turbulent flow, for instance when a narrower vessel widens to a larger one. Furthermore, an atheroma may be the cause of turbulent flow, and as such detecting turbulence with a stethoscope may be a sign of such a condition.

Reynolds number in viscous fluids



Creeping flow past a sphere: streamlines, drag force F_d and force by gravity F_g.

Where the viscosity is naturally high, such as polymer solutions and polymer melts, flow is normally laminar. The Reynolds number is very small and Stokes' Law can be used to measure the viscosity of the fluid. Spheres are allowed to fall through the fluid and they reach the terminal velocity quickly, from which the viscosity can be determined.

The laminar flow of polymer solutions is exploited by animals such as fish and dolphins, who exude viscous solutions from their skin to aid flow over their bodies while swimming. It has been used in yacht racing by owners who want to gain a speed advantage by pumping a polymer solution such as low molecular weight polyoxyethylene in water, over the wetted surface of the hull. It is however, a problem for mixing of polymers, because turbulence is needed to distribute fine filler (for example) through the material. Inventions such as the "cavity transfer mixer" have been developed to produce multiple folds into a moving melt so as to improve mixing efficiency. The device can be fitted onto extruders to aid mixing.

Where does it come from?

The Reynolds number can be obtained when one uses the nondimensional form of the incompressible Navier-Stokes equations:

$$\varrho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{f}.$$

Each term in the above equation has the units of a volume force or, equivalently, an acceleration times a density. Each term is thus dependent on the exact measurements of a flow. When one renders the equation nondimensional, that is that we multiply it by a factor with inverse units of the base equation, we obtain a form which does not depend directly on the physical sizes. One possible way to obtain a nondimensional equation is to multiply the whole equation by the following factor:

$$\frac{D}{\varrho V^2}$$

where the symbols are the same as those used in the definition of the Reynolds number. If we now set:

$$\mathbf{v}'=\frac{\mathbf{v}}{V},\ p'=p\frac{1}{\varrho V^2},\ \mathbf{f}'=\mathbf{f}\frac{D}{\varrho V^2},\ \frac{\partial}{\partial t'}=\frac{D}{V}\frac{\partial}{\partial t},\ \nabla'=D\nabla$$

we can rewrite the Navier-Stokes equation without dimensions:

$$\frac{\partial \mathbf{v}'}{\partial t'}+\mathbf{v}'\cdot\nabla'\mathbf{v}'=-\nabla'p'+\frac{\mu}{\varrho DV}\nabla'^2\mathbf{v}'+\mathbf{f}'$$

where the term : $\dfrac{\mu}{\varrho DV}=\dfrac{1}{\mathrm{Re}}.$

Finally, dropping the primes for ease of reading:

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{v}+\mathbf{f}.$$

This is why mathematically all flows with the same Reynolds number are comparable. Notice also, in the above equation, as : $\mathrm{Re} \to \infty$ the viscous terms vanish. Thus, high Reynolds number flows are approximately inviscid in the free-stream.

Chapter- 15

Affine Space

In mathematics, an **affine space** is a geometric structure that generalizes the affine properties of Euclidean space. In an affine space, one can subtract points to get vectors, or add a vector to a point to get another point, but one cannot add points. In particular, there is no distinguished point that serves as an origin. The solution set of an inhomogeneous linear equation is either empty or an affine space.

Informal descriptions

The following characterization may be easier to understand than the usual formal definition: an affine space is what is left of a vector space after you've forgotten which point is the origin (or, in the words of the great French mathematician Marcel Berger, "An affine space is nothing more than a vector space whose origin we try to forget about, by adding translations to the linear maps") . Imagine that Smith knows that a certain point is the true origin, and Jones believes that another point — call it $\boldsymbol{p}$ — is the origin. Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are to be added. Jones draws an arrow from $\boldsymbol{p}$ to $\boldsymbol{a}$ and another arrow from $\boldsymbol{p}$ to $\boldsymbol{b}$, and completes the parallelogram to find what Jones thinks is $\boldsymbol{a} + \boldsymbol{b}$, but Smith knows that it is actually

$$\boldsymbol{p} + (\boldsymbol{a} - \boldsymbol{p}) + (\boldsymbol{b} - \boldsymbol{p}).$$

Similarly, Jones and Smith may evaluate any linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$, or of any finite set of vectors, and will generally get different answers. However, importantly if the sum of the coefficients in a linear combination is 1, then Smith and Jones will arrive at the same answer.

Extending our previous example: Smith, knowing the true origin, can deduce that the point that Jones thinks is

$$\lambda\boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$$

is actually

$$\boldsymbol{p} + \lambda(\boldsymbol{a} - \boldsymbol{p}) + (1 - \lambda)(\boldsymbol{b} - \boldsymbol{p}) = \lambda\boldsymbol{a} + (1 - \lambda)\boldsymbol{b}.$$

Since λ + (1 − λ) = 1, Smith and Jones describe the same point with the same linear combination in their respective frames of reference.

Here is the punch line: Smith knows the "linear structure", but both Smith and Jones know the "affine structure"—i.e. the values of affine combinations, defined as linear combinations in which the sum of the coefficients is 1. An underlying set with an affine structure is an affine space.

Definition

An affine space is a set *A* together with a group action of a vector space *V*, considered as Abelian Lie group acting on the left such that the only vector acting with a fixpoint is 0 and there is a single orbit (i.e. the action is free and transitive). It is an example of (left) *G*-set, where *G* is the additive group *V*.

Explicitly, an affine space is a point set *A* together with a map

$l: V \times A \to A$, written as $(v, a) \mapsto v + a$

with the following properties:

1. for every *a* in *A* one has $0 + a = a$,
2. for every *v*, *w* in *V* and *a* in *A* one has $v + (w + a) = (v + w) + a$,
3. for every *a* in *A* the map $V \to A: v \mapsto v + a$ is a bijection.

By choosing an origin *a* one can thus identify *A* with *V*, hence turn *A* into a vector space.

Conversely, any vector space *V* is an affine space over itself.

If *o*, *a* and *b* are points in *A* and λ is a scalar, then

$$\lambda a + (1 - \lambda)b = o + \lambda(a - o) + (1 - \lambda)(b - o)$$

is independent of *o*. Instead of arbitrary linear combinations, only such affine combinations of points have meaning.

By noting that one can define subtraction of points of an affine space as follows:

$a - b$ is the unique vector in *V* such that $(a - b) + b = a$,

one can equivalently define an affine space as a point set *A*, together with a vector space *V*, and a subtraction map ϕ which sends (*a*,*b*) in $A \times A$ to $b - a$ in V

written as $(a, b) \mapsto b - a \equiv \overrightarrow{ab}$ with the following properties:

1. for every point $p \in A$ and vector $v \in V$ there is a unique point $q \in A$ such that $q - p = v$ and
2. for every *p*, *q* and *r* in *A* one has $(q - p) + (r - q) = r - p$.

Examples

- Any coset of a subspace *V* of a vector space is an affine space over *V*.
- If *A* is a matrix and *b* lies in its column space, the set of solutions of the equation *Ax* = *b* is an affine space over the subspace of solutions of *Ax* = 0.
- The solutions of an inhomogeneous linear differential equation form an affine space over the solutions of the corresponding homogeneous linear equation.
- Generalizing all of the above, if $T \colon V \to W$ is a linear mapping and *y* lies in its image, the set of solutions $x \in V$ to the equation *T*(*x*) = *y* is a coset of the kernel of *T*, and is therefore an affine space over Ker(*T*).

Affine subspaces

An **affine subspace** (sometimes called a **linear manifold** or **linear variety**) of a vector space *V* is a subset closed under affine combinations of vectors in the space. For example, the set

$$A = \Big\{ \sum_i^N \alpha_i \mathbf{v}_i \Big| \sum_i^N \alpha_i = 1 \Big\}$$

is an affine space, where $\{\mathbf{v}_i\}_i$ is a family of vectors in *V* – this space is the *affine span* of these point. To see that this is indeed an affine space, observe that this set carries a transitive action of the vector subspace *W* of *V*

$$W = \Big\{ \sum_i^N \beta_i \mathbf{v}_i \Big| \sum_i^N \beta_i = 0 \Big\}.$$

This affine subspace can be equivalently described as the coset of the *W*-action

$$S = \mathbf{p} + W,$$

where **p** is any element of *A*, or equivalently as any level set of the quotient map $V \to V/W$. A choice of **p** gives a base point of *A* and an identification of *W* with *A,* but there is no natural choice, nor a natural identification of *W* with *A*.

A linear transformation is a function that preserves all linear combinations; an affine transformation is a function that preserves all affine combinations. A linear subspace is an affine subspace containing the origin, or, equivalently, a subspace that is closed under linear combinations.

For example, in R^3, the origin, lines and planes through the origin and the whole space are linear subspaces, while points, lines and planes in general as well as the whole space are the affine subspaces.

Affine combinations and affine dependence

An **affine combination** is a linear combination in which the sum of the coefficients is 1. Just as members of a set of vectors are linearly independent if none is a linear combination of the others, so also they are **affinely independent** if none is an affine combination of the others. The set of linear combinations of a set of vectors is their "linear span" and is always a linear subspace; the set of all affine combinations is their "affine span" and is always an affine subspace. For example, the affine span of a set of two points is the line that contains both; the affine span of a set of three non-collinear points is the plane that contains all three. Vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

are linearly dependent if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, for which

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{0} \qquad (1)$$

Similarly they are **affinely dependent** if the same is true and also

$$\sum_{i=1}^{n} a_i = 0.$$

Equation (**1**) is an **affine relation** among the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Axioms

Affine space is usually studied as analytic geometry using coordinates, or equivalently vector spaces. It can also be studied as synthetic geometry by writing down axioms, though this approach is much less common. There are several different systems of axioms for affine space.

Coxeter (1969, p.192) axiomatizes affine geometry (over the reals) as ordered geometry together with an affine form of Desargues's theorem and an axiom stating that in a plane there is at most one line through a given point not meeting a given line.

Affine planes satisfy the following axioms (Cameron 1991, chapter 2): (in which two lines are called parallel if they are equal or disjoint):

- Any two distinct points lie on a unique line.
- Given a point and line there is a unique line which contains the point and is parallel to the line
- There exist three non-collinear points.

As well as affine planes over fields (or division rings), there are also many non-Desarguesian planes satisfying these axioms. (Cameron 1991, chapter 3) gives axioms for higher dimensional affine spaces.

Relation to projective spaces

$$\begin{tikzcd} & K^{n+1} \arrow[d, two heads] \\ \mathbf{A}^n \arrow[r, hook] & \mathbf{P}^n \end{tikzcd}$$

An affine space is a subspace of projective space, which is in turn a quotient of a vector space.

Affine spaces are subspaces of projective spaces: an affine plane can be obtained from any projective plane by removing a line and all the points on it, and conversely any affine plane can be used to construct a projective plane as a closure by adding a "line at infinity" whose points correspond to equivalence classes of parallel lines.

Further, transformations of projective space that preserve affine space (equivalently, that preserve the points at infinity as a set) yield transformations of affine space, and conversely any affine linear transformation extends uniquely to a projective linear transformations, so affine transformations are a subset of projective transforms. Most familiar is that Möbius transformations (transformations of the projective line, or Riemann sphere) are affine (transformations of the complex plane) if and only if they fix the point at infinity.

However, one cannot take the projectivization of an affine space, so projective spaces are not naturally *quotients* of affine spaces: one can only take the projectivization of a *vector* space, since the projective space is lines *through a given point,* and there is no distinguished point in an affine space. If one chooses a base point (as zero), then an affine space becomes a vector space, which one may then projectivize, but this requires a choice.