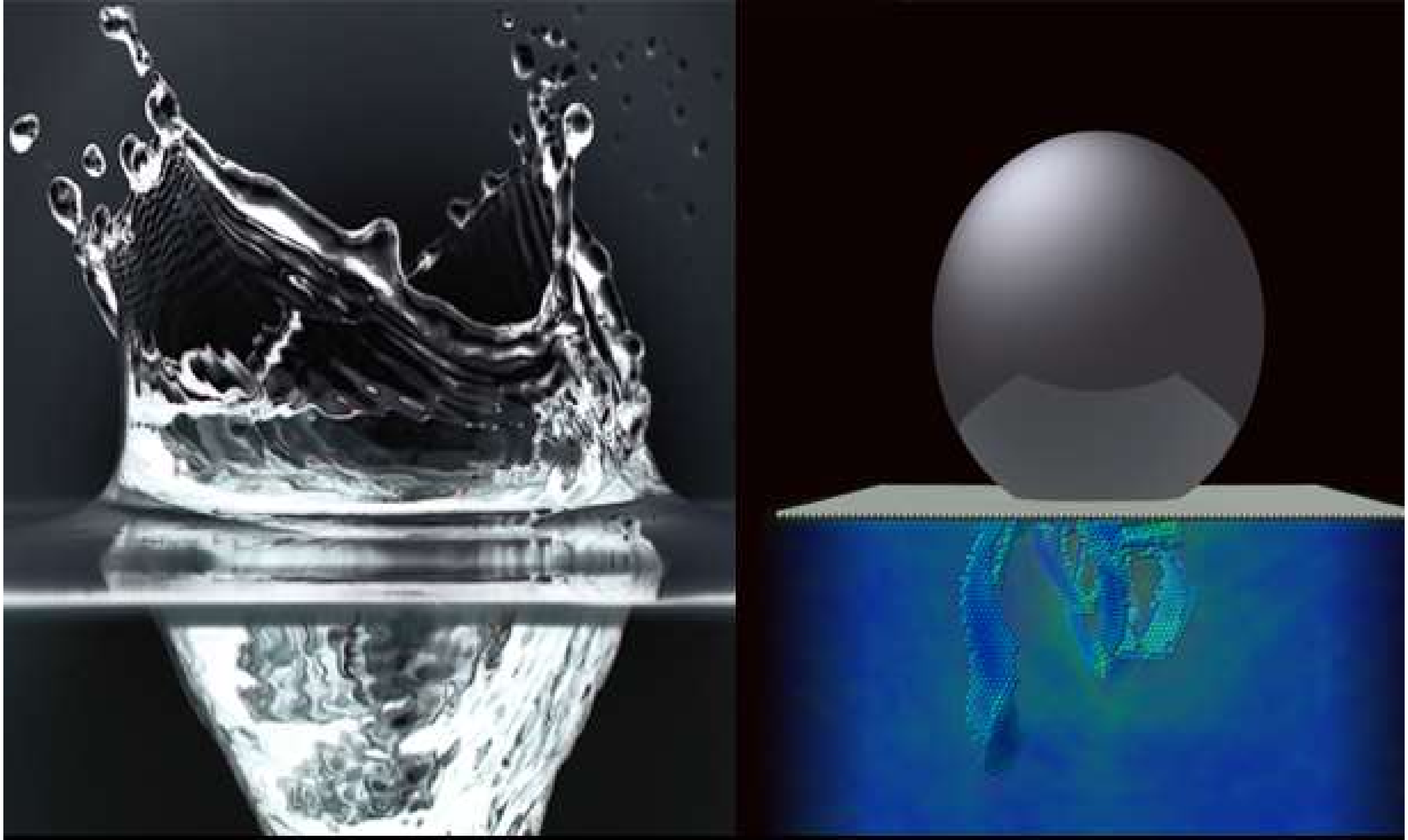


Fluid and Solid Mechanics

(Concepts and Applications)

Janetta Barham

Andrew Stubblefield

First Edition, 2012

ISBN 978-81-323-0883-6

Published by:
Academic Studio
4735/22 Prakashdeep Bldg,
Ansari Road, Darya Ganj,
Delhi - 110002
Email: info@wtbooks.com

Table of Contents

Chapter- 1

Fluid Mechanics

Fluid mechanics is the study of fluids and the forces on them. (Fluids include liquids, gases, and plasmas.) Fluid mechanics can be divided into fluid kinematics, the study of fluid motion, and fluid dynamics, the study of the effect of forces on fluid motion, which can further be divided into fluid statics, the study of fluids at rest, and fluid kinetics, the study of fluids in motion. It is a branch of continuum mechanics, a subject which models matter without using the information that it is made out of atoms, that is, it models matter from a macroscopic viewpoint rather than from a microscopic viewpoint. Fluid mechanics, especially fluid dynamics, is an active field of research with many unsolved or partly solved problems. Fluid mechanics can be mathematically complex. Sometimes it can best be solved by numerical methods, typically using computers. A modern discipline, called computational fluid dynamics (CFD), is devoted to this approach to solving fluid mechanics problems. Also taking advantage of the highly visual nature of fluid flow is particle image velocimetry, an experimental method for visualizing and analyzing fluid flow.

Brief history

The study of fluid mechanics goes back at least to the days of ancient Greece, when Archimedes investigated fluid statics and buoyancy and formulated his famous law known now as the Archimedes Principle. Rapid advancement in fluid mechanics began with Leonardo da Vinci (observation and experiment), Evangelista Torricelli (barometer), Isaac Newton (viscosity) and Blaise Pascal (hydrostatics), and was continued by Daniel Bernoulli with the introduction of mathematical fluid dynamics in *Hydrodynamica* (1738). Inviscid flow was further analyzed by various mathematicians (Leonhard Euler, d'Alembert, Lagrange, Laplace, Poisson) and viscous flow was explored by a multitude of engineers including Poiseuille and Gotthilf Heinrich Ludwig Hagen. Further mathematical justification was provided by Claude-Louis Navier and George Gabriel Stokes in the Navier–Stokes equations, and boundary layers were investigated (Ludwig Prandtl), while various scientists (Osborne Reynolds, Andrey Kolmogorov, Geoffrey Ingram Taylor) advanced the understanding of fluid viscosity and turbulence.

Relationship to continuum mechanics

Fluid mechanics is a subdiscipline of continuum mechanics, as illustrated in the following table.

<table>
<tr><td rowspan="4">Continuum mechanics
The study of the physics of continuous materials</td><td rowspan="2">Solid mechanics
The study of the physics of continuous materials with a defined rest shape.</td><td colspan="2">Elasticity
Describes materials that return to their rest shape after an applied stress.</td></tr>
<tr><td>Plasticity
Describes materials that permanently deform after a sufficient applied stress.</td><td rowspan="2">Rheology
The study of materials with both solid and fluid characteristics.</td></tr>
<tr><td rowspan="2">**Fluid mechanics**
The study of the physics of continuous materials which take the shape of their container.</td><td>Non-Newtonian fluids</td></tr>
<tr><td colspan="2">Newtonian fluids</td></tr>
</table>

In a mechanical view, a fluid is a substance that does not support shear stress; that is why a fluid at rest has the shape of its containing vessel. A fluid at rest has no shear stress.

Assumptions

Like any mathematical model of the real world, fluid mechanics makes some basic assumptions about the materials being studied. These assumptions are turned into equations that must be satisfied if the assumptions are to be held true. For example, consider an incompressible fluid in three dimensions. The assumption that mass is conserved means that for any fixed closed surface (such as a sphere) the rate of mass passing from *outside* to *inside* the surface must be the same as rate of mass passing the other way. (Alternatively, the mass *inside* remains constant, as does the mass *outside*). This can be turned into an integral equation over the surface.

Fluid mechanics assumes that every fluid obeys the following:

- Conservation of mass
- Conservation of energy
- Conservation of momentum
- The *continuum hypothesis*, detailed below.

Further, it is often useful (at subsonic conditions) to assume a fluid is incompressible – that is, the density of the fluid does not change. Liquids can often be modelled as incompressible fluids, whereas gases cannot.

Similarly, it can sometimes be assumed that the viscosity of the fluid is zero (the fluid is *inviscid*). Gases can often be assumed to be inviscid. If a fluid is viscous, and its flow contained in some way (e.g. in a pipe), then the flow at the boundary must have zero velocity. For a viscous fluid, if the boundary is not porous, the shear forces between the fluid and the boundary results also in a zero velocity for the fluid at the boundary. This is called the no-slip condition. For a porous media otherwise, in the frontier of the containing vessel, the slip condition is not zero velocity, and the fluid has a discontinuous velocity field between the free fluid and the fluid in the porous media (this is related to the Beavers and Joseph condition).

The continuum hypothesis

Fluids are composed of molecules that collide with one another and solid objects. The continuum assumption, however, considers fluids to be continuous. That is, properties such as density, pressure, temperature, and velocity are taken to be well-defined at "infinitely" small points, defining a REV (Reference Element of Volume), at the geometric order of the distance between two adjacent molecules of fluid. Properties are assumed to vary continuously from one point to another, and are averaged values in the REV. The fact that the fluid is made up of discrete molecules is ignored.

The continuum hypothesis is basically an approximation, in the same way planets are approximated by point particles when dealing with celestial mechanics, and therefore results in approximate solutions. Consequently, assumption of the continuum hypothesis can lead to results which are not of desired accuracy. That said, under the right circumstances, the continuum hypothesis produces extremely accurate results.

Those problems for which the continuum hypothesis does not allow solutions of desired accuracy are solved using statistical mechanics. To determine whether or not to use conventional fluid dynamics or statistical mechanics, the Knudsen number is evaluated for the problem. The Knudsen number is defined as the ratio of the molecular mean free path length to a certain representative physical length scale. This length scale could be, for example, the radius of a body in a fluid. (More simply, the Knudsen number is how many times its own diameter a particle will travel on average before hitting another particle). Problems with Knudsen numbers at or above unity are best evaluated using statistical mechanics for reliable solutions.

Navier–Stokes equations

The **Navier–Stokes equations** (named after Claude-Louis Navier and George Gabriel Stokes) are the set of equations that describe the motion of fluid substances such as liquids and gases. These equations state that changes in momentum (force) of fluid particles depend only on the external pressure and internal viscous forces (similar to friction) acting on the fluid. Thus, the Navier–Stokes equations describe the balance of forces acting at any given region of the fluid.

The Navier–Stokes equations are differential equations which describe the motion of a fluid. Such equations establish relations among the rates of change of the variables of interest. For example, the Navier–Stokes equations for an ideal fluid with zero viscosity states that acceleration (the rate of change of velocity) is proportional to the derivative of internal pressure.

This means that solutions of the Navier–Stokes equations for a given physical problem must be sought with the help of calculus. In practical terms only the simplest cases can be solved exactly in this way. These cases generally involve non-turbulent, steady flow (flow does not change with time) in which the Reynolds number is small.

For more complex situations, such as global weather systems like El Niño or lift in a wing, solutions of the Navier–Stokes equations can currently only be found with the help of computers. This is a field of sciences by its own called computational fluid dynamics.

General form of the equation

The general form of the Navier–Stokes equations for the conservation of momentum is:

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbb{P} + \rho\mathbf{f}$$

where

- ρ is the fluid density,
- $\frac{D}{Dt}$ is the substantive derivative (also called the material derivative),
- $\mathbf{v}$ is the velocity vector,
- $\mathbf{f}$ is the body force vector, and
- $\mathbb{P}$ is a tensor that represents the surface forces applied on a fluid particle (the stress tensor).

Unless the fluid is made up of spinning degrees of freedom like vortices, $\mathbb{P}$ is a symmetric tensor. In general, (in three dimensions) $\mathbb{P}$ has the form:

$$\mathbb{P} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

where

- σ are normal stresses,
- τ are tangential stresses (shear stresses).

The above is actually a set of three equations, one per dimension. By themselves, these aren't sufficient to produce a solution. However, adding conservation of mass and appropriate boundary conditions to the system of equations produces a solvable set of equations.

Newtonian versus non-Newtonian fluids

A **Newtonian fluid** (named after Isaac Newton) is defined to be a fluid whose shear stress is linearly proportional to the velocity gradient in the direction perpendicular to the plane of shear. This definition means regardless of the forces acting on a fluid, it *continues to flow*. For example, water is a Newtonian fluid, because it continues to display fluid properties no matter how much it is stirred or mixed. A slightly less rigorous definition is that the drag of a small object being moved slowly through the fluid is proportional to the force applied to the object. (Compare friction). Important fluids, like water as well as most gases, behave — to good approximation — as a Newtonian fluid under normal conditions on Earth.

By contrast, stirring a non-Newtonian fluid can leave a "hole" behind. This will gradually fill up over time – this behaviour is seen in materials such as pudding, oobleck, or sand (although sand isn't strictly a fluid). Alternatively, stirring a non-Newtonian fluid can cause the viscosity to decrease, so the fluid appears "thinner" (this is seen in non-drip paints). There are many types of non-Newtonian fluids, as they are defined to be something that fails to obey a particular property — for example, most fluids with long molecular chains can react in a non-Newtonian manner.

Equations for a Newtonian fluid

The constant of proportionality between the shear stress and the velocity gradient is known as the viscosity. A simple equation to describe Newtonian fluid behaviour is

$$\tau = -\mu \frac{dv}{dy}$$

where

τ is the shear stress exerted by the fluid ("drag")
μ is the fluid viscosity – a constant of proportionality
$\frac{dv}{dy}$ is the velocity gradient perpendicular to the direction of shear.

For a Newtonian fluid, the viscosity, by definition, depends only on temperature and pressure, not on the forces acting upon it. If the fluid is incompressible and viscosity is constant across the fluid, the equation governing the shear stress (in Cartesian coordinates) is

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

where

τ_{ij} is the shear stress on the i^{th} face of a fluid element in the j^{th} direction
v_i is the velocity in the i^{th} direction
x_j is the j^{th} direction coordinate.

If a fluid does not obey this relation, it is termed a non-Newtonian fluid, of which there are several types.

Among fluids, two rough broad divisions can be made: ideal and non-ideal fluids. An ideal fluid really does not exist, but in some calculations, the assumption is justifiable. An Ideal fluid is non viscous- offers no resistance whatsoever to a shearing force.

One can group real fluids into Newtonian and non-Newtonian. Newtonian fluids agree with Newton's law of viscosity. Non-Newtonian fluids can be either plastic, bingham plastic, pseudoplastic, dilatant, thixotropic, rheopectic, viscoelatic.

Chapter- 2

Navier–Stokes Equations

In physics the **Navier–Stokes equations**, named after Claude-Louis Navier and George Gabriel Stokes, describe the motion of fluid substances. These equations arise from applying Newton's second law to fluid motion, together with the assumption that the fluid stress is the sum of a diffusing viscous term (proportional to the gradient of velocity), plus a pressure term.

The equations are useful because they describe the physics of many things of academic and economic interest. They may be used to model the weather, ocean currents, water flow in a pipe and air flow around a wing. The Navier–Stokes equations in their full and simplified forms help with the design of aircraft and cars, the study of blood flow, the design of power stations, the analysis of pollution, and many other things. Coupled with Maxwell's equations they can be used to model and study magnetohydrodynamics.

The Navier–Stokes equations are also of great interest in a purely mathematical sense. Somewhat surprisingly, given their wide range of practical uses, mathematicians have not yet proven that in three dimensions solutions always exist (existence), or that if they do exist, then they do not contain any singularity (smoothness). These are called the Navier–Stokes existence and smoothness problems. The Clay Mathematics Institute has called this one of the seven most important open problems in mathematics and has offered a US$1,000,000 prize for a solution or a counter-example.

The Navier–Stokes equations dictate not position but rather velocity. A solution of the Navier–Stokes equations is called a velocity field or flow field, which is a description of the velocity of the fluid at a given point in space and time. Once the velocity field is solved for, other quantities of interest (such as flow rate or drag force) may be found. This is different from what one normally sees in classical mechanics, where solutions are typically trajectories of position of a particle or deflection of a continuum. Studying velocity instead of position makes more sense for a fluid; however for visualization purposes one can compute various trajectories.

Properties

Nonlinearity

The Navier–Stokes equations are nonlinear partial differential equations in almost every real situation. In some cases, such as one-dimensional flow and Stokes flow (or creeping flow), the equations can be simplified to linear equations. The nonlinearity makes most problems difficult or impossible to solve and is the main contributor to the turbulence that the equations model.

The nonlinearity is due to convective acceleration, which is an acceleration associated with the change in velocity over position. Hence, any convective flow, whether turbulent or not, will involve nonlinearity. An example of convective but laminar (nonturbulent) flow would be the passage of a viscous fluid (for example, oil) through a small converging nozzle. Such flows, whether exactly solvable or not, can often be thoroughly studied and understood.

Turbulence

Turbulence is the time dependent chaotic behavior seen in many fluid flows. It is generally believed that it is due to the inertia of the fluid as a whole: the culmination of time dependent and convective acceleration; hence flows where inertial effects are small tend to be laminar (the Reynolds number quantifies how much the flow is affected by inertia). It is believed, though not known with certainty, that the Navier–Stokes equations describe turbulence properly.

The numerical solution of the Navier–Stokes equations for turbulent flow is extremely difficult, and due to the significantly different mixing-length scales that are involved in turbulent flow, the stable solution of this requires such a fine mesh resolution that the computational time becomes significantly infeasible for calculation. Attempts to solve turbulent flow using a laminar solver typically result in a time-unsteady solution, which fails to converge appropriately. To counter this, time-averaged equations such as the Reynolds-averaged Navier-Stokes equations (RANS), supplemented with turbulence models, are used in practical computational fluid dynamics (CFD) applications when modeling turbulent flows. Some models include the Spalart-Allmaras, k-ω (k-omega), k-ε (k-epsilon), and SST models which add a variety of additional equations to bring closure to the RANS equations. Another technique for solving numerically the Navier–Stokes equation is the Large eddy simulation (LES). This approach is computationally more expensive than the RANS method (in time and computer memory), but produces better results since the larger turbulent scales are explicitly resolved.

Applicability

Together with supplemental equations (for example, conservation of mass) and well formulated boundary conditions, the Navier–Stokes equations seem to model fluid motion accurately; even turbulent flows seem (on average) to agree with real world observations.

The Navier–Stokes equations assume that the fluid being studied is a continuum not moving at relativistic velocities. At very small scales or under extreme conditions, real fluids made out of discrete molecules will produce results different from the continuous fluids modeled by the Navier–Stokes equations. Depending on the Knudsen number of the problem, statistical mechanics or possibly even molecular dynamics may be a more appropriate approach.

Another limitation is very simply the complicated nature of the equations. Time tested formulations exist for common fluid families, but the application of the Navier–Stokes equations to less common families tends to result in very complicated formulations which are an area of current research. For this reason, these equations are usually written for Newtonian fluids. Studying such fluids is "simple" because the viscosity model ends up being linear; truly general models for the flow of other kinds of fluids (such as blood) do not, as of 2011, exist.

Derivation and description

The derivation of the Navier–Stokes equations begins with an application of Newton's second law: conservation of momentum (often alongside mass and energy conservation) being written for an arbitrary portion of the fluid. In an inertial frame of reference, the general form of the equations of fluid motion is:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \nabla\cdot\mathbb{T} + \mathbf{f},$$

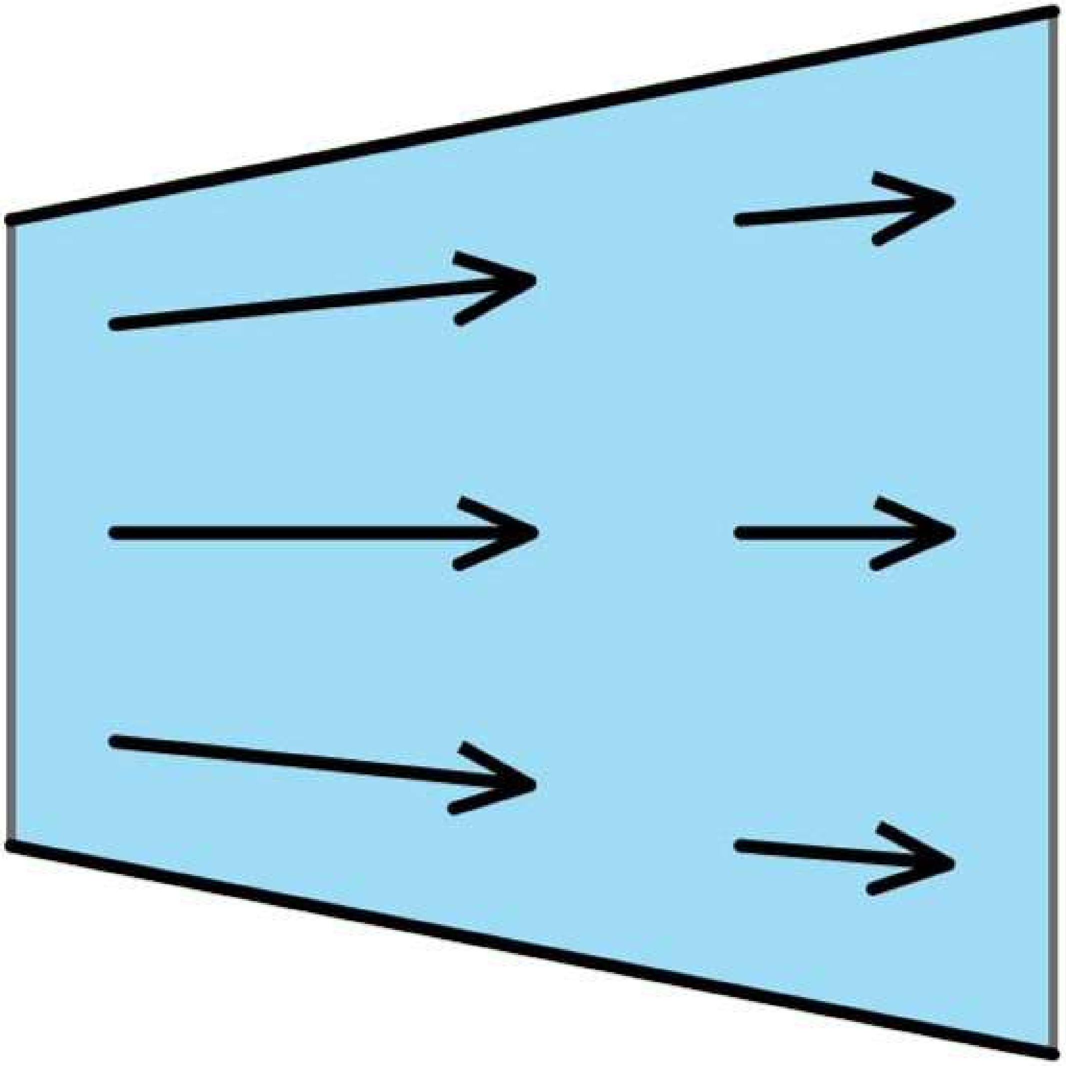
where $\mathbf{v}$is the flow velocity, ρ is the fluid density, *p* is the pressure, $\mathbb{T}$is the (deviatoric) stress tensor, and $\mathbf{f}$represents body forces (per unit volume) acting on the fluid and $\nabla$is the del operator. This is a statement of the conservation of momentum in a fluid and it is an application of Newton's second law to a continuum; in fact this equation is applicable to any non-relativistic continuum and is known as the Cauchy momentum equation.

This equation is often written using the material derivative *D***v**/*Dt*, making it more apparent that this is a statement of Newton's second law:

$$\rho\frac{D\mathbf{v}}{Dt} = -\nabla p + \nabla\cdot\mathbb{T} + \mathbf{f}.$$

The left side of the equation describes acceleration, and may be composed of time dependent or convective effects (also the effects of non-inertial coordinates if present). The right side of the equation is in effect a summation of body forces (such as gravity) and divergence of stress (pressure and shear stress).

Convective acceleration



An example of convection. Though the flow may be steady (time independent), the fluid decelerates as it moves down the diverging duct (assuming incompressible flow), hence there is an acceleration happening over position.

A very significant feature of the Navier–Stokes equations is the presence of convective acceleration: the effect of time independent acceleration of a fluid with respect to space. While individual fluid particles are indeed experiencing time dependent acceleration, the convective acceleration of the flow field is a spatial effect, one example being fluid speeding up in a nozzle. Convective acceleration is represented by the nonlinear quantity:

$$\mathbf{v} \cdot \nabla \mathbf{v},$$

which may be interpreted either as $\left(\mathbf{v}\cdot\nabla\right)\mathbf{v}$ or as $\mathbf{v}\cdot\left(\nabla\mathbf{v}\right)$, with $\nabla\mathbf{v}$ the tensor derivative of the velocity vector $\mathbf{v}$. Both interpretations give the same result, independent of the coordinate system — provided ∇ is interpreted as the covariant derivative.

Interpretation as (v·∇)v

The convection term is often written as

$$(\mathbf{v}\cdot\nabla)\mathbf{v},$$

where the advection operator $\mathbf{v}\cdot\nabla$ is used. Usually this representation is preferred because it is simpler than the one in terms of the tensor derivative $\nabla\mathbf{v}$.

Interpretation as v·(∇v)

Here $\nabla\mathbf{v}$ is the tensor derivative of the velocity vector, equal in Cartesian coordinates to the component by component gradient. The convection term may, by a vector calculus identity, be expressed without a tensor derivative:

$$\mathbf{v}\cdot\nabla\mathbf{v}=\nabla\left(\frac{\|\mathbf{v}\|^2}{2}\right)+\left(\nabla\times\mathbf{v}\right)\times\mathbf{v}.$$

The form has use in irrotational flow, where the curl of the velocity (called vorticity) $\omega=\nabla\times\mathbf{v}$ is equal to zero.

Regardless of what kind of fluid is being dealt with, convective acceleration is a nonlinear effect. Convective acceleration is present in most flows (exceptions include one-dimensional incompressible flow), but its dynamic effect is disregarded in creeping flow (also called Stokes flow) .

Stresses

The effect of stress in the fluid is represented by the ∇p and $\nabla\cdot\mathbb{T}$ terms; these are gradients of surface forces, analogous to stresses in a solid. ∇p is called the pressure gradient and arises from the isotropic part of the stress tensor. This part is given by normal stresses that turn up in almost all situations, dynamic or not. The anisotropic part of the stress tensor gives rise to $\nabla\cdot\mathbb{T}$, which conventionally describes viscous forces; for incompressible flow, this is only a shear effect. Thus, $\mathbb{T}$ is the deviatoric stress tensor, and the stress tensor is equal to:

$$\sigma=-p\mathbb{I}+\mathbb{T}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Interestingly, only the *gradient* of pressure matters, not the pressure itself. The effect of the pressure gradient is that fluid flows from high pressure to low pressure.

The stress terms p and $\mathbb{T}$are yet unknown, so the general form of the equations of motion is not usable to solve problems. Besides the equations of motion—Newton's second law—a force model is needed relating the stresses to the fluid motion. For this reason, assumptions on the specific behavior of a fluid are made (based on natural observations) and applied in order to specify the stresses in terms of the other flow variables, such as velocity and density.

The Navier–Stokes equations result from the following assumptions on the deviatoric stress tensor $\mathbb{T}$:

- the deviatoric stress vanishes for a fluid at rest, and – by Galilean invariance – also does not depend directly on the flow velocity itself, but only on spatial derivatives of the flow velocity
- in the Navier–Stokes equations, the deviatoric stress is expressed as the product of the tensor gradient $\nabla\mathbf{v}$of the flow velocity with a viscosity tensor $\mathbb{A}$, i.e. : $\mathbb{T} = \mathbb{A}(\nabla\mathbf{v})$
- the fluid is assumed to be isotropic, as valid for gases and simple liquids, and consequently $\mathbb{A}$is an isotropic tensor; furthermore, since the deviatoric stress tensor is symmetric, it turns out that it can be expressed in terms of two scalar dynamic viscosities μ and μ”: $\mathbb{T} = 2\mu\mathbb{E} + \mu''\Delta\mathbb{I}$,where $\mathbb{E} = \frac{1}{2}(\nabla\mathbf{v}) + \frac{1}{2}(\nabla\mathbf{v})^{\mathrm{T}}$ is the rate-of-strain tensor and $\Delta = \nabla\cdot\mathbf{v}$is the rate of expansion of the flow
- the deviatoric stress tensor has zero trace, so for a three-dimensional flow $2\mu + 3\mu'' = 0$

As a result, in the Navier–Stokes equations the deviatoric stress tensor has the following form:

$$\mathbb{T} = 2\mu\left(\mathbb{E} - \tfrac{1}{3}\Delta\mathbb{I}\right),$$

with the quantity between brackets the non-isotropic part of the rate-of-strain tensor $\mathbb{E}$.The dynamic viscosity μ does not need to be constant – in general it depends on conditions like temperature and pressure, and in turbulence modelling the concept of eddy viscosity is used to approximate the average deviatoric stress.

The pressure p is modelled by use of an equation of state. For the special case of an incompressible flow, the pressure constrains the flow in such a way that the volume of fluid elements is constant: isochoric flow resulting in a solenoidal velocity field with $\nabla\cdot\mathbf{v}=0$.

Other forces

The vector field $\mathbf{f}$represents body forces. Typically these consist of only gravity forces, but may include other types (such as electromagnetic forces). In a non-inertial coordinate system, other "forces" such as that associated with rotating coordinates may be inserted.

Often, these forces may be represented as the gradient of some scalar quantity. Gravity in the z direction, for example, is the gradient of $-\rho g z$. Since pressure shows up only as a gradient, this implies that solving a problem without any such body force can be mended to include the body force by modifying pressure.

Other equations

The Navier–Stokes equations are strictly a statement of the conservation of momentum. In order to fully describe fluid flow, more information is needed (how much depends on the assumptions made), this may include boundary data (no-slip, capillary surface, etc.), the conservation of mass, the conservation of energy, and/or an equation of state.

Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation, given in its most general form as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

or, using the substantive derivative:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{v}) = 0.$$

Incompressible flow of Newtonian fluids

A simplification of the resulting flow equations is obtained when considering an incompressible flow of a Newtonian fluid. The assumption of incompressibility rules out the possibility of sound or shock waves to occur; so this simplification is invalid if these phenomena are important. The incompressible flow assumption typically holds well even when dealing with a "compressible" fluid — such as air at room temperature — at low Mach numbers (even when flowing up to about Mach 0.3). Taking the incompressible flow assumption into account and assuming constant viscosity, the Navier–Stokes equations will read, in vector form:

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}.$$

Here **f** represents "other" body forces (forces per unit volume), such as gravity or centrifugal force. The shear stress term $\nabla\mathbb{T}$becomes the useful quantity $\mu\nabla^2\mathbf{v}$(∇^2 is the vector Laplacian) when the fluid is assumed incompressible, homogeneous and Newtonian, where $\mu$is the (constant) dynamic viscosity.

It's well worth observing the meaning of each term (compare to the Cauchy momentum equation):

$$\rho\Big(\overbrace{\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\text{Unsteady acceleration}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\text{Convective acceleration}}}^{\text{Inertia (per volume)}}\Big) = \overbrace{\underbrace{-\nabla p}_{\text{Pressure gradient}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{f}}_{\text{Other body forces}}.$$

Note that only the convective terms are nonlinear for incompressible Newtonian flow. The convective acceleration is an acceleration caused by a (possibly steady) change in velocity over *position*, for example the speeding up of fluid entering a converging nozzle. Though individual fluid particles are being accelerated and thus are under unsteady motion, the flow field (a velocity distribution) will not necessarily be time dependent.

Another important observation is that the viscosity is represented by the vector Laplacian of the velocity field (interpreted here as the difference between the velocity at a point and the mean velocity in a small volume around). This implies that Newtonian viscosity is **diffusion of momentum**, this works in much the same way as the diffusion of heat seen in the heat equation (which also involves the Laplacian).

If temperature effects are also neglected, the only "other" equation (apart from initial/boundary conditions) needed is the mass continuity equation. Under the incompressible assumption, density is a constant and it follows that the equation will simplify to:

$$\nabla \cdot \mathbf{v} = 0.$$

This is more specifically a statement of the conservation of volume.

These equations are commonly used in 3 coordinates systems: Cartesian, cylindrical, and spherical. While the Cartesian equations seem to follow directly from the vector equation above, the vector form of the Navier–Stokes equation involves some tensor calculus which means that writing it in other coordinate systems is not as simple as doing so for scalar equations (such as the heat equation).

Cartesian coordinates

Writing the vector equation explicitly,

$$\begin{aligned}
\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x \\
\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y \\
\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z
\end{aligned}$$

Note that gravity has been accounted for as a body force, and the values of g_x,g_y,g_z will depend on the orientation of gravity with respect to the chosen set of coordinates.

The continuity equation reads:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0.$$

When the flow is at steady-state, ρ does not change with respect to time. The continuity equation is reduced to:

$$\frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0.$$

When the flow is incompressible, ρ is constant and does not change with respect to space. The continuity equation is reduced to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The velocity components (the dependent variables to be solved for) are typically named *u*, *v*, *w*. This system of four equations comprises the most commonly used and studied form. Though comparatively more compact than other representations, this is still a nonlinear system of partial differential equations for which solutions are difficult to obtain.

Cylindrical coordinates

A change of variables on the Cartesian equations will yield the following momentum equations for *r*, φ, and *z*:

$$r:\ \rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_r}{\partial \phi} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\phi^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right] + \rho g_r$$

$$\phi:\ \rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_\phi}{\partial \phi} + u_z\frac{\partial u_\phi}{\partial z} + \frac{u_r u_\phi}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{\partial^2 u_\phi}{\partial z^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi} - \frac{u_\phi}{r^2}\right] + \rho g_\phi$$

$$z:\ \rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + \frac{u_\phi}{r}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \phi^2} + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z.$$

The gravity components will generally not be constants, however for most applications either the coordinates are chosen so that the gravity components are constant or else it is assumed that gravity is counteracted by a pressure field (for example, flow in horizontal pipe is treated normally without gravity and without a vertical pressure gradient). The continuity equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} + \frac{\partial u_z}{\partial z} = 0.$$

This cylindrical representation of the incompressible Navier–Stokes equations is the second most commonly seen (the first being Cartesian above). Cylindrical coordinates are chosen to take advantage of symmetry, so that a velocity component can disappear. A very common case is axisymmetric flow with the assumption of no tangential velocity ($u_\varphi = 0$), and the remaining quantities are independent of φ:

$$\begin{aligned}
&\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right) + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2}\right] + \rho g_r \\
&\rho\left(\frac{\partial u_z}{\partial t} + u_r\frac{\partial u_z}{\partial r} + u_z\frac{\partial u_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + \frac{\partial^2 u_z}{\partial z^2}\right] + \rho g_z \\
&\frac{1}{r}\frac{\partial}{\partial r}\left(ru_r\right) + \frac{\partial u_z}{\partial z} = 0.
\end{aligned}$$

Spherical coordinates

In spherical coordinates, the *r*, φ, and θ momentum equations are (note the convention used: θ is polar angle, or colatitude, 0 ≤ θ ≤ π):

$$\begin{aligned}
&\rho\left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial u_r}{\partial \phi} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\phi^2 + u_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \rho g_r + \\
&\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_r}{\partial r}\right) + \frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_r}{\partial \phi^2} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_r}{\partial \theta}\right) - 2\frac{u_r + \frac{\partial u_\theta}{\partial \theta} + u_\theta\cot(\theta)}{r^2} - \frac{2}{r^2\sin(\theta)}\frac{\partial u_\phi}{\partial \phi}\right] \\
&\rho\left(\frac{\partial u_\phi}{\partial t} + u_r\frac{\partial u_\phi}{\partial r} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\frac{\partial u_\phi}{\partial \theta} + \frac{u_r u_\phi + u_\phi u_\theta\cot(\theta)}{r}\right) = -\frac{1}{r\sin(\theta)}\frac{\partial p}{\partial \phi} + \rho g_\phi + \\
&\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\phi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_\phi}{\partial \theta}\right) + \frac{2\sin(\theta)\frac{\partial u_r}{\partial \phi} + 2\cos(\theta)\frac{\partial u_\theta}{\partial \phi} - u_\phi}{r^2\sin(\theta)^2}\right] \\
&\rho\left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\phi}{r\sin(\theta)}\frac{\partial u_\theta}{\partial \phi} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta - u_\phi^2\cot(\theta)}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \\
&\mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_\theta}{\partial r}\right) + \frac{1}{r^2\sin(\theta)^2}\frac{\partial^2 u_\theta}{\partial \phi^2} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial u_\theta}{\partial \theta}\right) + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta + 2\cos(\theta)\frac{\partial u_\phi}{\partial \phi}}{r^2\sin(\theta)^2}\right].
\end{aligned}$$

Mass continuity will read:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin(\theta)}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)u_\theta\right) = 0.$$

These equations could be (slightly) compacted by, for example, factoring $1 / r^2$ from the viscous terms. However, doing so would undesirably alter the structure of the Laplacian and other quantities.

Stream function formulation

Taking the curl of the Navier–Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed (*w* = 0 and no dependence of anything on *z*), where the equations reduce to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\rho g_x,$$
$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+\rho g_y.$$

Differentiating the first with respect to *y*, the second with respect to *x* and subtracting the resulting equations will eliminate pressure and any conservative force. Defining the stream function ψ through

$$u=\frac{\partial\psi}{\partial y}\quad;\quad v=-\frac{\partial\psi}{\partial x}$$

results in mass continuity being unconditionally satisfied (given the stream function is continuous), and then incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right)-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right)=\nu\nabla^4\psi,$$

where ∇^4 is the (2D) biharmonic operator and ν is the kinematic viscosity, $\nu=\frac{\mu}{\rho}$. We can also express this compactly using the Jacobian determinant:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right)+\frac{\partial\left(\psi,\nabla^2\psi\right)}{\partial\left(y,x\right)}=\nu\nabla^4\psi.$$

This single equation together with appropriate boundary conditions describes 2D fluid flow, taking only kinematic viscosity as a parameter. Note that the equation for creeping flow results when the left side is assumed zero.

In axisymmetric flow another stream function formulation, called the Stokes stream function, can be used to describe the velocity components of an incompressible flow with one scalar function.

Compressible flow of Newtonian fluids

There are some phenomena that are closely linked with fluid compressibility. One of the obvious examples is sound. Description of such phenomena requires more general presentation of the Navier–Stokes equation that takes into account fluid compressibility. If viscosity is assumed a constant, one additional term appears, as shown here:

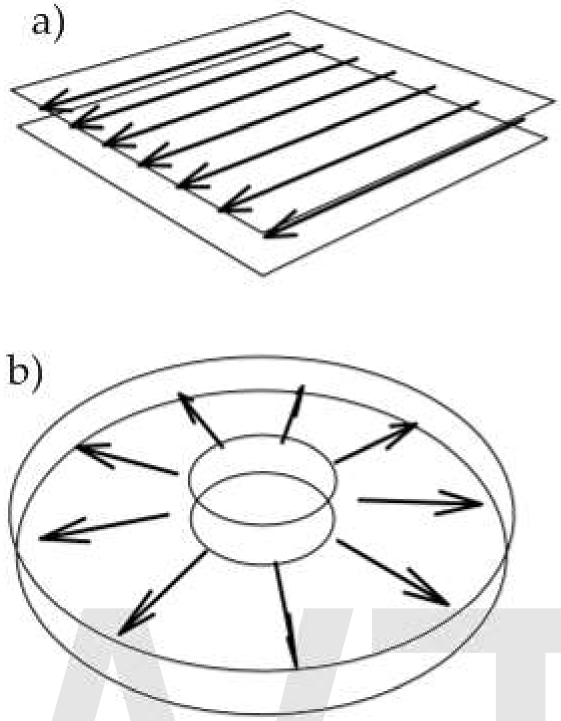
$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\mu\nabla^2\mathbf{v}+\left(\tfrac{1}{3}\mu+\mu^v\right)\nabla(\nabla\cdot\mathbf{v})+\mathbf{f}$$

where μ^v is the volume viscosity coefficient, also known as *second viscosity coefficient* or *bulk viscosity*. This additional term disappears for an *incompressible fluid*, when the divergence of the flow equals zero.

Application to specific problems

The Navier–Stokes equations, even when written explicitly for specific fluids, are rather generic in nature and their proper application to specific problems can be very diverse. This is partly because there is an enormous variety of problems that may be modeled, ranging from as simple as the distribution of static pressure to as complicated as multiphase flow driven by surface tension.

Generally, application to specific problems begins with some flow assumptions and initial/boundary condition formulation, this may be followed by scale analysis to further simplify the problem. For example, after assuming steady, parallel, one dimensional, nonconvective pressure driven flow between parallel plates, the resulting scaled (dimensionless) boundary value problem is:



Visualization of a) parallel flow and b) radial flow.

$$\frac{d^2u}{dy^2} = -1 \quad ; \quad u(0) = u(1) = 0.$$

The boundary condition is the no slip condition. This problem is easily solved for the flow field:

$$u(y) = \frac{y - y^2}{2}.$$

From this point onward more quantities of interest can be easily obtained, such as viscous drag force or net flow rate.

Difficulties may arise when the problem becomes slightly more complicated. A seemingly modest twist on the parallel flow above would be the *radial* flow between parallel plates; this involves convection and thus nonlinearity. The velocity field may be represented by a function $f(z)$ that must satisfy:

$$\frac{d^2f}{dz^2} + Rf^2 = -1 \quad ; \quad f(-1) = f(1) = 0.$$

This ordinary differential equation is what is obtained when the Navier–Stokes equations are written and the flow assumptions applied (additionally, the pressure gradient is solved for). The nonlinear term makes this a very difficult problem to solve analytically (a lengthy implicit solution may be found which involves elliptic integrals and roots of cubic polynomials). Issues with the actual existence of solutions arise for R > 1.41 (approximately; this is not the square root of 2), the parameter R being the Reynolds number with appropriately chosen scales. This is an example of flow assumptions losing their applicability, and an example of the difficulty in "high" Reynolds number flows.

Exact solutions of the Navier–Stokes equations

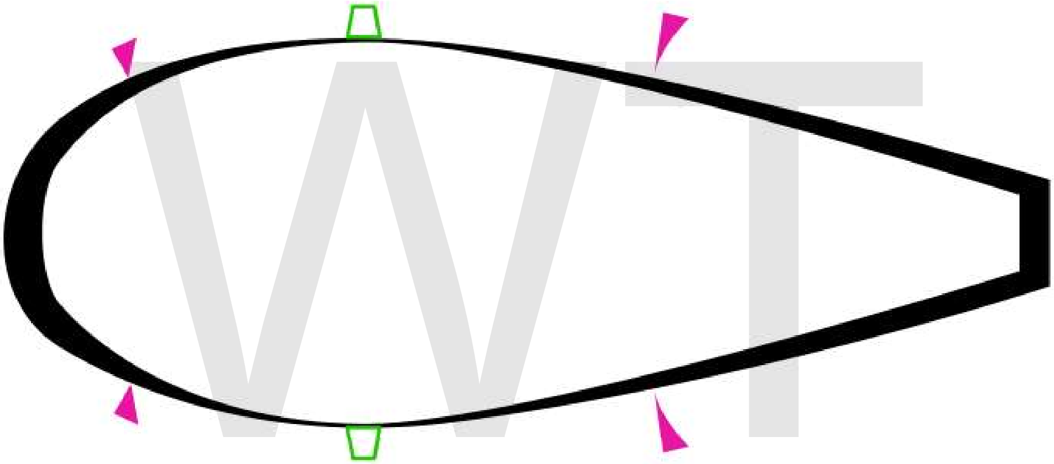
Some exact solutions to the Navier–Stokes equations exist. Examples of degenerate cases — with the non-linear terms in the Navier–Stokes equations equal to zero — are Poiseuille flow, Couette flow and the oscillatory Stokes boundary layer. But also more interesting examples, solutions to the full non-linear equations, exist; for example the Taylor–Green vortex. Note that the existence of these exact solutions does not imply they are stable: turbulence may develop at higher Reynolds numbers.

Wyld diagrams

Wyld diagrams are bookkeeping graphs that correspond to the Navier–Stokes equations via a perturbation expansion of the fundamental continuum mechanics. Similar to the Feynman diagrams in quantum field theory, these diagrams are an extension of Keldysh's technique for nonequilibrium processes in fluid dynamics. In other words, these diagrams assign graphs to the (often) turbulent phenomena in turbulent fluids by allowing correlated and interacting fluid particles to obey stochastic processes associated to pseudo-random functions in probability distributions.

Chapter- 3

Fluid Dynamics



Typical aerodynamic teardrop shape, assuming a viscous medium passing from left to right, the diagram shows the pressure distribution as the thickness of the black line and shows the velocity in the boundary layer as the violet triangles. The green vortex generators prompt the transition to turbulent flow and prevent back-flow also called flow separation from the high pressure region in the back. The surface in front is as smooth as possible or even employs shark like skin, as any turbulence here will reduce the energy of the airflow. The truncation on the right, known as a Kammback, also prevents back flow from the high pressure region in the back across the spoilers to the convergent part.

In physics, **fluid dynamics** is a sub-discipline of fluid mechanics that deals with **fluid flow**—the natural science of fluids (liquids and gases) in motion. It has several subdisciplines itself, including aerodynamics (the study of air and other gases in motion) and **hydrodynamics** (the study of liquids in motion). Fluid dynamics has a wide range of applications, including calculating forces and moments on aircraft, determining the mass flow rate of petroleum through pipelines, predicting weather patterns, understanding nebulae in interstellar space and reportedly modeling fission weapon detonation. Some of its principles are even used in traffic engineering, where traffic is treated as a continuous fluid.

Fluid dynamics offers a systematic structure that underlies these practical disciplines, that embraces empirical and semi-empirical laws derived from flow measurement and used to solve practical problems. The solution to a fluid dynamics problem typically involves calculating various properties of the fluid, such as velocity, pressure, density, and temperature, as functions of space and time.

Historically, *hydrodynamics* meant something different than it does today. Before the twentieth century, hydrodynamics was synonymous with fluid dynamics. This is still reflected in names of some fluid dynamics topics, like magnetohydrodynamics and hydrodynamic stability—both also applicable in, as well as being applied to, gases.

Equations of fluid dynamics

The foundational axioms of fluid dynamics are the conservation laws, specifically, conservation of mass, conservation of linear momentum (also known as Newton's Second Law of Motion), and conservation of energy (also known as First Law of Thermodynamics). These are based on classical mechanics and are modified in quantum mechanics and general relativity. They are expressed using the Reynolds Transport Theorem.

In addition to the above, fluids are assumed to obey the *continuum assumption*. Fluids are composed of molecules that collide with one another and solid objects. However, the continuum assumption considers fluids to be continuous, rather than discrete. Consequently, properties such as density, pressure, temperature, and velocity are taken to be well-defined at infinitesimally small points, and are assumed to vary continuously from one point to another. The fact that the fluid is made up of discrete molecules is ignored.

For fluids which are sufficiently dense to be a continuum, do not contain ionized species, and have velocities small in relation to the speed of light, the momentum equations for Newtonian fluids are the Navier-Stokes equations, which is a non-linear set of differential equations that describes the flow of a fluid whose stress depends linearly on velocity gradients and pressure. The unsimplified equations do not have a general closed-form solution, so they are primarily of use in Computational Fluid Dynamics. The equations can be simplified in a number of ways, all of which make them easier to solve. Some of them allow appropriate fluid dynamics problems to be solved in closed form.

In addition to the mass, momentum, and energy conservation equations, a thermodynamical equation of state giving the pressure as a function of other thermodynamic variables for the fluid is required to completely specify the problem. An example of this would be the perfect gas equation of state:

$$p = \frac{\rho R_u T}{M}$$

where p is pressure, ρ is density, R_u is the gas constant, M is the molar mass and T is temperature.

Compressible vs incompressible flow

All fluids are compressible to some extent, that is changes in pressure or temperature will result in changes in density. However, in many situations the changes in pressure and temperature are sufficiently small that the changes in density are negligible. In this case the flow can be modeled as an incompressible flow. Otherwise the more general compressible flow equations must be used.

Mathematically, incompressibility is expressed by saying that the density ρ of a fluid parcel does not change as it moves in the flow field, i.e.,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0\,,$$

where D / D*t* is the substantial derivative, which is the sum of local and convective derivatives. This additional constraint simplifies the governing equations, especially in the case when the fluid has a uniform density.

For flow of gases, to determine whether to use compressible or incompressible fluid dynamics, the Mach number of the flow is to be evaluated. As a rough guide, compressible effects can be ignored at Mach numbers below approximately 0.3. For liquids, whether the incompressible assumption is valid depends on the fluid properties (specifically the critical pressure and temperature of the fluid) and the flow conditions (how close to the critical pressure the actual flow pressure becomes). Acoustic problems always require allowing compressibility, since sound waves are compression waves involving changes in pressure and density of the medium through which they propagate.

Viscous vs inviscid flow

Viscous problems are those in which fluid friction has significant effects on the fluid motion.

The Reynolds number, which is a ratio between inertial and viscous forces, can be used to evaluate whether viscous or inviscid equations are appropriate to the problem.

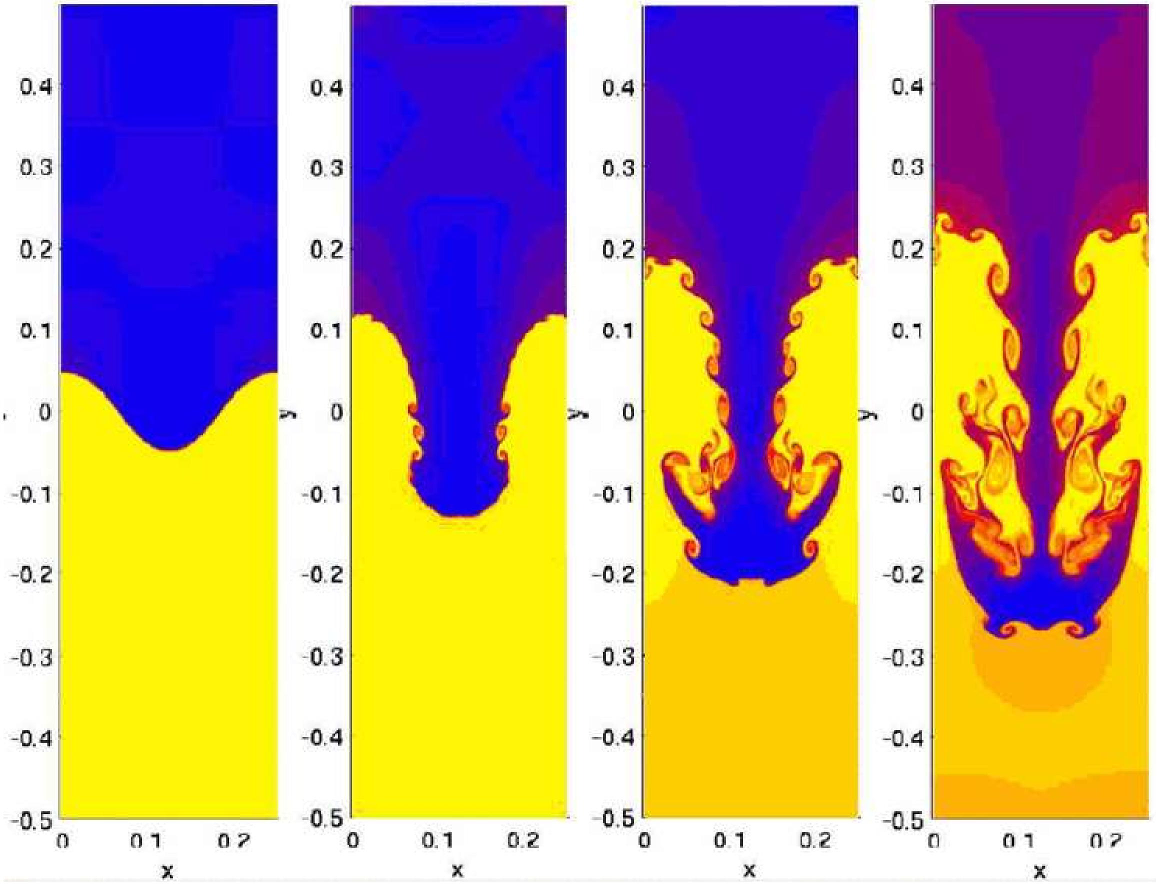
Stokes flow is flow at very low Reynolds numbers, $Re \ll 1$, such that inertial forces can be neglected compared to viscous forces.

On the contrary, high Reynolds numbers indicate that the inertial forces are more significant than the viscous (friction) forces. Therefore, we may assume the flow to be an inviscid flow, an approximation in which we neglect viscosity completely, compared to inertial terms.

This idea can work fairly well when the Reynolds number is high. However, certain problems such as those involving solid boundaries, may require that the viscosity be included. Viscosity often cannot be neglected near solid boundaries because the no-slip condition can generate a thin region of large strain rate (known as Boundary layer) which enhances the effect of even a small amount of viscosity, and thus generating vorticity. Therefore, to calculate net forces on bodies (such as wings) we should use viscous flow equations. As illustrated by d'Alembert's paradox, a body in an inviscid fluid will experience no drag force. The standard equations of inviscid flow are the Euler equations. Another often used model, especially in computational fluid dynamics, is to use the Euler equations away from the body and the boundary layer equations, which incorporates viscosity, in a region close to the body.

The Euler equations can be integrated along a streamline to get Bernoulli's equation. When the flow is everywhere irrotational and inviscid, Bernoulli's equation can be used throughout the flow field. Such flows are called potential flows.

Steady vs unsteady flow



Hydrodynamics simulation of the Rayleigh–Taylor instability

When all the time derivatives of a flow field vanish, the flow is considered to be a **steady flow**. Steady-state flow refers to the condition where the fluid properties at a point in the system do not change over time. Otherwise, flow is called unsteady. Whether a particular flow is steady or unsteady, can depend on the chosen frame of reference. For instance, laminar flow over a sphere is steady in the frame of reference that is stationary with respect to the sphere. In a frame of reference that is stationary with respect to a background flow, the flow is unsteady.

Turbulent flows are unsteady by definition. A turbulent flow can, however, be statistically stationary. According to Pope:

The random field $U(x,t)$ is statistically stationary if all statistics are invariant under a shift in time.

This roughly means that all statistical properties are constant in time. Often, the mean field is the object of interest, and this is constant too in a statistically stationary flow.

Steady flows are often more tractable than otherwise similar unsteady flows. The governing equations of a steady problem have one dimension fewer (time) than the governing equations of the same problem without taking advantage of the steadiness of the flow field.

Laminar vs turbulent flow

Turbulence is flow characterized by recirculation, eddies, and apparent randomness. Flow in which turbulence is not exhibited is called laminar. It should be noted, however, that the presence of eddies or recirculation alone does not necessarily indicate turbulent flow—these phenomena may be present in laminar flow as well. Mathematically, turbulent flow is often represented via a Reynolds decomposition, in which the flow is broken down into the sum of an average component and a perturbation component.

It is believed that turbulent flows can be described well through the use of the Navier–Stokes equations. Direct numerical simulation (DNS), based on the Navier–Stokes equations, makes it possible to simulate turbulent flows at moderate Reynolds numbers. Restrictions depend on the power of the computer used and the efficiency of the solution algorithm. The results of DNS have been found to agree well with experimental data for some flows.

Most flows of interest have Reynolds numbers much too high for DNS to be a viable option, given the state of computational power for the next few decades. Any flight vehicle large enough to carry a human (L > 3 m), moving faster than 72 km/h (20 m/s) is well beyond the limit of DNS simulation (Re = 4 million). Transport aircraft wings (such as on an Airbus A300 or Boeing 747) have Reynolds numbers of 40 million (based on the wing chord). In order to solve these real-life flow problems, turbulence models will be a necessity for the foreseeable future. Reynolds-averaged Navier–Stokes equations (RANS) combined with turbulence modeling provides a model of the effects of the turbulent flow. Such a modeling mainly provides the additional momentum transfer by the Reynolds stresses, although the turbulence also enhances the heat and mass transfer. Another promising methodology is large eddy simulation (LES), especially in the guise of detached eddy simulation (DES)—which is a combination of RANS turbulence modeling and large eddy simulation.

Newtonian vs non-Newtonian fluids

Sir Isaac Newton showed how stress and the rate of strain are very close to linearly related for many familiar fluids, such as water and air. These Newtonian fluids are modeled by a coefficient called viscosity, which depends on the specific fluid.

However, some of the other materials, such as emulsions and slurries and some visco-elastic materials (e.g. blood, some polymers), have more complicated *non-Newtonian* stress-strain behaviours. These materials include *sticky liquids* such as latex, honey, and lubricants which are studied in the sub-discipline of rheology.

Subsonic vs transonic, supersonic and hypersonic flows

While many terrestrial flows (e.g. flow of water through a pipe) occur at low mach numbers, many flows of practical interest (e.g. in aerodynamics) occur at high fractions of the Mach Number M=1 or in excess of it (supersonic flows). New phenomena occur at these Mach number regimes (e.g. shock waves for supersonic flow, transonic instability in a regime of flows with M nearly equal to 1, non-equilibrium chemical behavior due to ionization in hypersonic flows) and it is necessary to treat each of these flow regimes separately.

Magnetohydrodynamics

Magnetohydrodynamics is the multi-disciplinary study of the flow of electrically conducting fluids in electromagnetic fields. Examples of such fluids include plasmas, liquid metals, and salt water. The fluid flow equations are solved simultaneously with Maxwell's equations of electromagnetism.

Other approximations

There are a large number of other possible approximations to fluid dynamic problems. Some of the more commonly used are listed below.

- The **Boussinesq approximation** neglects variations in density except to calculate buoyancy forces. It is often used in free convection problems where density changes are small.
- **Lubrication theory** and **Hele-Shaw flow** exploits the large aspect ratio of the domain to show that certain terms in the equations are small and so can be neglected.

- **Slender-body theory** is a methodology used in Stokes flow problems to estimate the force on, or flow field around, a long slender object in a viscous fluid.
- The **shallow-water equations** can be used to describe a layer of relatively inviscid fluid with a free surface, in which surface gradients are small.
- The **Boussinesq equations** are applicable to surface waves on thicker layers of fluid and with steeper surface slopes.
- **Darcy's law** is used for flow in porous media, and works with variables averaged over several pore-widths.
- In rotating systems, the **quasi-geostrophic approximation** assumes an almost perfect balance between pressure gradients and the Coriolis force. It is useful in the study of atmospheric dynamics.

Terminology in fluid dynamics

The concept of pressure is central to the study of both fluid statics and fluid dynamics. A pressure can be identified for every point in a body of fluid, regardless of whether the fluid is in motion or not. Pressure can be measured using an aneroid, Bourdon tube, mercury column, or various other methods.

Some of the terminology that is necessary in the study of fluid dynamics is not found in other similar areas of study. In particular, some of the terminology used in fluid dynamics is not used in fluid statics.

Terminology in incompressible fluid dynamics

The concepts of total pressure and dynamic pressure arise from Bernoulli's equation and are significant in the study of all fluid flows. (These two pressures are not pressures in the usual sense—they cannot be measured using an aneroid, Bourdon tube or mercury column.) To avoid potential ambiguity when referring to pressure in fluid dynamics, many authors use the term static pressure to distinguish it from total pressure and dynamic pressure. Static pressure is identical to pressure and can be identified for every point in a fluid flow field.

In *Aerodynamics*, L.J. Clancy writes: *To distinguish it from the total and dynamic pressures, the actual pressure of the fluid, which is associated not with its motion but with its state, is often referred to as the static pressure, but where the term pressure alone is used it refers to this static pressure.*

A point in a fluid flow where the flow has come to rest (i.e. speed is equal to zero adjacent to some solid body immersed in the fluid flow) is of special significance. It is of such importance that it is given a special name—a stagnation point. The static pressure at the stagnation point is of special significance and is given its own name—stagnation pressure. In incompressible flows, the stagnation pressure at a stagnation point is equal to the total pressure throughout the flow field.

Terminology in compressible fluid dynamics

In a compressible fluid, such as air, the temperature and density are essential when determining the state of the fluid. In addition to the concept of total pressure (also known as stagnation pressure), the concepts of total (or stagnation) temperature and total (or stagnation) density are also essential in any study of compressible fluid flows. To avoid potential ambiguity when referring to temperature and density, many authors use the terms static temperature and static density. Static temperature is identical to temperature; and static density is identical to density; and both can be identified for every point in a fluid flow field.

The temperature and density at a stagnation point are called stagnation temperature and stagnation density.

A similar approach is also taken with the thermodynamic properties of compressible fluids. Many authors use the terms total (or stagnation) enthalpy and total (or stagnation) entropy. The terms static enthalpy and static entropy appear to be less common, but where they are used they mean nothing more than enthalpy and entropy respectively, and the prefix "static" is being used to avoid ambiguity with their 'total' or 'stagnation' counterparts. Because the 'total' flow conditions are defined by isentropically bringing the fluid to rest, the total (or stagnation) entropy is by definition always equal to the "static" entropy.

Chapter- 4

Fluid Power and Capillary Surface

Fluid power

'Fluid power' is the technology of exploiting the properties of fluids to generate, control, and transmit power as a result of the pressurization of fluids.

As the term "fluid" refers either to gases or to liquids, fluid power is also subdivided into the categories of "hydraulics" and "pneumatics". The differences being that with hydraulics the medium used is a liquid (i.e. mineral oil or water) and for pneumatics it is a gas (i.e. air or another inert gas).

Practical use

Transport energy

A fluid power system with a pump driven by a prime mover such as an electric motor or IC engine that converts mechanical energy into fluid energy. This fluid flow is used to actuate a device specifically designed to operate with fluid energy such as:

- Cylinder (hydraulic or pneumatic): Provides force in a linear fashion
- Motor (hydraulic or pneumatic): Provides continuous rotational motion
- Rotary actuator: Provides rotational motion of less than 360 degrees.

Application

Hydraulics and pneumatics are similar in many ways, but there are clear reasons for using one over the other.

- **Cost**: Pneumatics are considerably cheaper to build and operate. For one, air is used as the compressed medium, so no reservoir is needed to store fluid, nor is there any need to provide means to drain or recover fluid. With increasing working pressures, pneumatics require larger parts than hydraulics.

- **Precision**: Unlike liquids, gases change volume significantly when pressurized making it difficult to achieve precision.
- **Safety**: Compressed gases tend to expand at high velocities when decompressed, thus pneumatics are typically limited in utilities with a working pressure up to around 100 psi (7 bar).

Capillary surface

In fluid mechanics and mathematics, a **capillary surface** is a surface that represents the interface between two different fluids. As a consequence of being a surface, a capillary surface has no thickness in slight contrast with most real fluid interfaces.

Capillary surfaces are of interest in mathematics because the problems involved are very nonlinear and have interesting properties, such as discontinuous dependence on boundary data at isolated points. In particular, static capillary surfaces with gravity absent have constant mean curvature, so that a minimal surface is a special case of static capillary surface.

They are also of practical interest for fluid management in space (or other environments free of body forces), where both flow and static configuration are often dominated by capillary effects.

The stress balance equation

The defining equation for a capillary surface is called the stress balance equation, which can be derived by considering the forces and stresses acting on a small volume that is partly bounded by a capillary surface. For a fluid meeting another fluid (the "other" fluid notated with bars) at a surface *S*, the equation reads

$$(\sigma_{ij} - \bar{\sigma}_{ij})\hat{\mathbf{n}} = -\gamma\hat{\mathbf{n}}(\nabla_S \cdot \hat{\mathbf{n}}) + \nabla_S\gamma \qquad ; \quad \nabla_S\gamma = \nabla\gamma - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \nabla\gamma)$$

where $\hat{\mathbf{n}}$ is the unit normal pointing toward the "other" fluid (the one whose quantities are notated with bars), σ_{ij} is the stress tensor (note that on the left is a tensor-vector product), γ is the surface tension associated with the interface, and ∇_S is the surface gradient. Note that the quantity $-\nabla_S \cdot \hat{\mathbf{n}}$ is twice the mean curvature of the surface.

In fluid mechanics, this equation serves as a boundary condition for interfacial flows, typically complementing the Navier–Stokes equations. It describes the discontinuity in stress that is balanced by forces at the surface. As a boundary condition, it is somewhat unusual in that it introduces a new variable: the surface *S* that defines the interface. It's not too surprising then that the stress balance equation normally mandates its own boundary conditions.

For best use, this vector equation is normally turned into 3 scalar equations via dot product with the unit normal and two selected unit tangents:

$$
\begin{aligned}
&((\sigma_{ij}-\bar{\sigma}_{ij})\hat{\mathbf{n}})\cdot\hat{\mathbf{n}} = -\gamma\nabla_S\cdot\hat{\mathbf{n}}\\
&((\sigma_{ij}-\bar{\sigma}_{ij})\hat{\mathbf{n}})\cdot\hat{\mathbf{t}}_{\mathbf{1}} = \nabla_S\gamma\cdot\hat{\mathbf{t}}_{\mathbf{1}}\\
&((\sigma_{ij}-\bar{\sigma}_{ij})\hat{\mathbf{n}})\cdot\hat{\mathbf{t}}_{\mathbf{2}} = \nabla_S\gamma\cdot\hat{\mathbf{t}}_{\mathbf{2}}
\end{aligned}
$$

Note that the products lacking dots are tensor products of tensors with vectors (resulting in vectors similar to a matrix-vector product), those with dots are dot products. The first equation is called the **normal stress equation**, or the normal stress boundary condition. The second two equations are called **tangential stress equations**.

The stress tensor

The stress tensor is related to velocity and pressure. Its actual form will depend on the specific fluid being dealt with, for the common case of incompressible Newtonian flow the stress tensor is given by

$$
\begin{aligned}
\sigma_{ij} &= -\begin{pmatrix} p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & p\end{pmatrix} + \mu\begin{pmatrix} 2\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\\ \frac{\partial v}{\partial x}+\frac{\partial u}{\partial y} & 2\frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\\ \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z}\end{pmatrix}\\
&= -pI + \mu(\nabla\mathbf{v}+(\nabla\mathbf{v})^T)
\end{aligned}
$$

where p is the pressure in the fluid, $\mathbf{v}$is the velocity, and μ is the viscosity.

Static interfaces

In the absence of motion, the stress tensors yield only hydrostatic pressure so that $\sigma_{ij}=-pI$, regardless of fluid type or compressibility. Considering the normal and tangential equations,

$$
\begin{aligned}
&\bar{p}-p = \gamma\nabla\cdot\hat{\mathbf{n}}\\
&0 = \nabla\gamma\cdot\hat{\mathbf{t}}
\end{aligned}
$$

The first equation establishes that curvature forces are balanced by pressure forces. The second equation implies that a static interface cannot exist in the presence of nonzero surface tension gradient.

If gravity is the only body force present, the Navier–Stokes equations simplify significantly:

$$0 = -\nabla p + \rho\mathbf{g}$$

If coordinates are chosen so that gravity is nonzero only in the z direction, this equation degrades to a particularly simple form:

$$\frac{dp}{dz} = \rho g \quad \Rightarrow \quad p = \rho g z + p_0$$

where p_0 is an integration constant that represents some reference pressure at $z = 0$. Substituting this into the normal stress equation yields what is known as the Young-Laplace equation:

$$\bar{\rho} g z + \bar{p}_0 - (\rho g z + p_0) = \gamma \nabla \cdot \hat{\mathbf{n}} \quad \Rightarrow \quad \Delta\rho g z + \Delta p = \gamma \nabla \cdot \hat{\mathbf{n}}$$

where Δp is the (constant) pressure difference across the interface, and $\Delta\rho$ is the difference in density. Note that, since this equation defines a surface, z is the z coordinate of the capillary surface. This nonlinear partial differential equation when supplied with the right boundary conditions will define the static interface.

The pressure difference above is a constant, but its value will change if the z coordinate is shifted. The linear solution to pressure implies that, **unless the gravity term is absent**, it is always possible to define the z coordinate so that $\Delta p = 0$. Nondimensionalized, the Young-Laplace equation is usually studied in the form

$$\kappa z + \lambda = \nabla \cdot \hat{\mathbf{n}}$$

where (if gravity is in the negative z direction) κ is positive if the denser fluid is "inside" the interface, negative if it is "outside", and zero if there is no gravity or if there is no difference in density between the fluids.

This nonlinear equation has some rich properties, especially in terms of existence of unique solutions. For example, the nonexistence of solution to some boundary value problem implies that, physically, the problem can't be static. If a solution does exist, normally it'll exist for very specific values of λ, which is representative of the pressure jump across the interface. This is interesting because there isn't another physical equation to determine the pressure difference. In a capillary tube, for example, implementing the contact angle boundary condition will yield a unique solution for exactly one value of λ. Solutions often aren't unique, this implies that there are multiple static interfaces possible; while they may all solve the same boundary value problem, the minimization of energy will normally favor one. Different solutions are called *configurations* of the interface.

Energy consideration

A deep property of capillary surfaces is the surface energy that is imparted by surface tension:

$$E_S = \gamma_S A_S$$

where *A* is the area of the surface being considered, and the total energy is the summation of all energies. Note that *every* interface imparts energy. For example, if there are two different fluids (say liquid and gas) inside a solid container with gravity and other energy potentials absent, the energy of the system is

$$E = \sum \gamma_S A_S = \gamma_{LG} A_{LG} + \gamma_{SG} A_{SG} + \gamma_{SL} A_{SL}$$

where the subscripts *LG*, *SG*, and *SL* respectively indicate the liquid-gas, solid-gas, and solid-liquid interfaces. Note that inclusion of gravity would require consideration of the volume enclosed by the capillary surface and the solid walls.

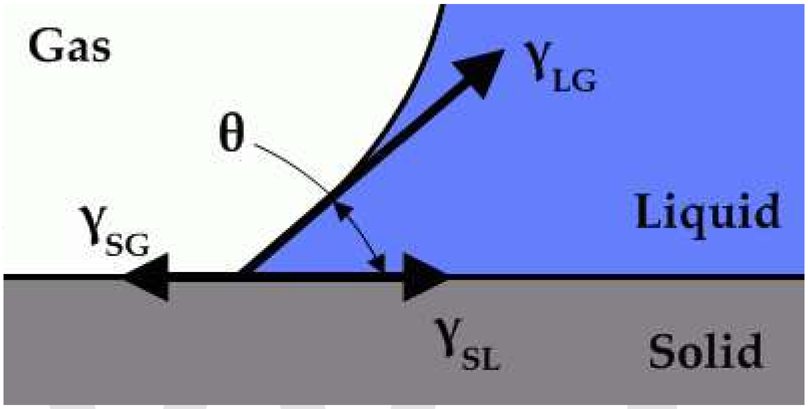


Illustration of distributed forces at the contact line, with the contact line perpendicular to the image. The vertical part of γ_{LG} is balanced by a slight deformation of the solid (not shown and inconsequential to this context).

Typically the surface tension values between the solid-gas and solid-liquid interfaces are not known. This does not pose a problem; since only changes in energy are of primary interest. If the net solid area $A_{SG} + A_{SL}$ is a constant, and the contact angle is known, it may be shown that (again, for two different fluids in a solid container)

$$E = \gamma_{SL}(A_{SL} + A_{SG}) + \gamma_{LG} A_{LG} + \gamma_{LG} A_{SG} \cos(\theta)$$

so that

$$\frac{\Delta E}{\gamma_{LG}} = \Delta A_{LG} + \Delta A_{SG} \cos(\theta) = \Delta A_{LG} - \Delta A_{SL} \cos(\theta)$$

where θ is the contact angle and the capital delta indicates the change from one configuration to another. To obtain this result, it's necessary to sum (distributed) forces at the contact line (where solid, gas, and liquid meet) in a direction tangent to the solid interface and perpendicular to the contact line:

$$
\begin{aligned}
0 &= \sum F_{\text{Contact line}} \\
&= \gamma_{LG}\cos(\theta) + \gamma_{SL} - \gamma_{SG}
\end{aligned}
$$

where the sum is zero because of the static state. When solutions to the Young-Laplace equation aren't unique, the most physically favorable solution is the one of minimum energy, though experiments (especially low gravity) show that metastable surfaces can be surprisingly persistent, and that the most stable configuration can become metastable through mechanical jarring without too much difficulty. On the other hand, a metastable surface can sometimes spontaneously achieve lower energy without any input (seemingly at least) given enough time.

Boundary conditions

Boundary conditions for stress balance describe the capillary surface at the **contact line**: the line where a solid meets the capillary interface; also, volume constraints can serve as boundary conditions (a suspended drop, for example, has no contact line but clearly must admit a unique solution).

For static surfaces, the most common contact line boundary condition is the implementation of the contact angle, which specifies the angle that one of the fluids meets the solid wall. The contact angle condition on the surface *S* is normally written as:

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{v}} = \cos(\theta)$$

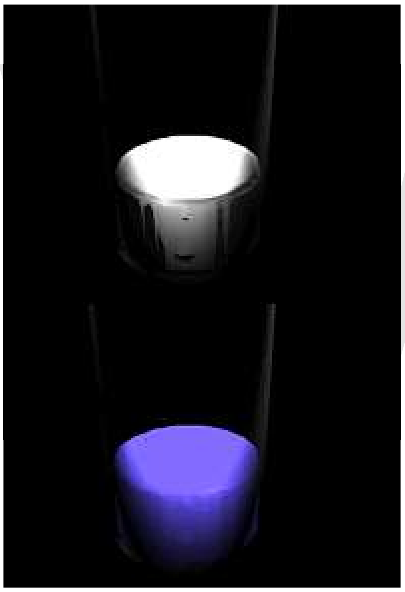
where θ is the contact angle. This condition is imposed on the boundary (or boundaries) ∂S of the surface. $\hat{v}$ is the unit outward normal to the solid surface, and $\hat{n}$ is a unit normal to *S*. Choice of $\hat{n}$ depends on which fluid the contact angle is specified for.

For dynamic interfaces, the boundary condition showed above works well if the contact line velocity is low. If the velocity is high, the contact angle will change ("dynamic contact angle"), and as of 2007 the mechanics of the moving contact line (or even the validity of the contact angle as a parameter) is not known and an area of active research.

Chapter- 5

Viscosity

Viscosity



Clear liquid above has lower viscosity than the substance below

SI symbol:	μ, η
SI unit:	Pa·s = kg/(s·m)
Derivations from other quantities:	μ = G·t

Viscosity is a measure of the resistance of a fluid which is being deformed by either shear stress or tensile stress. In everyday terms (and for fluids only), viscosity is "thickness" or "internal friction". Thus, water is "thin", having a lower viscosity, while honey is "thick", having a higher viscosity. Put simply, the less viscous the fluid is, the greater its ease of movement (fluidity).

Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. For example, high-viscosity felsic magma will create a tall, steep stratovolcano, because it cannot flow far before it cools, while low-viscosity mafic lava will create a wide, shallow-sloped shield volcano. All real fluids (except superfluids) have some resistance to stress and therefore are **viscous**, but a fluid which has no resistance to shear stress is known as an **ideal fluid** or **inviscid fluid**.

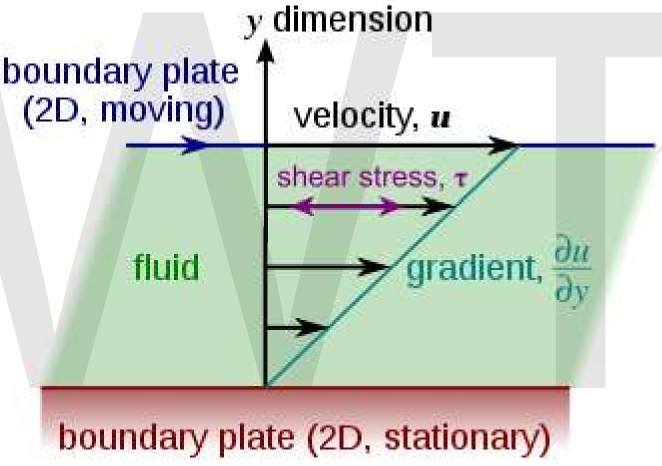
The study of flowing matter is known as rheology, which includes viscosity and related concepts.

Etymology

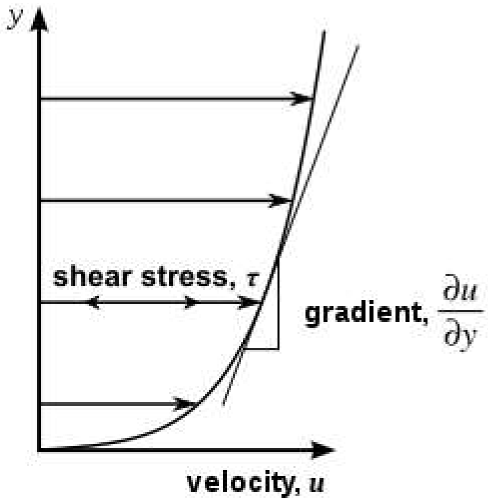
The word "viscosity" derives from the Latin word "viscum alba" for mistletoe. A viscous glue called birdlime was made from mistletoe berries and used for lime-twigs to catch birds.

Properties and behavior

Overview



Laminar shear of fluid between two plates. Friction between the fluid and the moving boundaries causes the fluid to shear. The force required for this action is a measure of the fluid's viscosity. This type of flow is known as a Couette flow.



Laminar shear, the non-constant gradient, is a result of the geometry the fluid is flowing through (e.g. a pipe).

In general, in any flow, layers move at different velocities and the fluid's viscosity arises from the shear stress between the layers that ultimately opposes any applied force.

The relationship between the shear stress and the velocity gradient can be obtained by considering two plates closely spaced at a distance *y*, and separated by a homogeneous substance. Assuming that the plates are very large, with a large area *A*, such that edge effects may be ignored, and that the lower plate is fixed, let a force *F* be applied to the upper plate. If this force causes the substance between the plates to undergo shear flow with a velocity gradient *u* (as opposed to just shearing elastically until the shear stress in the substance balances the applied force), the substance is called a fluid.

The applied force is proportional to the area and velocity gradient in the fluid and inversely proportional to the distance between the plates. Combining these three relations results in the equation:

$$F = \mu A \frac{u}{y},$$

where μ is the proportionality factor called *viscosity*.

This equation can be expressed in terms of shear stress $\tau = \frac{F}{A}$. Thus as expressed in differential form by Isaac Newton for straight, parallel and uniform flow, the shear stress between layers is proportional to the velocity gradient in the direction perpendicular to the layers:

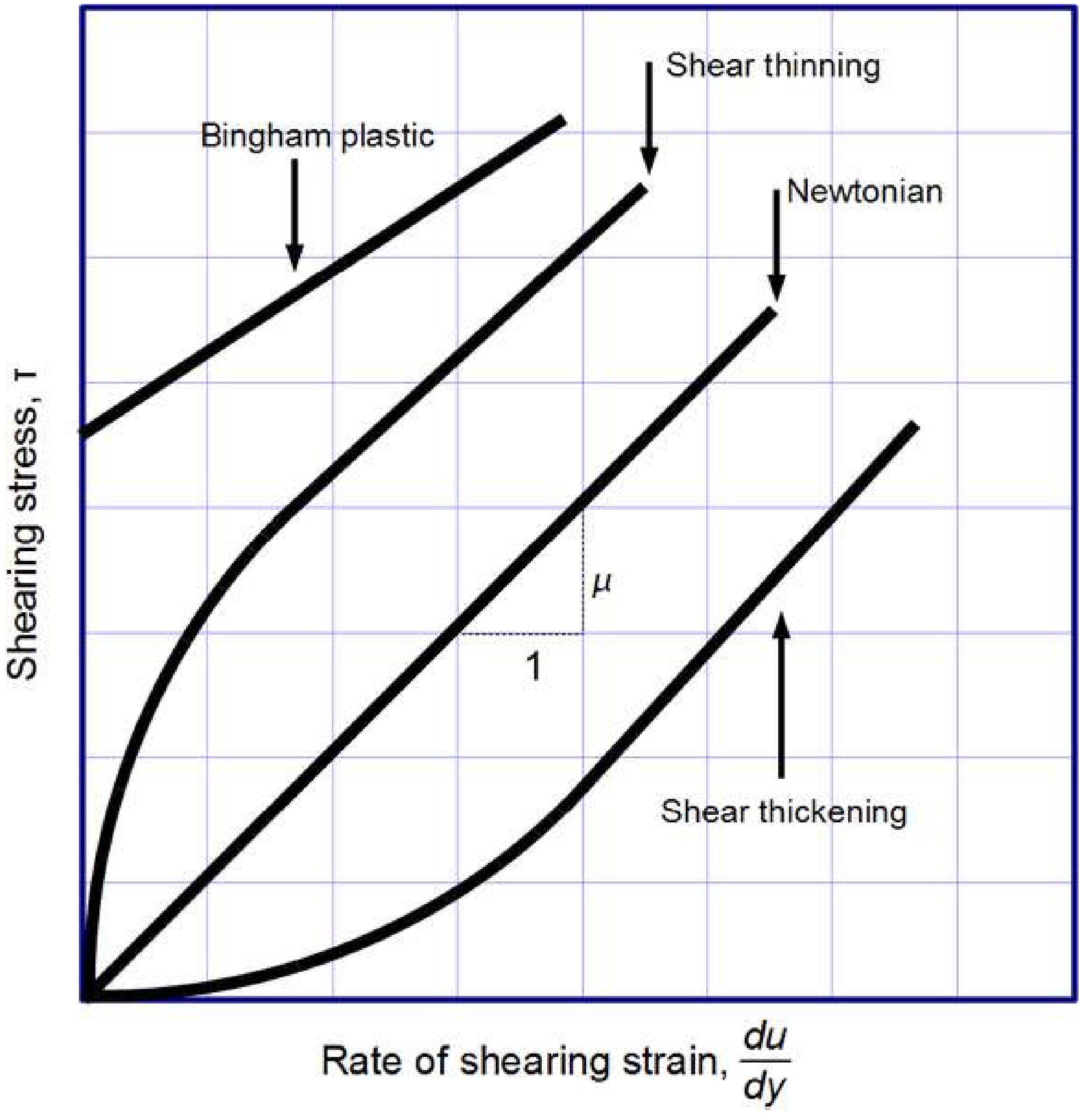
$$\tau = \mu \frac{\partial u}{\partial y}$$

Hence, through this method, the relation between the shear stress and the velocity gradient can be obtained.

Note that the *rate of shear deformation* is $\frac{u}{y}$ which can be also written as a *shear velocity*, $\frac{du}{dy}$.

James Clerk Maxwell called viscosity *fugitive elasticity* because of the analogy that elastic deformation opposes shear stress in solids, while in viscous fluids, shear stress is opposed by *rate* of deformation.

Types of viscosity



Viscosity, the slope of each line, varies among materials

Newton's law of viscosity, given above, is a constitutive equation (like Hooke's law, Fick's law, Ohm's law). It is not a fundamental law of nature but an approximation that holds in some materials and fails in others. Non-Newtonian fluids exhibit a more complicated relationship between shear stress and velocity gradient than simple linearity. Thus there exist a number of forms of viscosity:

- **Newtonian**: fluids, such as water and most gases which have a constant viscosity.
- **Shear thickening**: viscosity *increases* with the rate of shear.
- **Shear thinning**: viscosity *decreases* with the rate of shear. Shear thinning liquids are very commonly, but misleadingly, described as thixotropic.
- **Thixotropic**: materials which become *less* viscous over time when shaken, agitated, or otherwise stressed.
- **Rheopectic**: materials which become *more* viscous over time when shaken, agitated, or otherwise stressed.
- A **Bingham plastic** is a material that behaves as a solid at low stresses but flows as a viscous fluid at high stresses.
- A **magnetorheological fluid** is a type of "smart fluid" which, when subjected to a magnetic field, greatly increases its apparent viscosity, to the point of becoming a viscoelastic solid.

Viscosity coefficients

Viscosity coefficients can be defined in two ways:

- **Dynamic viscosity**, also **absolute viscosity**, the more usual one (typical units Pa·s, Poise, P);
- **Kinematic viscosity** is the *dynamic viscosity* divided by the density (typical units m^2/s, Stokes, St).

Viscosity is a tensorial quantity that can be decomposed in different ways into two independent components. The most usual decomposition yields the following viscosity coefficients:

- **Shear viscosity**, the most important one, often referred to as simply **viscosity**, describing the reaction to applied shear stress; simply put, it is the ratio between the pressure exerted on the surface of a fluid, in the lateral or horizontal direction, to the change in velocity of the fluid as you move down in the fluid (this is what is referred to as a velocity gradient).
- **Volume viscosity** (also called **bulk viscosity** or **second viscosity**) becomes important only for such effects where fluid compressibility is essential. Examples would include shock waves and sound propagation. It appears in the Stokes' law (sound attenuation) that describes propagation of sound in Newtonian liquid.

Alternatively,

- **Extensional viscosity**, a linear combination of shear and bulk viscosity, describes the reaction to elongation, widely used for characterizing polymers. For example, at room temperature, water has a dynamic shear viscosity of about 1.0×10^{-3} Pa·s and motor oil of about 250×10^{-3} Pa·s.

Viscosity measurement

Viscosity is measured with various types of viscometers and rheometers. A rheometer is used for those fluids which cannot be defined by a single value of viscosity and therefore require more parameters to be set and measured than is the case for a viscometer. Close temperature control of the fluid is essential to accurate measurements, particularly in materials like lubricants, whose viscosity can double with a change of only 5 °C.

For some fluids, viscosity is a constant over a wide range of shear rates (Newtonian fluids). The fluids without a constant viscosity (non-Newtonian fluids) cannot be described by a single number. Non-Newtonian fluids exhibit a variety of different correlations between shear stress and shear rate.

One of the most common instruments for measuring kinematic viscosity is the glass capillary viscometer.

In paint industries, viscosity is commonly measured with a Zahn cup, in which the efflux time is determined and given to customers. The efflux time can also be converted to kinematic viscosities (centistokes, cSt) through the conversion equations.

Also used in paint, a Stormer viscometer uses load-based rotation in order to determine viscosity. The viscosity is reported in Krebs units (KU), which are unique to Stormer viscometers.

A Ford viscosity cup measures the rate of flow of a liquid. This, under ideal conditions, is proportional to the kinematic viscosity.

Vibrating viscometers can also be used to measure viscosity. These models such as the *Dynatrol* use vibration rather than rotation to measure viscosity.

Extensional viscosity can be measured with various rheometers that apply extensional stress.

Volume viscosity can be measured with an acoustic rheometer.

Apparent viscosity is a calculation derived from tests performed on drilling fluid used in oil or gas well development. These calculations and tests help engineers develop and maintain the properties of the drilling fluid to the specifications required.

Units

Dynamic viscosity

The usual symbol for dynamic viscosity used by mechanical and chemical engineers — as well as fluid dynamicists — is the Greek letter mu (μ). The symbol η is also used by chemists, physicists, and the IUPAC.

The SI physical unit of dynamic viscosity is the pascal-second (Pa·s), (equivalent to N·s/m^2, or kg/(m·s)). If a fluid with a viscosity of one Pa·s is placed between two plates, and one plate is pushed sideways with a shear stress of one pascal, it moves a distance equal to the thickness of the layer between the plates in one second.

The cgs physical unit for dynamic viscosity is the *poise* (P), named after Jean Louis Marie Poiseuille. It is more commonly expressed, particularly in ASTM standards, as *centipoise* (cP). Water at 20 °C has a viscosity of 1.0020 cP or 0.001002 kg/(m·s).

$1\ \mathrm{P} = 1\ \mathrm{g{\cdot}cm^{-1}{\cdot}s^{-1}}.$
$1\ \mathrm{Pa{\cdot}s} = 1\ \mathrm{kg{\cdot}m^{-1}{\cdot}s^{-1}} = 10\ \mathrm{P}.$

The relation to the SI unit is

$1\ \mathrm{P} = 0.1\ \mathrm{Pa{\cdot}s},$
$1\ \mathrm{cP} = 1\ \mathrm{mPa{\cdot}s} = 0.001\ \mathrm{Pa{\cdot}s}.$

Kinematic viscosity

In many situations, we are concerned with the ratio of the inertial force to the viscous force (i.e. the Reynolds number, $Re = VD / \nu$) , the former characterized by the fluid density ρ. This ratio is characterized by the *kinematic viscosity* (Greek letter nu, ν), defined as follows:

$$\nu = \frac{\mu}{\rho}$$

The SI unit of ν is m^2/s. The SI unit of ρ is kg/m^3.

The cgs physical unit for kinematic viscosity is the *stokes* (St), named after George Gabriel Stokes. It is sometimes expressed in terms of *centiStokes* (cSt). In U.S. usage, *stoke* is sometimes used as the singular form.

$1\ \mathrm{St} = 1\ \mathrm{cm^2{\cdot}s^{-1}} = 10^{-4}\ \mathrm{m^2{\cdot}s^{-1}}.$
$1\ \mathrm{cSt} = 1\ \mathrm{mm^2{\cdot}s^{-1}} = 10^{-6}\mathrm{m^2{\cdot}s^{-1}}.$

Water at 20 °C has a kinematic viscosity of about 1 cSt.

The kinematic viscosity is sometimes referred to as **diffusivity of momentum**, because it has the same unit as and is comparable to diffusivity of heat and diffusivity of mass. It is therefore used in dimensionless numbers which compare the ratio of the diffusivities.

Fluidity

The reciprocal of viscosity is *fluidity*, usually symbolized by $\varphi = 1 / \mu$ or $F = 1 / \mu$, depending on the convention used, measured in *reciprocal poise* ($\mathrm{cm \cdot s \cdot g^{-1}}$), sometimes called the *rhe*. *Fluidity* is seldom used in engineering practice.

The concept of fluidity can be used to determine the viscosity of an ideal solution. For two components *a* and *b*, the fluidity when *a* and *b* are mixed is

$$F \approx \chi_a F_a + \chi_b F_b,$$

which is only slightly simpler than the equivalent equation in terms of viscosity:

$$\mu \approx \frac{1}{\chi_a/\mu_a + \chi_b/\mu_b},$$

where χ_a and χ_b is the mole fraction of component *a* and *b* respectively, and μ_a and μ_b are the components pure viscosities.

Non-standard units

The Reyn is a British unit of dynamic viscosity.

Viscosity index is a measure for the change of kinematic viscosity with temperature. It is used to characterise lubricating oil in the automotive industry.

At one time the petroleum industry relied on measuring kinematic viscosity by means of the Saybolt viscometer, and expressing kinematic viscosity in units of *Saybolt Universal Seconds* (SUS). Other abbreviations such as SSU (*Saybolt Seconds Universal*) or SUV (*Saybolt Universal Viscosity*) are sometimes used. Kinematic viscosity in centistoke can be converted from SUS according to the arithmetic and the reference table provided in ASTM D 2161.

Molecular origins

Pitch has a viscosity approximately 230 billion (2.3×10^{11}) times that of water.

The viscosity of a system is determined by how molecules constituting the system interact. There are no simple but correct expressions for the viscosity of a fluid. The simplest exact expressions are the Green–Kubo relations for the linear shear viscosity or the Transient Time Correlation Function expressions derived by Evans and Morriss in 1985. Although these expressions are each exact in order to calculate the viscosity of a dense fluid, using these relations requires the use of molecular dynamics computer simulations.

Gases

Viscosity in gases arises principally from the molecular diffusion that transports momentum between layers of flow. The kinetic theory of gases allows accurate prediction of the behavior of gaseous viscosity.

Within the regime where the theory is applicable:

- Viscosity is independent of pressure and
- Viscosity increases as temperature increases.

James Clerk Maxwell published a famous paper in 1866 using the kinetic theory of gases to study gaseous viscosity. To understand why the viscosity is independent of pressure consider two adjacent boundary layers (A and B) moving with respect to each other. The internal friction (the viscosity) of the gas is determined by the probability a particle of layer A enters layer B with a corresponding transfer of momentum. Maxwell's calculations showed him that the viscosity coefficient is proportional to both the density, the mean free path and the mean velocity of the atoms. On the other hand, the *mean free path* is inversely proportional to the density. So an increase of pressure doesn't result in any change of the viscosity.

Relation to mean free path of diffusing particles

In relation to diffusion, the kinematic viscosity provides a better understanding of the behavior of mass transport of a dilute species. Viscosity is related to shear stress and the rate of shear in a fluid, which illustrates its dependence on the mean free path, λ, of the diffusing particles.

From fluid mechanics, for a Newtonian fluid, the shear stress, τ, on a unit area moving parallel to itself, is found to be proportional to the rate of change of velocity with distance perpendicular to the unit area:

$$\tau = \mu \frac{du_x}{dy}$$

for a unit area parallel to the x-z plane, moving along the x axis. We will derive this formula and show how μ is related to λ.

Interpreting shear stress as the time rate of change of momentum, p, per unit area A (rate of momentum flux) of an arbitrary control surface gives

$$\tau = \frac{\dot{p}}{A} = \frac{\dot{m}\langle u_x \rangle}{A}.$$

where $\langle u_x \rangle$ is the average velocity along x of fluid molecules hitting the unit area, with respect to the unit area.

Further manipulation will show

$$\dot{m} = \rho \bar{u} A$$

$$\langle u_x \rangle = \frac{1}{2} \lambda \frac{du_x}{dy}$$

, assuming that molecules hitting the unit area come from all distances between 0 and λ (equally distributed), and that their average velocities change linearly with distance (always true for small enough λ). From this follows:

$$\tau = \underbrace{\frac{1}{2} \rho \bar{u} \lambda}_{\mu} \cdot \frac{du_x}{dy} \quad \Rightarrow \quad \nu = \frac{\mu}{\rho} = \tfrac{1}{2}\, \bar{u} \lambda,$$

where

$\dot{m}$ is the rate of fluid mass hitting the surface,
ρ is the density of the fluid,
$\bar{u}$ is the average molecular speed ($\bar{u} = \sqrt{\langle u^2 \rangle}$),
μ is the dynamic viscosity.

Effect of temperature on the viscosity of a gas

Sutherland's formula can be used to derive the dynamic viscosity of an ideal gas as a function of the temperature:

$$\mu = \mu_0 \frac{T_0 + C}{T + C} \left(\frac{T}{T_0} \right)^{3/2}$$

This in turn is equal to

$$\lambda \cdot \frac{T^{3/2}}{T + C}, \text{where } \lambda = \frac{\mu_0 (T_0 + C)}{T_0^{3/2}} \text{ which is a constant.}$$

in Sutherland's formula:

- μ = dynamic viscosity in (Pa·s) at input temperature T,
- μ_0 = reference viscosity in (Pa·s) at reference temperature T_0,
- T = input temperature in kelvins,
- T_0 = reference temperature in kelvins,
- C = Sutherland's constant for the gaseous material in question.

Valid for temperatures between $0 < T < 555$ K with an error due to pressure less than 10% below 3.45 MPa.

Sutherland's constant and reference temperature for some gases

Gas	C [K]	T_0 [K]	μ_0 [μPa s]
air	120	291.15	18.27
nitrogen	111	300.55	17.81
oxygen	127	292.25	20.18
carbon dioxide	240	293.15	14.8
carbon monoxide	118	288.15	17.2
hydrogen	72	293.85	8.76
ammonia	370	293.15	9.82
sulfur dioxide	416	293.65	12.54
helium	79.4	273	19

Viscosity of a dilute gas

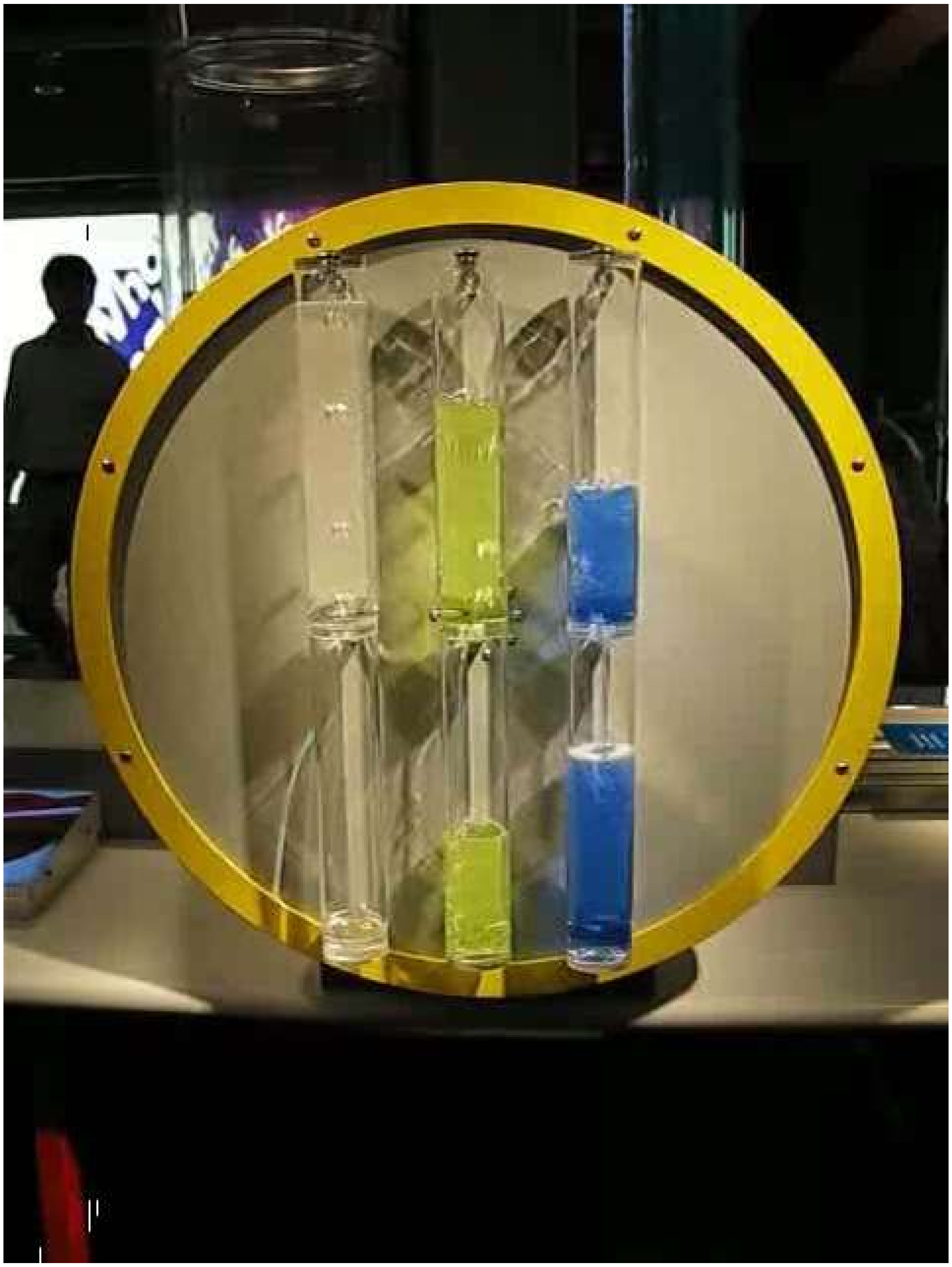
The Chapman-Enskog equation may be used to estimate viscosity for a dilute gas. This equation is based on a semi-theoretical assumption by Chapman and Enskog. The equation requires three empirically determined parameters: the collision diameter (σ), the maximum energy of attraction divided by the Boltzmann constant (ϵ/κ) and the collision integral ($\omega(T^*)$).

$$\mu_0 \times 10^6 = 2.6693 \frac{(MT)^{1/2}}{\sigma^2 \omega(T^*)},$$

with

- $T^* = \kappa T/\varepsilon$ — reduced temperature (dimensionless),
- μ_0 = viscosity for dilute gas (μPa.s),
- M = molecular mass (g/mol),
- T = temperature (K),
- σ = the collision diameter (Å),
- ε / κ = the maximum energy of attraction divided by the Boltzmann constant (K),
- ω_μ = the collision integral.

Liquids



Three liquids with different Viscosities

In liquids, the additional forces between molecules become important. This leads to an additional contribution to the shear stress though the exact mechanics of this are still controversial. Thus, in liquids:

- Viscosity is independent of pressure (except at very high pressure); and
- Viscosity tends to fall as temperature increases (for example, water viscosity goes from 1.79 cP to 0.28 cP in the temperature range from 0 °C to 100 °C.

The dynamic viscosities of liquids are typically several orders of magnitude higher than dynamic viscosities of gases.

Viscosity of blends of liquids

The viscosity of the blend of two or more liquids can be estimated using the Refutas equation. The calculation is carried out in three steps.

The first step is to calculate the Viscosity Blending Number (VBN) (also called the Viscosity Blending Index) of each component of the blend:

$$\text{VBN} = 14.534 \times \ln\left[\ln(v + 0.8)\right] + 10.975 \qquad (1)$$

where v is the kinematic viscosity in centistokes (cSt). It is important that the kinematic viscosity of each component of the blend be obtained at the same temperature.

The next step is to calculate the VBN of the blend, using this equation:

$$\text{VBN}_{\text{Blend}} = [x_A \times \text{VBN}_A] + [x_B \times \text{VBN}_B] + \cdots + [x_N \times \text{VBN}_N] \qquad (2)$$

where x_X is the mass fraction of each component of the blend.

Once the viscosity blending number of a blend has been calculated using equation (2), the final step is to determine the kinematic viscosity of the blend by solving equation (1) for v:

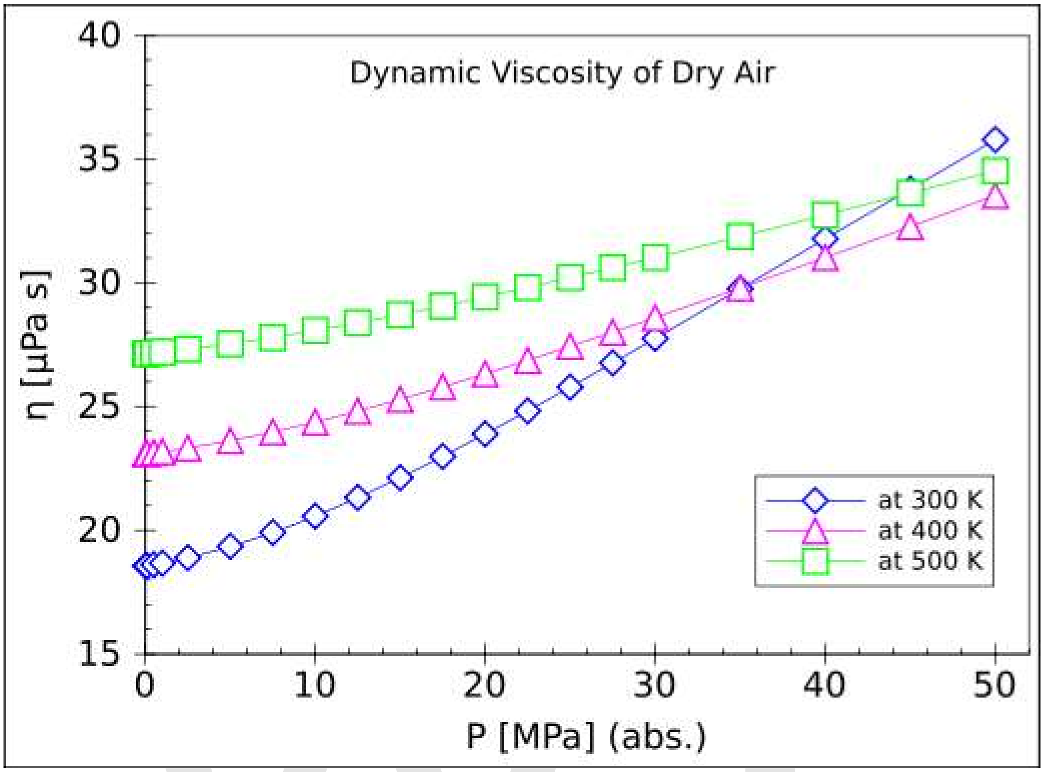
$$v = \exp\left(\exp\left(\frac{\text{VBN}_{\text{Blend}} - 10.975}{14.534}\right)\right) - 0.8, \qquad (3)$$

where VBN_{Blend} is the viscosity blending number of the blend.

Viscosity of selected substances

The viscosity of air and water are by far the two most important materials for aviation aerodynamics and shipping fluid dynamics. Temperature plays the main role in determining viscosity.

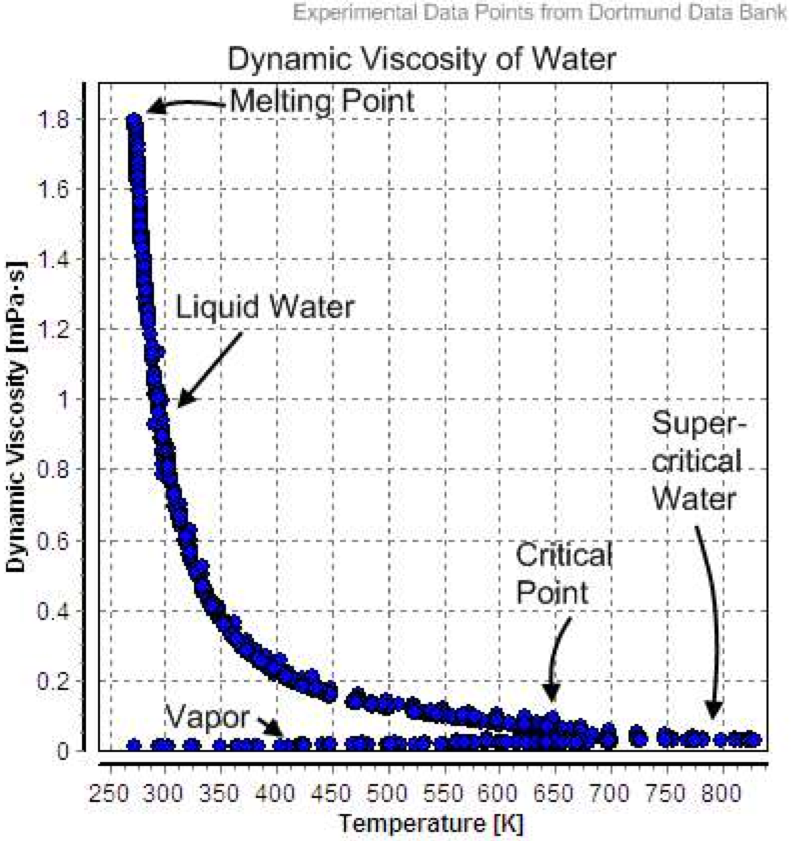
Viscosity of air



Pressure dependence of the dynamic viscosity of dry air at the temperatures of 300, 400 and 500 K

The viscosity of air depends mostly on the temperature. At 15.0 °C, the viscosity of air is 1.78×10^{-5} kg/(m·s), 17.8 μPa.s or 1.78×10^{-5} Pa.s.. One can get the viscosity of air as a function of temperature from the Gas Viscosity Calculator

Viscosity of water



Dynamic Viscosity of Water

The dynamic viscosity of water is 8.90×10^{-4} Pa·s or 8.90×10^{-3} dyn·s/cm^2 or 0.890 cP at about 25 °C.
Water has a viscosity of 0.0091 poise at 25 °C, or 1 centipoise at 20 °C.
As a function of temperature T (K): (Pa·s) = $A \times 10^{B/(T-C)}$
where A=2.414×10^{-5} Pa·s ; B = 247.8 K ; and C = 140 K.

Viscosity of liquid water at different temperatures up to the normal boiling point is listed below.

Temperature [°C]	Viscosity [mPa·s]
10	1.308
20	1.002
30	0.7978
40	0.6531
50	0.5471

60	0.4668
70	0.4044
80	0.3550
90	0.3150
100	0.2822

Viscosity of various materials

Example of the viscosity of milk and water. Liquids with higher viscosities will not make such a splash when poured at the same velocity.

Honey being drizzled.

Peanut butter is a semi-solid and can therefore hold peaks.

Some dynamic viscosities of Newtonian fluids are listed below:

Viscosity of selected gases at 100 kPa, [μPa·s]

Gas	at 0 °C (273 K)	at 27 °C (300 K)
air	17.4	18.6
hydrogen	8.4	9.0
helium		20.0
argon		22.9
xenon	21.2	23.2
carbon dioxide		15.0
methane		11.2
ethane		9.5

Viscosity of liquids at 25 °C		
Liquid ():	**Viscosity** [Pa·s]	**Viscosity** [cP=mPa·s]
acetone	3.06×10^{-4}	0.306
benzene	6.04×10^{-4}	0.604
blood (37 °C)	$(3–4)\times10^{-3}$	3–4
castor oil	0.985	985
corn syrup	1.3806	1380.6
ethanol	1.074×10^{-3}	1.074
ethylene glycol	1.61×10^{-2}	16.1
glycerol	1.2 (at 20 °C)	1200
HFO-380	2.022	2022
mercury	1.526×10^{-3}	1.526
methanol	5.44×10^{-4}	0.544
Motor oil SAE 10 (20 °C)	0.065	65
Motor oil SAE 40 (20 °C)	0.319	319
nitrobenzene	1.863×10^{-3}	1.863
liquid nitrogen @ 77K	1.58×10^{-4}	0.158
propanol	1.945×10^{-3}	1.945
olive oil	.081	81
pitch	2.3e8	2.3e11
quark–gluon plasma	5e11	5e14
sulfuric acid	2.42×10^{-2}	24.2
water	8.94×10^{-4}	0.894

Viscosity of fluids with variable compositions		
Fluid	**Viscosity** [Pa·s]	**Viscosity** [cP]
honey	2–10	2,000–10,000
molasses	5–10	5,000–10,000
molten glass	10–1,000	10,000–1,000,000
chocolate syrup	10–25	10,000–25,000
molten chocolate*	45–130	45,000–130,000
ketchup*	50–100	50,000–100,000
peanut butter*	c. 250	c. 250,000
shortening*	c. 250	250,000

* These materials are highly non-Newtonian.

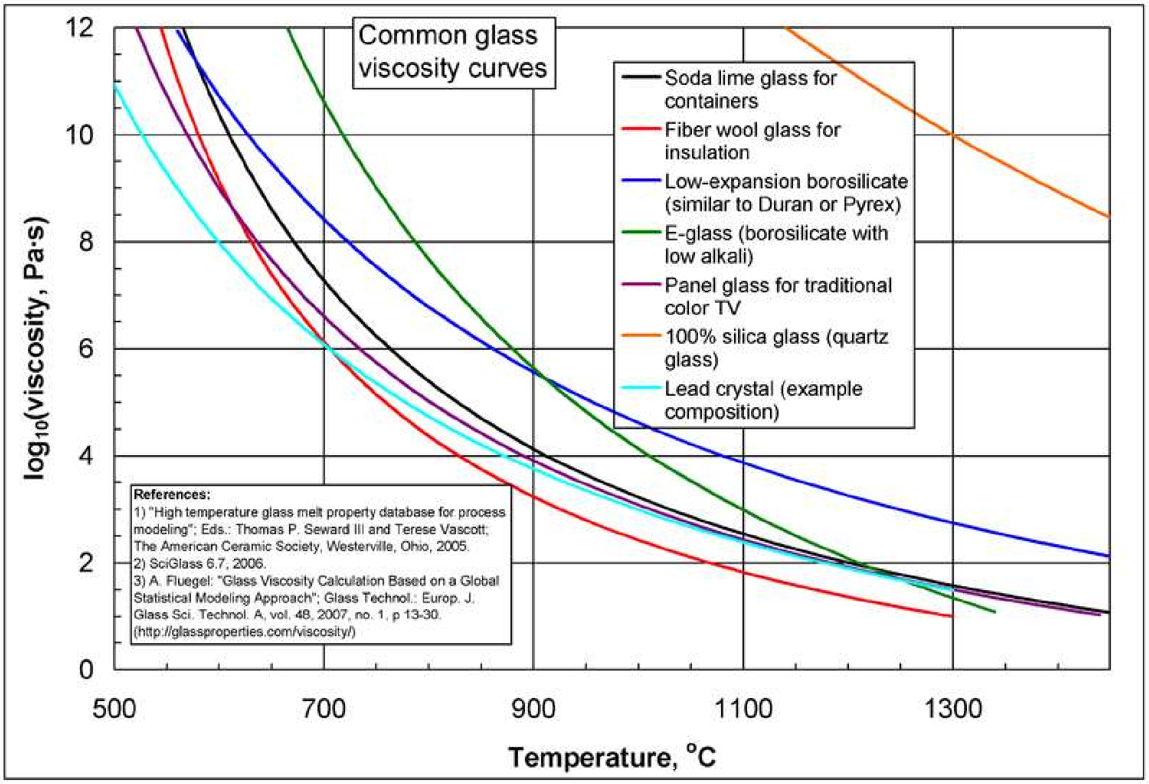
Viscosity of solids

On the basis that all solids such as granite flow to a small extent in response to small shear stress, some researchers have contended that substances known as amorphous solids, such as glass and many polymers, may be considered to have viscosity. This has led some to the view that solids are simply "liquids" with a very high viscosity, typically greater than 10^{12} Pa·s. This position is often adopted by supporters of the widely held misconception that glass flow can be observed in old buildings. This distortion is the result of the undeveloped glass making process of earlier eras, and not due to the viscosity of glass.

However, others argue that solids are, in general, elastic for small stresses while fluids are not. Even if solids flow at higher stresses, they are characterized by their low-stress behavior. This distinction is muddled if measurements are continued over long time periods, such as the Pitch drop experiment. Viscosity may be an appropriate characteristic for solids in a plastic regime. The situation becomes somewhat confused as the term *viscosity* is sometimes used for solid materials, for example Maxwell materials, to describe the relationship between stress and the rate of change of strain, rather than rate of shear.

These distinctions may be largely resolved by considering the constitutive equations of the material in question, which take into account both its viscous and elastic behaviors. Materials for which both their viscosity and their elasticity are important in a particular range of deformation and deformation rate are called *viscoelastic*. In geology, earth materials that exhibit viscous deformation at least three times greater than their elastic deformation are sometimes called rheids.

Viscosity of amorphous materials



References:
1) "High temperature glass melt property database for process modeling"; Eds.: Thomas P. Seward III and Terese Vascott; The American Ceramic Society, Westerville, Ohio, 2005.
2) SciGlass 6.7, 2006.
3) A. Fluegel: "Glass Viscosity Calculation Based on a Global Statistical Modeling Approach"; Glass Technol.: Europ. J. Glass Sci. Technol. A, vol. 48, 2007, no. 1, p 13-30. (http://glassproperties.com/viscosity/)

Common glass viscosity curves.

Viscous flow in amorphous materials (e.g. in glasses and melts) is a thermally activated process:

$$\mu = A \cdot e^{Q/RT},$$

where Q is activation energy, T is temperature, R is the molar gas constant and A is approximately a constant.

The viscous flow in amorphous materials is characterized by a deviation from the Arrhenius-type behavior: Q changes from a high value Q_H at low temperatures (in the glassy state) to a low value Q_L at high temperatures (in the liquid state). Depending on this change, amorphous materials are classified as either

- strong when: $Q_H - Q_L < Q_L$ or
- fragile when: $Q_H - Q_L \geq Q_L$.

The fragility of amorphous materials is numerically characterized by the Doremus' fragility ratio:

$$R_D = \frac{Q_H}{Q_L}$$

and strong material have $R_D < 2$ whereas fragile materials have $R_D \geq 2$.

The viscosity of amorphous materials is quite exactly described by a two-exponential equation:

$$\mu = A_1 \cdot T \cdot \left[1 + A_2 \cdot e^{B/RT}\right] \cdot \left[1 + C \cdot e^{D/RT}\right],$$

with constants A_1, A_2, B, C and D related to thermodynamic parameters of joining bonds of an amorphous material.

Not very far from the glass transition temperature, T_g, this equation can be approximated by a Vogel-Fulcher-Tammann (VFT) equation.

If the temperature is significantly lower than the glass transition temperature, $T < T_g$, then the two-exponential equation simplifies to an Arrhenius type equation:

$$\mu = A_L T \cdot e^{Q_H/RT}$$

with:

$$Q_H = H_d + H_m,$$

where H_d is the enthalpy of formation of broken bonds (termed configuron s) and H_m is the enthalpy of their motion. When the temperature is less than the glass transition temperature, $T < T_g$, the activation energy of viscosity is high because the amorphous materials are in the glassy state and most of their joining bonds are intact.

If the temperature is highly above the glass transition temperature, $T > T_g$, the two-exponential equation also simplifies to an Arrhenius type equation:

$$\mu = A_H T \cdot e^{Q_L/RT},$$

with:

$$Q_L = H_m.$$

When the temperature is higher than the glass transition temperature, $T > T_g$, the activation energy of viscosity is low because amorphous materials are melt and have most of their joining bonds broken which facilitates flow.

Eddy viscosity

In the study of turbulence in fluids, a common practical strategy for calculation is to ignore the small-scale *vortices* (or *eddies*) in the motion and to calculate a large-scale motion with an *eddy viscosity* that characterizes the transport and dissipation of energy in the smaller-scale flow. Values of eddy viscosity used in modeling ocean circulation may be from 5×10^4 to 10^6 Pa·s depending upon the resolution of the numerical grid.

The linear viscous stress tensor

Viscous forces in a fluid are a function of the rate at which the fluid velocity is changing over distance. The velocity at any point ***r*** is specified by the velocity field ***v***(***r***). The velocity at a small distance ***dr*** from point ***r*** may be written as a Taylor series:

$$\mathbf{v}(\mathbf{r}+d\mathbf{r}) = \mathbf{v}(\mathbf{r}) + \frac{d\mathbf{v}}{d\mathbf{r}}d\mathbf{r} + \ldots,$$

where $d\mathbf{v} / d\mathbf{r}$ is shorthand for the dyadic product of the del operator and the velocity:

$$\frac{d\mathbf{v}}{d\mathbf{r}} = \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix}.$$

This is just the Jacobian of the velocity field.

Viscous forces are the result of relative motion between elements of the fluid, and so are expressible as a function of the velocity field. In other words, the forces at ***r*** are a function of ***v***(***r***) and all derivatives of ***v***(***r***) at that point. In the case of linear viscosity, the viscous force will be a function of the Jacobian tensor alone. For almost all practical situations, the linear approximation is sufficient.

If we represent x, y, and z by indices 1, 2, and 3 respectively, the i,j component of the Jacobian may be written as $\partial_i v_j$ where ∂_i is shorthand for $\partial/\partial x_i$. Note that when the first and higher derivative terms are zero, the velocity of all fluid elements is parallel, and there are no viscous forces.

Any matrix may be written as the sum of an antisymmetric matrix and a symmetric matrix, and this decomposition is independent of coordinate system, and so has physical significance. The velocity field may be approximated as:

$$v_j(\mathbf{r}+d\mathbf{r}) = v_j(\mathbf{r}) + \frac{1}{2}\left(\partial_i v_j - \partial_j v_i\right) dr_i + \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) dr_i,$$

where Einstein notation is now being used in which repeated indices in a product are implicitly summed. The second term from the right is the asymmetric part of the first derivative term, and it represents a rigid rotation of the fluid about ***r*** with angular velocity *ω* where:

$$\omega = \frac{1}{2}\nabla \times \mathbf{v} = \frac{1}{2}\begin{bmatrix}\partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1\end{bmatrix}.$$

For such a rigid rotation, there is no change in the relative positions of the fluid elements, and so there is no viscous force associated with this term. The remaining symmetric term is responsible for the viscous forces in the fluid. Assuming the fluid is isotropic (i.e. its properties are the same in all directions), then the most general way that the symmetric term (the rate-of-strain tensor) can be broken down in a coordinate-independent (and therefore physically real) way is as the sum of a constant tensor (the rate-of-expansion tensor) and a traceless symmetric tensor (the rate-of-shear tensor):

$$\frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) = \underbrace{\frac{1}{3}\partial_k v_k \delta_{ij}}_{\text{rate-of-expansion tensor}} + \underbrace{\left(\frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right) - \frac{1}{3}\partial_k v_k \delta_{ij}\right)}_{\text{rate-of-shear tensor}},$$

where δ_{ij} is the unit tensor. The most general linear relationship between the stress tensor $\boldsymbol{\sigma}$ and the rate-of-strain tensor is then a linear combination of these two tensors:

$$\sigma_{\text{visc};ij} = \zeta \partial_k v_k \delta_{ij} + \mu\left(\partial_i v_j + \partial_j v_i - \frac{2}{3}\partial_k v_k \delta_{ij}\right),$$

where ς is the coefficient of bulk viscosity (or "second viscosity") and μ is the coefficient of (shear) viscosity.

The forces in the fluid are due to the velocities of the individual molecules. The velocity of a molecule may be thought of as the sum of the fluid velocity and the thermal velocity. The viscous stress tensor described above gives the force due to the fluid velocity only. The force on an area element in the fluid due to the thermal velocities of the molecules is just the hydrostatic pressure. This pressure term ($-p\ \delta_{ij}$) must be added to the viscous stress tensor to obtain the total stress tensor for the fluid.

$$\sigma_{ij} = -p\delta_{ij} + \sigma_{\text{visc};ij}.$$

The infinitesimal force dF_i on an infinitesimal area dA_i is then given by the usual relationship:

$$dF_i = \sigma_{ij}\, dA_j.$$

Chapter- 6

Compressible Flow

Compressible flow is the area of fluid mechanics that deals with fluids in which the fluid density varies significantly in response to a change in pressure. Compressibility effects are typically considered significant if the Mach number (the ratio of the flow velocity to the local speed of sound) of the flow exceeds 0.3, or if the fluid undergoes very large pressure changes. The most distinct differences between the compressible and incompressible flow models are that the compressible flow model allows for the existence of shock waves and choked flow.

Definition

Compressible flow describes the behaviour of fluids that experience significant variations in density. For flows in which the density does not vary significantly, the analysis of the behaviour of such flows may be simplified greatly by assuming a constant density. This is an idealization, which leads to the theory of incompressible flow. However, in the many cases dealing with gases (especially at higher velocities) and those cases dealing with liquids with large pressure changes, the significant variations in density can occur, and the flow should be analysed as a compressible flow if accurate results are to be obtained.

Allowing for a change in density brings an additional variable into the analysis. In contrast to incompressible flows, which can usually be solved by considering only the equations from conservation of mass and conservation of momentum. Usually, the principle of conservation of energy is included. However, this introduces another variable (temperature), and so a fourth equation (such as the ideal gas equation) is required to relate the temperature to the other thermodynamic properties in order to fully describe the flow.

When defining what is meant by a compressible flow, it is useful to compare the density to a reference value, such as the stagnation density, ρ_0, which is the density of the fluid if it were to be slowed down isentropically to stationary. As a general rule of thumb, if the change in density relative to the stagnation density is greater than 5%, then the fluid should be analysed as a compressible flow. For an ideal gas with a ratio of specific heats of 1.4, this occurs at a Mach number greater than approximately 0.3. Below this value, however, whether or not a specific case should be treated as compressible or incompressible depends largely on the level of accuracy that is required .

Compressible Flow Phenomena

Two of the most distinctive phenomena which occur in compressible are the possibility of choked flow and the presence of acoustic waves, which may also be referred to as either compression or expansion waves, depending on whether the lead to an increase or decrease in pressure.

Shock Waves

Shock waves are one of the most common examples of compressible flow phenomena. A shock is characterised by a discontinuous change in the thermodynamic properties. In one dimensional flows, shock waves can form when a series of compression waves coalesce, or when a membrane separating two regions of differing pressure is suddenly removed. This is the technique often used to produce shock waves in shock tubes.

In two and three dimensional supersonic flows, oblique shock waves occur as a result of a change in direction of the flow. A classic example of these shock waves are those shock waves that form off the nose of a supersonic aircraft.

Aerodynamics

Aerodynamics is a subfield of fluid dynamics and gas dynamics, and is primarily concerned with obtaining the forces that air exerts on an object. For Mach numbers greater than about 0.3, density changes are significant, and the flow should be considered compressible for an accurate representation of reality.

Subsonic Aerodynamics

Due to the complexities of compressible flow theory, it is often easier to calculate the incompressible flow characteristics first, and then employ a correction factor to obtain the actual flow properties. Several correction factors exist with varying degrees of complexity and accuracy.

Prandtl–Glauert transformation

The Prandtl-Glauert transformation is found by linearizing the potential equations associated with compressible, inviscid flow. The Prandtl–Glauert transformation or Prandtl–Glauert rule (also Prandtl–Glauert–Ackeret rule) is an approximation function which allows comparison of aerodynamical processes occurring at different Mach numbers. It was discovered that the linearized pressures in such a flow were equal to those found from incompressible flow theory multiplied by a correction factor. This correction factor is given below. :

$$c_p = \frac{c_{p0}}{\sqrt{1 - M^2}}.$$

where

- c_p is the compressible pressure coefficient
- c_{p0} is the incompressible pressure coefficient
- M is the Mach number.

This correction factor works well for all Mach numbers between 0.3 and 0.7. It should be noted that since this correction factor is derived from linearized equations, the pressures calculated is always less in magnitude than the actual pressures within the fluid.

Karman-Tsien correction factor

The Karman-Tsien transformation is a nonlinear correction factor to find the pressure coefficient of a compressible, inviscid flow. It is an empirically derived correction factor that tends to slightly overestimate the magnitude of the fluid's pressure. In order to employ this correction factor, the incompressible, inviscid fluid pressure must be known from previous investigation.

$$C_P = \frac{C_{P0}}{\sqrt{1 - M^2} + \frac{C_{P0}}{2}\left(1 - \sqrt{1 - M^2}\right)}$$

where

- c_p is the compressible pressure coefficient
- c_{p0} is the incompressible pressure coefficient
- M is the Mach number.

This correction factor is valid for M<0.8.

Supersonic Aerodynamics

As with subsonic aerodynamics, a compressibility correction factor can be derived by linearising the governing equations. The supersonic correction factor is similar to the Prandtl-Glauert transformation, but the terms under the square root sign are reversed.

$$c_p = \frac{c_{p0}}{\sqrt{M^2 - 1}}.$$

where

- c_p is the compressible pressure coefficient
- c_{p0} is the incompressible pressure coefficient

- M is the Mach number.

This is considered a valid approximation for Mach numbers greater than 1.3.

Transonic Aerodynamics

Transonic flow typically occurs in flows with Mach numbers between 0.8 and 1.2. Under these conditions, some of the flow is supersonic and some is subsonic. At these velocities, the correction factors derived using linearized theory breaks down due to a singularity that occurs at a Mach number of 1. In addition, severe instabilities caused by the formation of local shock waves and the existence of of both subsonic and supersonic flow (which behave completely differently) makes the solution of the governing equations rather difficult. However, the analysis of compressible flows in the transonic regime has led to some developments which help reduce the increases in drag caused by compressibility effects, including the use of swept wings and the Whitcomb area rule.

Internal Flows

If the flow of a fluid the fluid is confined by a surface, it is referred to as an internal flow. This includes the flow of fluids through pipes and ducts, and often arise in industrial and manufacturing processes, and is vital in the analysis of propulsion systems.

One example is in die casting or injection molding processes. This involves injecting a liquid material (such as a thermosetting plastic for injection molding or molten metal for die casting) at very high pressures into a cavity. The air that is already in the cavity is displaced very rapidly, and compressibility needs to be considered in the design of the die if problems with air entrapment are to be avoided.

Effect of area changes

Compressible flows play a big role in determining the behaviour of nozzles. Subsonic and supersonic flow react differently to changes in cross sectional area. While subsonic flow flowing through a converging duct (narrowing down from a wide diameter to a smaller diameter in the direction of the flow) will experience an increase in velocity, a supersonic flow through an identical duct will experience a decrease in velocity. In general, flow through a converging nozzle will always tend towards Mach 1. If the area convergence is great enough that the sound speed is reached, a phenomenon known as "choking" occurs. In this case, the flow is choked, and either the flow rate of the fluid entering the pipe is limited, or shock waves form in the nozzle such that the Mach number at the point of minimum area (called the throat) remains unity. Similarly, subsonic flow through a diverging nozzle will always be slowed, and supersonic flow will accelerate. The Mach number of the flow can be directly related to the area by the relation

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{\left(\frac{\gamma+1}{2}\right)}{1+\left(\frac{\gamma-1}{2}\right)M^2}\right]^{\frac{\gamma+1}{2-2\gamma}}$$

where *A* and *M* are the area and Mach number at a point in the nozzle, γ is the ratio of specific heats, and A^* is the area that would cause the flow velocity to reach a Mach number of 1 (i.e. the area at the throat, provided that the nozzle is choked).

Thus, for a subsonic flow to be accelerated to supersonic velocities, the nozzle needs to have a converging section in which the flow is subsonic, a throat, at which the flow velocity is the local speed of sound, and a diverging section with supersonic flow. Such an arrangement is called a de Laval nozzle, and is commonly used in propulsion systems such as rocket and supersonic jet engines.

Note that Mach 1 can be a very high speed for a hot gas, since the speed of sound varies as the square root of absolute temperature. Thus the speed reached at a nozzle throat can be far higher than the speed of sound under standard atmospheric conditions. This fact is used extensively in rocketry where hypersonic flows are required, and where propellant mixtures are deliberately chosen to further increase the sonic speed.

Effect of friction

Friction has a similar effect as an area change on compressible flow. In a pipe of constant cross sectional area in which the walls exert a frictional force on the flow, the flow velocity will tend toward the speed of sound. In other words, subsonic flow through a pipe with friction will accelerate, and supersonic flow will decelerate. If the pipe length is long enough that the flow velocity would pass through unity, then the flow chokes such that the flow exiting the pipe is at Mach 1. As with the nozzle, this is achieved either through the flow rate at the inlet being limited, or the formation of shock waves in the pipe (for supersonic flows). For the adiabatic flow of an ideal gas model, the effects of friction may be calculated using the Fanno flow model. For a constant friction factor, the model is given by

$$4\frac{fL^*}{D_h} = \left(\frac{1-M^2}{\gamma M^2}\right) + \left(\frac{\gamma+1}{2\gamma}\right)\ln\left[\frac{M^2}{\left(\frac{2}{\gamma+1}\right)\left(1+\frac{\gamma-1}{2}M^2\right)}\right]$$

where *f* is the Fanning friction factor, L^* is the required pipe length passed the point being considered that would result in the flow choking, and D_h is the hydraulic diameter of the pipe.

Effect of heat transfer

Adding heat to a fluid flowing at subsonic velocities in a pipe will cause the flow to accelerate, and adding heat to supersonic flow in a pipe will cause the flow to decelerate. As with the cases of friction and area change discussed above, adding more heat than that required to reach a Mach number of 1 will result in the flow choking.

For an ideal gas in a constant area pipe, the effect of heat addition to the pipe may be calculated using the Rayleigh flow model, which describes how the Mach number varies with changes in the stagnation temperature. The stagnation temperature at a point is the temperature that the fluid would reach if it were to be slowed isentropically to stationary. As heat is added to the system, the stagnation temperature increases. The Rayleigh flow model is given by

$$\frac{T_0}{T_0^*} = \frac{2\left(\gamma+1\right)M^2}{\left(1+\gamma M^2\right)^2}\left(1+\frac{\gamma-1}{2}M^2\right)$$

where T_0 and T_0^* represent the stagnation temperatures at the point under consideration, and at the point at which the Mach number is 1 respectively.

Shock Tubes

In addition to measurements of rates of chemical kinetics, shock tubes have been used to measure dissociation energies and molecular relaxation rates, investigate shock wave behaviour, and they have been used in aerodynamic tests. The fluid flow in the driven gas (the gas behind the shock wave) can be used much as a wind tunnel, allowing higher temperatures and pressures replicating the conditions in the turbine sections of jet engines. However, test times are limited to a few milliseconds, either by the arrival of the contact surface or the reflected shock wave.

They have been further developed into shock tunnels, with an added nozzle and dump tank. The resultant high temperature hypersonic flow can be used to simulate atmospheric re-entry of spacecraft or hypersonic craft, again with limited testing times.

Chapter- 7

Incompressible Flow

In fluid mechanics or more generally continuum mechanics, incompressible (isochoric) flow refers to flow in which the material density is constant within an infinitesimal volume that moves with the velocity of the fluid. An equivalent statement that implies incompressible flow is that the divergence of the fluid velocity is zero.

Incompressible flow does not imply that the fluid itself is compressible. It is shown in the derivation below that even compressible fluids can undergo incompressible flow. Incompressible fluids must have a constant density everywhere, while incompressible flow only requires that the density remain constant within a parcel of fluid which moves with the fluid velocity.

Derivation

The fundamental requirement for incompressible flow is that the density, ρ, is constant within an infinitesimal volume, *dV*, which moves at the velocity of the fluid, **v**. Mathematically, this constraint implies that the material derivative (discussed below) of the density must vanish to ensure incompressible flow. Before introducing this constraint, we must apply the conservation of mass to generate the necessary relations. The mass is calculated by a volume integral of the density, ρ:

$$m = \iiint\limits_V \rho dV \quad .$$

The conservation of mass requires that the time derivative of the mass inside a control volume be equal to the mass flux, **J**, across its boundaries. Mathematically, we can represent this constraint in terms of a surface integral:

$$\frac{\partial m}{\partial t} = -\oint\!\!\!\!\oint\limits_S (\mathbf{J} \cdot d\mathbf{S}) \quad .$$

The negative sign in the above expression ensures that outward flow results in a decrease in the mass with respect to time, using the convention that the surface area vector points outward. Now, using the divergence theorem we can derive the relationship between the flux and the partial time derivative of the density:

$$\iiint_V \frac{\partial \rho}{\partial t} dV = - \iiint_V (\nabla \cdot \mathbf{J})\, dV ,$$

therefore:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J} .$$

The partial derivative of the density with respect to time need not vanish to ensure incompressible *flow*. When we speak of the partial derivative of the density with respect to time, we are referring to this rate of change within a control volume of *fixed position*. By allowing the partial time derivative of the density to be non-zero, we are not restricting ourselves to incompressible fluids because the density is allowed to change as observed from a fixed position as fluid flows through the control volume. This approach maintains generality, and not requiring that the partial time derivative of the density vanishes illustrates that compressible fluids can still undergo incompressible flow. What we are interested in now is the change in density of a control volume which moves along with the fluid velocity, **v**. The flux is related to the fluid velocity through the following function:

$$\mathbf{J} = \rho \mathbf{v} .$$

So that the conservation of mass implies that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v} + \rho (\nabla \cdot \mathbf{v}) = 0 .$$

The previous relation (where we have used the appropriate product rule) is known as the continuity equation. Now, we need the following relation about the total derivative of the density (where we apply the chain rule):

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt} .$$

So if we choose a control volume that is moving at the same rate as the fluid (i.e. (dx/dt,dy/dt,dz/dt)=**v**), then this expression simplifies to the material derivative:

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{v}.$$

And so using the continuity equation derived above, we see that:

$$\frac{D\rho}{Dt} = -\rho \left(\nabla \cdot \mathbf{v} \right).$$

A change in the density over time would imply that the fluid had either compressed or expanded (or that the mass contained in our constant volume, dV, had changed), which we have prohibited. We must then require that the material derivative of the density vanishes, and equivalently (for non-zero density) so must the divergence of the fluid velocity:

$$\nabla \cdot \mathbf{v} = 0.$$

And so beginning with the conservation of mass and the constraint that the density within a moving volume of fluid remains constant, it has been shown that an equivalent condition required for incompressible flow is that the divergence of the fluid velocity vanishes.

Relation to compressibility factor

In some fields, a measure of the incompressibility of a flow is the change in density as a result of the pressure variations. This is best expressed in terms of the compressibility factor

$$Z = \frac{1}{\rho} \frac{d\rho}{dp}.$$

If the compressibility factor is acceptably small, the flow is considered to be incompressible.

Relation to solenoidal field

An incompressible flow is described by a velocity field which is solenoidal. But a solenoidal field, besides having a zero divergence, also has the additional connotation of having non-zero curl (i.e., rotational component).

Otherwise, if an incompressible flow also has a curl of zero, so that it is also irrotational, then the velocity field is actually Laplacian.

Difference between incompressible flow and material

As defined earlier, an incompressible (isochoric) flow is the one in which

$$\nabla \cdot \mathbf{u} = 0.$$

This is equivalent to saying that

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0$$

i.e. the material derivative of the density is zero. Thus if we follow a material element, its mass density will remain constant. Note that the material derivative consists of two terms. The first term $\frac{\partial \rho}{\partial t}$ describes how the density of the material element changes with time. This term is also known as the *unsteady term*. The second term, $\mathbf{u} \cdot \nabla \rho$ describes the changes in the density as the material element moves from one point to another. This is the *convection* or the *advection term*. For a flow to be incompressible the sum of these terms should be zero.

On the other hand, a **homogeneous, incompressible material** is defined as one which has constant density throughout. For such a material, ρ = *constant*. This implies that,

$\frac{\partial \rho}{\partial t} = 0$ and
$\nabla \rho = 0$ *independently*.

From the continuity equation it follows that

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = 0 \implies \nabla \cdot \mathbf{u} = 0$$

Thus homogeneous materials always undergo flow that is incompressible, but the converse is not true.

It is common to find references where the author mentions incompressible flow and assumes that density is constant. Even though this is technically incorrect, it is an accepted practice. One of the advantages of using the incompressible material assumption over the incompressible flow assumption is in the momentum equation where the kinematic viscosity ($\nu = \frac{\mu}{\rho}$) can be assumed to be constant. The subtlety above is frequently a source of confusion. Therefore many people prefer to refer explicitly to *incompressible materials* or *isochoric flow* when being descriptive about the mechanics.

Related flow constraints

In fluid dynamics, a flow is considered to be incompressible if the divergence of the velocity is zero. However, related formulations can sometimes be used, depending on the flow system to be modelled. Some versions are described below:

1. *Incompressible flow*: $\nabla \cdot \mathbf{u} = 0$. This can assume either constant density (strict incompressible) or varying density flow. The varying density set accepts solutions involving small perturbations in density, pressure and/or temperature fields, and can allow for pressure stratification in the domain.
2. *Anelastic flow*: $\nabla \cdot (\rho_o \mathbf{u}) = 0$. Principally used in the field of atmospheric sciences, the anelastic constraint extend incompressible flow validity to stratified density and/or temperature as well as pressure. This allow the thermodynamic variables to relax to an 'atmospheric' base state seen in the lower atmosphere when used in the field of meteorology, for example. This condition can also be used for various astrophysical systems.
3. *Low Mach-number flow / Pseudo-incompressibility*: $\nabla \cdot (\alpha \mathbf{u}) = \beta$. The low Mach-number constraint can be derived from the compressible Euler equations using scale analysis of non-dimensional quantities. The restraint, like the previous in this section, allows for the removal of acoustic waves, but also allows for *large* perturbations in density and/or temperature. The assumption is that the flow remains within a Mach number limit (normally less than 0.3) for any solution using such a constraint to be valid. Again, in accordance with all incompressible flows the pressure deviation must be small in comparison to the pressure base state.

These methods make differing assumptions about the flow, but all take into account the general form of the constraint $\nabla \cdot (\alpha \mathbf{u}) = \beta$ for general flow dependent functions α and β.

Numerical approximations of incompressible flow

The stringent nature of the incompressible flow equations means that specific mathematical techniques have been devised to solve them. Some of these methods include:

1. The projection method (both approximate and exact)
2. Artificial compressibility technique (approximate)
3. Compressibility pre-conditioning

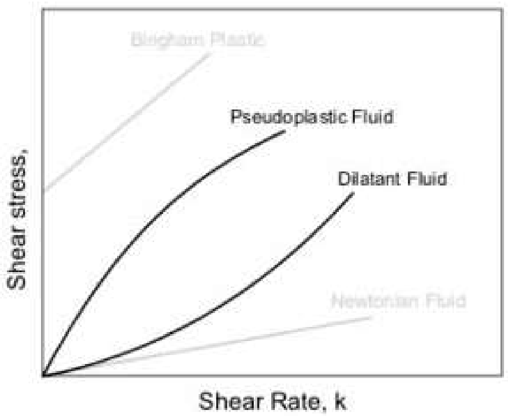
Chapter- 8

Viscoelasticity

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Viscous materials, like honey, resist shear flow and strain linearly with time when a stress is applied. Elastic materials strain instantaneously when stretched and just as quickly return to their original state once the stress is removed. Viscoelastic materials have elements of both of these properties and, as such, exhibit time dependent strain. Whereas elasticity is usually the result of bond stretching along crystallographic planes in an ordered solid, viscosity is the result of the diffusion of atoms or molecules inside an amorphous material.

Background

In the nineteenth century, physicists such as Maxwell, Boltzmann, and Kelvin researched and experimented with creep and recovery of glasses, metals, and rubbers. Viscoelasticity was further examined in the late twentieth century when synthetic polymers were engineered and used in a variety of applications. Viscoelasticity calculations depend heavily on the viscosity variable, η. The inverse of η is also known as fluidity, φ. The value of either can be derived as a function of temperature or as a given value (ie for a dashpot).

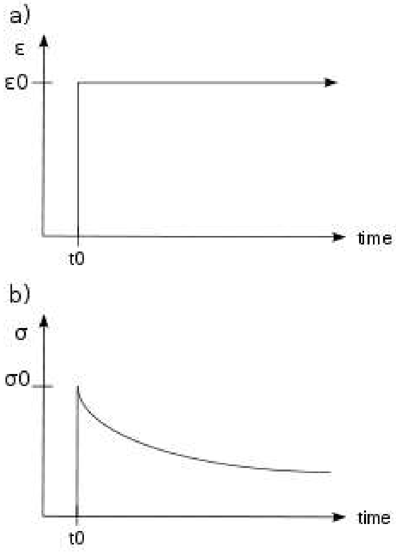


Different types of responses (σ) to a change in strain rate (dε/dt)

Depending on the change of strain rate versus stress inside a material the viscosity can be categorized as having a linear, non-linear, or plastic response. When a material exhibits a linear response it is categorized as a Newtonian material. In this case the stress is linearly proportional to the strain rate. If the material exhibits a non-linear response to the strain rate, it is categorized as Non-Newtonian fluid. There is also an interesting case where the viscosity decreases as the shear/strain rate remains constant. A material which exhibits this type of behavior is known as thixotropic. In addition, when the stress is independent of this strain rate, the material exhibits plastic deformation. Many viscoelastic materials exhibit rubber like behavior explained by the thermodynamic theory of polymer elasticity. In reality all materials deviate from Hooke's law in various ways, for example by exhibiting viscous-like as well as elastic characteristics. Viscoelastic materials are those for which the relationship between stress and strain depends on time. Anelastic solids represent a subset of viscoelastic materials: they have a unique equilibrium configuration and ultimately recover fully after removal of a transient load.

Some phenomena in viscoelastic materials are:

- if the stress is held constant, the strain increases with time (creep);
- if the strain is held constant, the stress decreases with time (relaxation);
- the effective stiffness depends on the rate of application of the load;
- if cyclic loading is applied, hysteresis (a phase lag) occurs, leading to a dissipation of mechanical energy;
- acoustic waves experience attenuation;
- rebound of an object following an impact is less than 100%;
- during rolling, frictional resistance occurs.



a) Applied strain and b) induced stress as functions of time for a viscoelastic material.

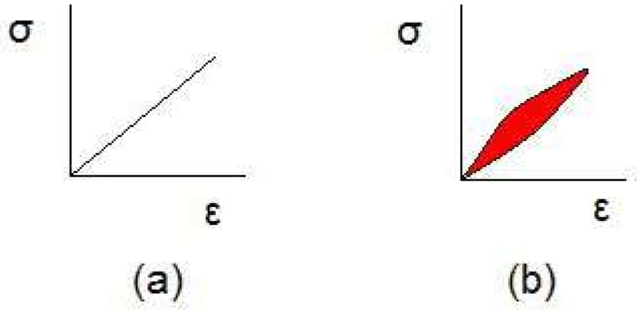
All materials exhibit some viscoelastic response. In common metals such as steel or aluminum, as well as in quartz, at room temperature and at small strain, the behavior does not deviate much from linear elasticity. Synthetic polymers, wood, and human tissue as well as metals at high temperature display significant viscoelastic effects. In some applications, even a small viscoelastic response can be significant. To be complete, an analysis or design involving such materials must incorporate their viscoelastic behavior. Knowledge of the viscoelastic response of a material is based on measurement

Some examples of viscoelastic materials include amorphous polymers, semicrystalline polymers, biopolymers, metals at very high temperatures, and bitumen materials. Cracking occurs when the strain is applied quickly and outside of the elastic limit.

A **viscoelastic** material has the following properties:

- hysteresis is seen in the stress-strain curve.
- stress relaxation occurs: step constant strain causes decreasing stress
- creep occurs: step constant stress causes increasing strain

Elastic behavior versus viscoelastic behavior



Stress-Strain Curves for a purely elastic material (a) and a viscoelastic material (b). The red area is a hysteresis loop and shows the amount of energy lost (as heat) in a loading and unloading cycle. It is equal to $\oint \sigma \, d\varepsilon$, where σ is stress and ε is strain.

Unlike purely elastic substances, a viscoelastic substance has an elastic component and a viscous component. The viscosity of a viscoelastic substance gives the substance a strain rate dependent on time. Purely elastic materials do not dissipate energy (heat) when a load is applied, then removed. However, a viscoelastic substance loses energy when a load is applied, then removed. Hysteresis is observed in the stress-strain curve, with the area of the loop being equal to the energy lost during the loading cycle. Since viscosity is the resistance to thermally activated plastic deformation, a viscous material will lose energy through a loading cycle. Plastic deformation results in lost energy, which is uncharacteristic of a purely elastic material's reaction to a loading cycle.

Specifically, viscoelasticity is a molecular rearrangement. When a stress is applied to a viscoelastic material such as a polymer, parts of the long polymer chain change position. This movement or rearrangement is called Creep. Polymers remain a solid material even when these parts of their chains are rearranging in order to accompany the stress, and as this occurs, it creates a back stress in the material. When the back stress is the same magnitude as the applied stress, the material no longer creeps. When the original stress is taken away, the accumulated back stresses will cause the polymer to return to its original form. The material creeps, which gives the prefix visco-, and the material fully recovers, which gives the suffix -elasticity.

Types of viscoelasticity

Linear viscoelasticity is when the function is separable in both creep response and load. All linear viscoelastic models can be represented by a Volterra equation connecting stress and strain:

$$\epsilon(t) = \frac{\sigma(t)}{E_{\text{inst,creep}}} + \int_0^t K(t-t')\dot{\sigma}(t')\,dt'$$

or

$$\sigma(t) = E_{\mathbf{inst,relax}}\epsilon(t) + \int_0^t F(t-t')\dot{\epsilon}(t')dt'$$

where

- t is time
- $\sigma(t)$ is stress
- $\varepsilon(t)$ is strain
- $E_{inst,creep}$ and $E_{inst,relax}$ are instantaneous elastic moduli for creep and relaxation
- $K(t)$ is the creep function
- $F(t)$ is the relaxation function

Linear viscoelasticity is usually applicable only for small deformations.

Nonlinear viscoelasticity is when the function is not separable. It is usually happens when the deformations are large or if the material changes its properties under deformations.

An **anelastic** material is a special case of a viscoelastic material: an anelastic material will fully recover to its original state on the removal of load.

Dynamic modulus

Viscoelasticity is studied using dynamic mechanical analysis, applying a small oscillatory strain and measuring the resulting stress.

- Purely elastic materials have stress and strain in phase, so that the response of one caused by the other is immediate.
- In purely viscous materials, strain lags stress by a 90 degree phase lag.
- Viscoelastic materials exhibit behavior somewhere in the middle of these two types of material, exhibiting some lag in strain.

Complex Dynamic modulus G can be used to represent the relations between the oscillating stress and strain:

$$G = G' + iG''$$

where $i^2 = -1$; G' is the *storage modulus* and G'' is the *loss modulus*:

$$G' = \frac{\sigma_0}{\varepsilon_0}\cos\delta$$

$$G'' = \frac{\sigma_0}{\varepsilon_0}\sin\delta$$

where σ_0 and ε_0 are the amplitudes of stress and strain and δ is the phase shift between them.

Constitutive models of linear viscoelasticity

Viscoelastic materials, such as amorphous polymers, semicrystalline polymers, and biopolymers, can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies. These models, which include the Maxwell model, the Kelvin-Voigt model, and the Standard Linear Solid Model, are used to predict a material's response under different loading conditions. Viscoelastic behavior has elastic and viscous components modeled as linear combinations of springs and dashpots, respectively. Each model differs in the arrangement of these elements, and all of these viscoelastic models can be equivalently modeled as electrical circuits. In an equivalent electrical circuit, stress is represented by voltage, and the derivative of strain (velocity) by current. The elastic modulus of a spring is analogous to a circuit's *capacitance* (it stores energy) and the viscosity of a dashpot to a circuit's *resistance* (it dissipates energy).

The elastic components, as previously mentioned, can be modeled as springs of elastic constant E, given the formula:

$$\sigma = E\varepsilon$$

where σ is the stress, E is the elastic modulus of the material, and ε is the strain that occurs under the given stress, similar to Hooke's Law.

The viscous components can be modeled as dashpots such that the stress-strain rate relationship can be given as,

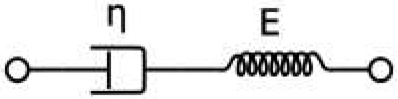
$$\sigma = \eta \frac{d\varepsilon}{dt}$$

where σ is the stress, η is the viscosity of the material, and $d\varepsilon/dt$ is the time derivative of strain.

The relationship between stress and strain can be simplified for specific stress rates. For high stress states/short time periods, the time derivative components of the stress-strain relationship dominate. A dashpots resists changes in length, and in a high stress state it can be approximated as a rigid rod. Since a rigid rod cannot be stretched past its original length, no strain is added to the system

Conversely, for low stress states/longer time periods, the time derivative components are negligible and the dashpot can be effectively removed from the system - an "open" circuit. As a result, only the spring connected in parallel to the dashpot will contribute to the total strain in the system

Maxwell model



Maxwell model

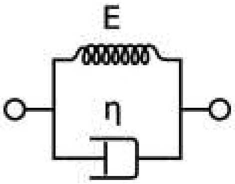
The Maxwell model can be represented by a purely viscous damper and a purely elastic spring connected in series, as shown in the diagram. The model can be represented by the following equation:

$$\frac{d\epsilon_{Total}}{dt} = \frac{d\epsilon_D}{dt} + \frac{d\epsilon_S}{dt} = \frac{\sigma}{\eta} + \frac{1}{E}\frac{d\sigma}{dt}.$$

Under this model, if the material is put under a constant strain, the stresses gradually relax, When a material is put under a constant stress, the strain has two components. First, an elastic component occurs instantaneously, corresponding to the spring, and relaxes immediately upon release of the stress. The second is a viscous component that grows with time as long as the stress is applied. The Maxwell model predicts that stress decays exponentially with time, which is accurate for most polymers. One limitation of this model is that it does not predict creep accurately. The Maxwell model for creep or constant-stress conditions postulates that strain will increase linearly with time. However, polymers for the most part show the strain rate to be decreasing with time.

Application to soft solids:thermoplastic polymers in the vicinity of their melting temperature, fresh concrete (neglecting its ageing), numerous metals at a temperature close to their melting point.

Kelvin–Voigt model



Schematic representation of Kelvin–Voigt model.

The Kelvin–Voigt model, also known as the Voigt model, consists of a Newtonian damper and Hookean elastic spring connected in parallel, as shown in the picture. It is used to explain the creep behaviour of polymers.

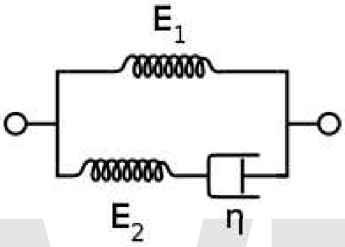
The constitutive relation is expressed as a linear first-order differential equation:

$$\sigma(t) = E\varepsilon(t) + \eta\frac{d\varepsilon(t)}{dt}$$

This model represents a solid undergoing reversible, viscoelastic strain. Upon application of a constant stress, the material deforms at a decreasing rate, asymptotically approaching the steady-state strain. When the stress is released, the material gradually relaxes to its undeformed state. At constant stress (creep), the Model is quite realistic as it predicts strain to tend to σ/E as time continues to infinity. Similar to the Maxwell model, the Kelvin–Voigt model also has limitations. The model is extremely good with modelling creep in materials, but with regards to relaxation the model is much less accurate.

Applications: organic polymers, rubber, wood when the load is not too high.

Standard linear solid model



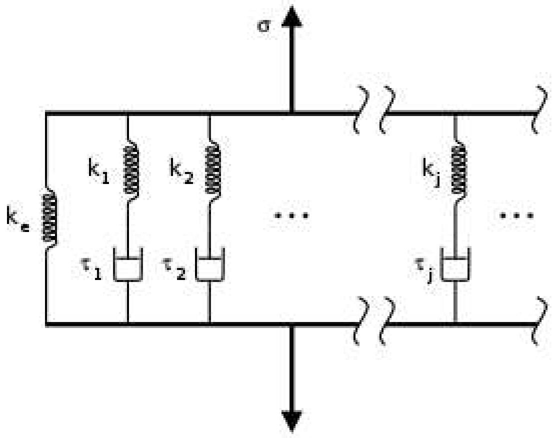
Schematic representation of the Standard Linear Solid model.

The Standard Linear Solid Model effectively combines the Maxwell Model and a Hookean spring in parallel. A viscous material is modeled as a spring and a dashpot in series with each other, both of which are in parallel with a lone spring. For this model, the governing constitutive relation is:

$$\frac{d\varepsilon}{dt} = \frac{\frac{E_2}{\eta}\left(\frac{\eta}{E_2}\frac{d\sigma}{dt} + \sigma - E_1\varepsilon\right)}{E_1 + E_2}$$

Under a constant stress, the modeled material will instantaneously deform to some strain, which is the elastic portion of the strain, and after that it will continue to deform and asymptotically approach a steady-state strain. This last portion is the viscous part of the strain. Although the Standard Linear Solid Model is more accurate than the Maxwell and Kelvin-Voigt models in predicting material responses, mathematically it returns inaccurate results for strain under specific loading conditions and is rather difficult to calculate.

Generalized Maxwell Model



Schematic of Maxwell-Wiechert Model

The Generalized Maxwell also known as the Maxwell-Weichert model (after James Clerk Maxwell and Dieter Weichert) is the most general form of the models described above. It takes into account that relaxation does not occur at a single time, but at a distribution of times. Due to molecular segments of different lengths with shorter ones contributing less than longer ones, there is a varying time distribution. The Weichert model shows this by having as many spring-dashpot Maxwell elements as are necessary to accurately represent the distribution. The Figure on the right represents a possible Wiechert model. Applications: metals and alloys at temperatures lower than one quarter of their absolute melting temperature (expressed in K).

Prony series

In a one-dimensional relaxation test, the material is subjected to a sudden strain that is kept constant over the duration of the test, and the stress is measured over time. The initial stress is due to the elastic response of the material. Then, the stress relaxes over time due to the viscous effects in the material. Typically, either a tensile, compressive, bulk compression, or shear strain is applied. The resulting stress vs. time data can be fitted with a number of equations, called models. Only the notation changes depending of the type of strain applied: tensile-compressive relaxation is denoted *E*, shear is denoted *G*, bulk is denoted *K*. The Prony series for the shear relaxation is

$$G(t) = G_\infty + \Sigma_{i=1}^{N} G_i \exp(-t/\tau_i)$$

where $G_\infty$is the long term modulus once the material is totally relaxed, τ_i are the relaxation times; the higher their values, the longer it takes for the stress to relax. The data is fitted with the equation by using a minimization algorithm that adjust the parameters (G_∞, G_i, τ_i) to minimize the error between the predicted and data values.

An alternative form is obtained noting that the elastic modulus is related to the long term modulus by

$$G(t=0) = G_0 = G_\infty + \Sigma_{i=1}^{N} G_i$$

Therefore,

$$G(t) = G_0 - \Sigma_{i=1}^{N} G_i [1 - \exp(-t/\tau_i)]$$

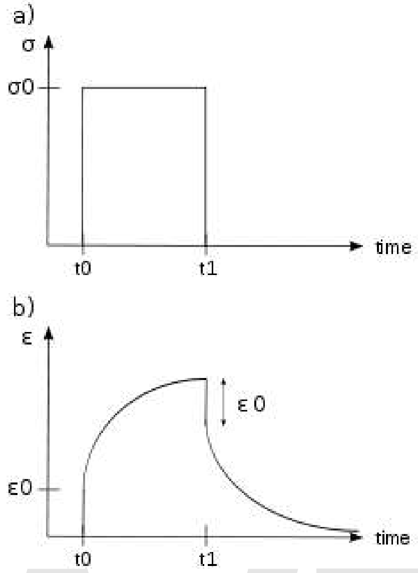
This form is convenient when the elastic shear modulus G_0 is obtained from data independent from the relaxation data, and/or for computer implementation, when it is desired to specify the elastic properties separately from the viscous properties, as in.

A creep experiment is usually easier to perform than a relaxation one, so most data is available as (creep) compliance vs. time. Unfortunately, there is no known closed form for the (creep) compliance in terms of the coefficient of the Prony series. So, if one has creep data, it is not easy to get the coefficients of the (relaxation) Prony series, which are needed for example in. An expedient way to obtain these coefficients is the following. First, fit the creep data with a model that has closed form solutions in both compliance and relaxation; for example the Maxwell-Kelvin model (eq. 7.18-7.19) in or the Standard Solid Model (eq. 7.20-7.21) in (section 7.1.3). Once the parameters of the creep model are known, produce relaxation pseudo-data with the conjugate relaxation model for the same times of the original data. Finally, fit the pseudo data with the Prony series.

Effect of temperature on viscoelastic behavior

The secondary bonds of a polymer constantly break and reform due to thermal motion. Application of a stress favors some conformations over others, so the molecules of the polymer will gradually "flow" into the favored conformations over time. Because thermal motion is one factor contributing to the deformation of polymers, viscoelastic properties change with increasing or decreasing temperature. In most cases, the creep modulus, defined as the ratio of applied stress to the time-dependent strain, decreases with increasing temperature. Generally speaking, an increase in temperature correlates to a logarithmic decrease in the time required to impart equal strain under a constant stress. In other words, it takes less work to stretch a viscoelastic material an equal distance at a higher temperature than it does at a lower temperature.

Viscoelastic creep



a) Applied stress and b) induced strain (b) as functions of time over a short period for a viscoelastic material.

When subjected to a step constant stress, viscoelastic materials experience a time-dependent increase in strain. This phenomenon is known as viscoelastic creep.

At a time t_0, a viscoelastic material is loaded with a constant stress that is maintained for a sufficiently long time period. The material responds to the stress with a strain that increases until the material ultimately fails. When the stress is maintained for a shorter time period, the material undergoes an initial strain until a time t_1, after which the strain immediately decreases (discontinuity) then gradually decreases at times $t > t_1$ to a residual strain.

Viscoelastic creep data can be presented by plotting the creep modulus (constant applied stress divided by total strain at a particular time) as a function of time. Below its critical stress, the viscoelastic creep modulus is independent of stress applied. A family of curves describing strain versus time response to various applied stress may be represented by a single viscoelastic creep modulus versus time curve if the applied stresses are below the material's critical stress value.

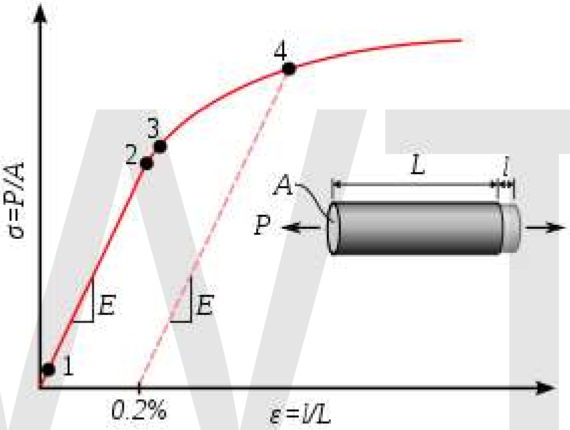
Viscoelastic creep is important when considering long-term structural design. Given loading and temperature conditions, designers can choose materials that best suit component lifetimes.

Measuring viscoelasticity

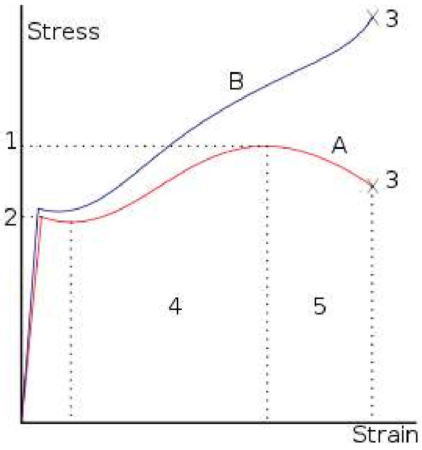
Though there are many instruments that test the mechanical and viscoelastic response of materials, broadband viscoelastic spectroscopy (BVS) and resonant ultrasound spectroscopy (RUS) are more commonly used to test viscoelastic behavior because they can be used above and below ambient temperatures and are more specific to testing viscoelasticity. These two instruments employ a damping mechanism at various frequencies and time ranges with no appeal to time-temperature superposition. Using BVS and RUS to study the mechanical properties of materials is important to understanding how a material exhibiting viscoelasticity will perform.

Chapter- 9

Plasticity



Stress-strain curve showing typical yield behavior for nonferrous alloys. Stress (σ) is shown as a function of strain (ε)
1: True elastic limit
2: Proportionality limit
3: Elastic limit
4: Offset yield strength



A stress–strain curve typical of structural steel
1. Ultimate Strength
2. Yield Strength
3. Rupture
4. Strain hardening region
5. Necking region.
A: Apparent stress (F/A_0)
B: Actual stress (F/A)

In physics and materials science, **plasticity** describes the deformation of a material undergoing non-reversible changes of shape in response to applied forces. For example, a solid piece of metal being bent or pounded into a new shape displays plasticity as permanent changes occur within the material itself. In engineering, the transition from elastic behavior to plastic behavior is called yield.

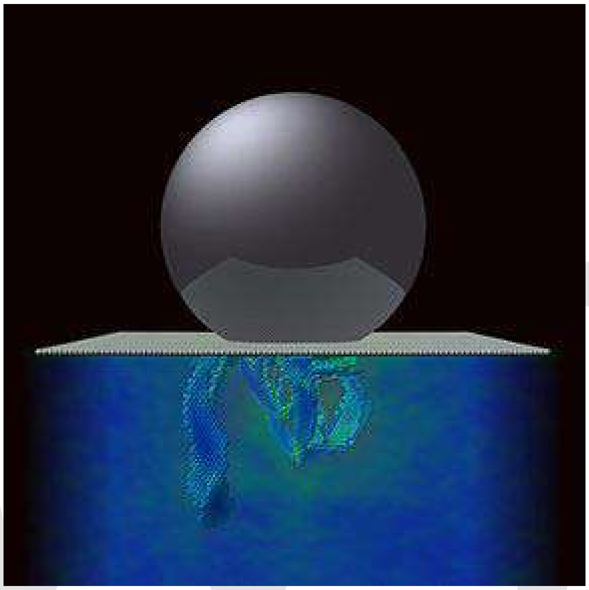
Plastic deformation is observed in most materials including metals, soils, rocks, concrete, foams, bone and skin. However, the physical mechanisms that cause plastic deformation can vary widely. At the crystal scale, plasticity in metals is usually a consequence of dislocations. In most crystalline materials such defects are relatively rare. But there are also materials where defects are numerous and are part of the very crystal structure, in such cases plastic crystallinity can result. In brittle materials such as rock, concrete, and bone, plasticity is caused predominantly by slip at microcracks.

For many ductile metals, tensile loading applied to a sample will cause it to behave in an elastic manner. Each increment of load is accompanied by a proportional increment in extension, and when the load is removed, the piece returns exactly to its original size. However, once the load exceeds some threshold (the yield strength), the extension increases more rapidly than in the elastic region, and when the load is removed, some amount of the extension remains.

However, elastic deformation is an approximation and its quality depends on the considered time frame and loading speed. If the deformation behavior includes elastic deformation as indicated in the adjacent graph it is also often referred to as elastic-plastic or elasto-plastic deformation.

Perfect plasticity is a property of materials to undergo irreversible deformation without any increase in stresses or loads. Plastic materials with hardening necessitate increasingly higher stresses to result in further plastic deformation. Generally plastic deformation is also dependent on the deformation speed, i.e. usually higher stresses have to be applied to increase the rate of deformation and such materials are said to deform visco-plastically.

Physical mechanisms



Plasticity under a spherical Nanoindenter in (111) Copper. All particles in ideal lattice positions are omitted and the color code refers to the von Mises stress field.

Plasticity in metals

Plasticity in a crystal of pure metal is primarily caused by two modes of deformation in the crystal lattice, slip and twinning. Slip is a shear deformation which moves the atoms through many interatomic distances relative to their initial positions. Twinning is the plastic deformation which takes place along two planes due to set of forces applied on a given metal piece.

Slip systems

Crystalline materials contain uniform planes of atoms organized with long-range order. Planes may slip past each other along their close-packed directions, as is shown on the slip systems page. The result is a permanent change of shape within the crystal and plastic deformation. The presence of dislocations increases the likelihood of planes slipping.

Reversible plasticity

On the nano scale the primary plastic deformation in simple fcc metals is reversible, as long as there is no material transport in form of cross-glide.

Shear banding

The presence of other defects within a crystal may entangle dislocations or otherwise prevent them from gliding. When this happens, plasticity is localized to particular regions in the material. For crystals, these regions of localized plasticity are called shear bands.

Plasticity in amorphous materials

Crazing

In amorphous materials, the discussion of “dislocations” is inapplicable, since the entire material lacks long range order. These materials can still undergo plastic deformation. Since amorphous materials, like polymers, are not well-ordered, they contain a large amount of free volume, or wasted space. Pulling these materials in tension opens up these regions and can give materials a hazy appearance. This haziness is the result of *crazing*, where fibrils are formed within the material in regions of high hydrostatic stress. The material may go from an ordered appearance to a "crazy" pattern of strain and stretch marks.

Plasticity in martensitic materials

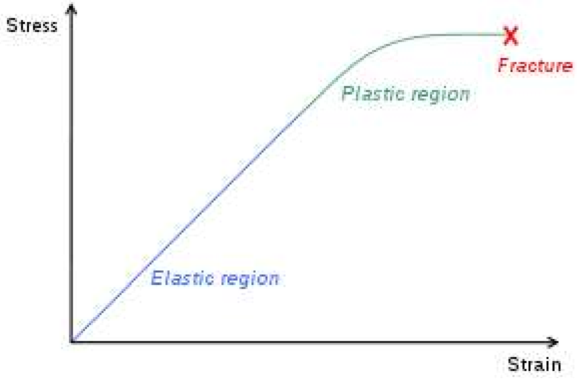
Some materials, especially those prone to Martensitic transformations, deform in ways that are not well described by the classic theories of plasticity and elasticity. One of the best-known examples of this is nitinol, which exhibits pseudoelasticity: deformations which are reversible in the context of mechanical design, but irreversible in terms of thermodynamics.

Plasticity in cellular materials

These materials plastically deform when the bending moment exceeds the fully plastic moment. This applies to open cell foams where the bending moment is exerted on the cell walls. The foams can be made of any material with a plastic yield point which includes rigid polymers and metals. This method of modeling the foam as beams is only valid if the ratio of the density of the foam to the density of the matter is less than 0.3. This is because beams yield axially instead of bending. In closed cell foams, the yield strength is increased if the material is under tension because of the membrane that spans the face of the cells.

Mathematical descriptions of plasticity

Deformation theory



An idealized uniaxial stress-strain curve showing elastic and plastic deformation regimes for the deformation theory of plasticity.

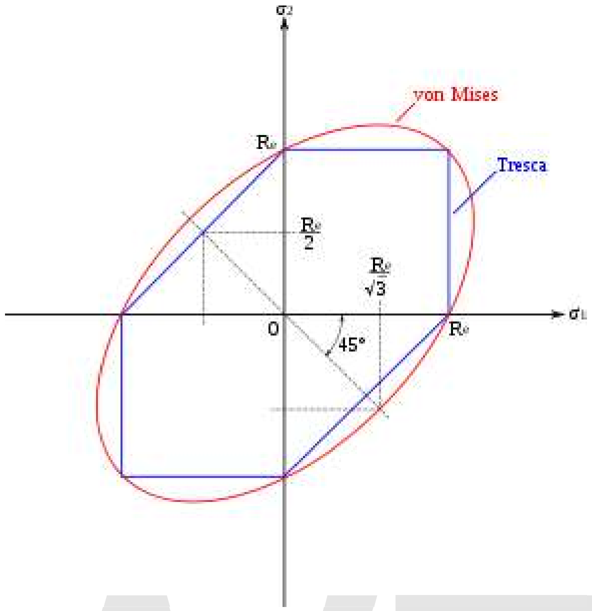
There are several mathematical descriptions of plasticity. One is deformation theory where the stress tensor (of order d in d dimensions) is a function of the strain tensor. Although this description is accurate when a small part of matter is subjected to increasing loading (such as strain loading), this theory cannot account for irreversibility.

Ductile materials can sustain large plastic deformations without fracture. However, even ductile metals will fracture when the strain becomes large enough - this is as a result of work-hardening of the material, which causes it to become brittle. Heat treatment such as annealing can restore the ductility of a worked piece, so that shaping can continue.

Flow plasticity theory

In 1934, Egon Orowan, Michael Polanyi and Geoffrey Ingram Taylor, roughly simultaneously, realized that the plastic deformation of ductile materials could be explained in terms of the theory of dislocations. The more correct mathematical theory of plasticity, flow plasticity theory, uses a set of non-linear, non-integrable equations to describe the set of changes on strain and stress with respect to a previous state and a small increase of deformation.

Yield criteria



Comparison of Tresca criterion to Von Mises criterion.

If the stress exceeds a critical value, as was mentioned above, the material will undergo plastic, or irreversible, deformation. This critical stress can be tensile or compressive. The Tresca and the von Mises criteria are commonly used to determine whether a material has yielded. However, these criteria have proved inadequate for a large range of materials and several other yield criteria are in widespread use.

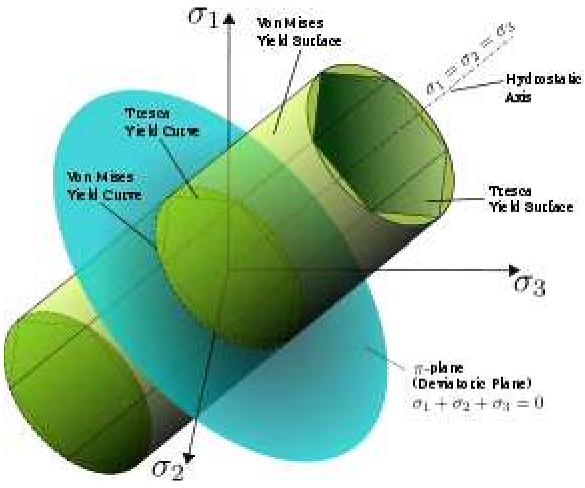
Tresca criterion

This criterion is based on the notion that when a material fails, it does so in shear, which is a relatively good assumption when considering metals. Given the principal stress state, we can use Mohr's circle to solve for the maximum shear stresses our material will experience and conclude that the material will fail if:

$$\sigma_1 - \sigma_3 \geq \sigma_0$$

Where σ_1 is the maximum normal stress, σ_3 is the minimum normal stress, and σ_0 is the stress under which the material fails in uniaxial loading. A yield surface may be constructed, which provides a visual representation of this concept. Inside of the yield surface, deformation is elastic. On the surface, deformation is plastic. It is impossible for a material to have stress states outside its yield surface.

Huber-von Mises criterion



The von Mises yield surfaces in principal stress coordinates circumscribes a cylinder around the hydrostatic axis. Also shown is Tresca's hexagonal yield surface.

This criterion is based on the Tresca criterion but takes into account the assumption that hydrostatic stresses do not contribute to material failure. M.T. Huber was first (1904, Lwów) who proposed the criterion of shear energy. Von Mises solves for an effective stress under uniaxial loading, subtracting out hydrostatic stresses, and claims that all effective stresses greater than that which causes material failure in uniaxial loading will result in plastic deformation.

$$\sigma_{\text{effective}}^2 = \frac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{11}-\sigma_{33})^2\right] + 3\left(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2\right)$$

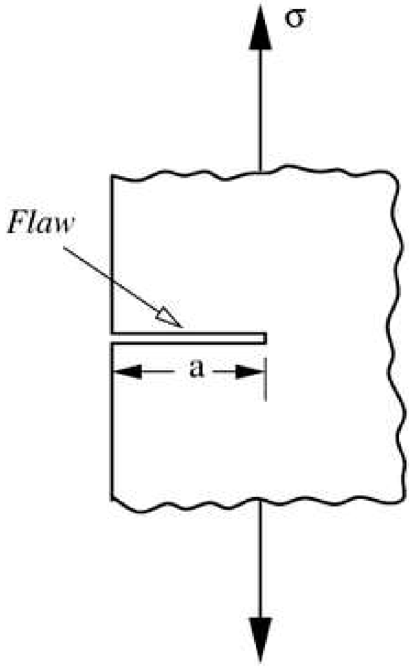
Again, a visual representation of the yield surface may be constructed using the above equation, which takes the shape of an ellipse. Inside the surface, materials undergo elastic deformation. Reaching the surface means the material undergoes plastic deformations. It is physically impossible for a material to go beyond its yield surface.

Chapter- 10

Fracture Mechanics

Fracture mechanics is the field of mechanics concerned with the study of the propagation of cracks in materials. It uses methods of analytical solid mechanics to calculate the driving force on a crack and those of experimental solid mechanics to characterize the material's resistance to fracture.

In modern materials science, fracture mechanics is an important tool in improving the mechanical performance of materials and components. It applies the physics of stress and strain, in particular the theories of elasticity and plasticity, to the microscopic crystallographic defects found in real materials in order to predict the macroscopic mechanical failure of bodies. Fractography is widely used with fracture mechanics to understand the causes of failures and also verify the theoretical failure predictions with real life failures.



An edge crack (flaw) of length *a* in a material.

Linear elastic fracture mechanics

Griffith's criterion

Fracture mechanics was developed during World War I by English aeronautical engineer, A. A. Griffith, to explain the failure of brittle materials. Griffith's work was motivated by two contradictory facts:

- The stress needed to fracture bulk glass is around 100 MPa (15,000 psi).
- The theoretical stress needed for breaking atomic bonds is approximately 10,000 MPa (1,500,000 psi).

A theory was needed to reconcile these conflicting observations. Also, experiments on glass fibers that Griffith himself conducted suggested that the fracture stress increases as the fiber diameter decreases. Hence the uniaxial tensile strength, which had been used extensively to predict material failure before Griffith, could not be a specimen-independent material property. Griffith suggested that the low fracture strength observed in experiments, as well as the size-dependence of strength, was due to the presence of microscopic flaws in the bulk material.

To verify the flaw hypothesis, Griffith introduced an artificial flaw in his experimental specimens. The artificial flaw was in the form of a surface crack which was much larger than other flaws in a specimen. The experiments showed that the product of the square root of the flaw length (a) and the stress at fracture (σ_f) was nearly constant, which is expressed by the equation:

$$\sigma_f \sqrt{a} \approx C$$

An explanation of this relation in terms of linear elasticity theory is problematic. Linear elasticity theory predicts that stress (and hence the strain) at the tip of a sharp flaw in a linear elastic material is infinite. To avoid that problem, Griffith developed a thermodynamic approach to explain the relation that he observed.

The growth of a crack requires the creation of two new surfaces and hence an increase in the surface energy. Griffith found an expression for the constant C in terms of the surface energy of the crack by solving the elasticity problem of a finite crack in an elastic plate. Briefly, the approach was:

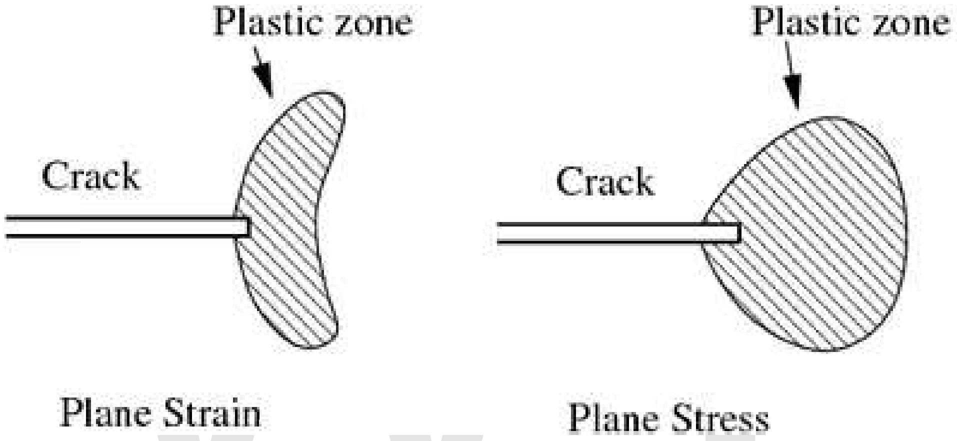
- Compute the potential energy stored in a perfect specimen under an uniaxial tensile load.
- Fix the boundary so that the applied load does no work and then introduce a crack into the specimen. The crack relaxes the stress and hence reduces the elastic energy near the crack faces. On the other hand, the crack increases the total surface energy of the specimen.
- Compute the change in the free energy (surface energy − elastic energy) as a function of the crack length. Failure occurs when the free energy attains a peak

value at a critical crack length, beyond which the free energy decreases by increasing the crack length, i.e. by causing fracture. Using this procedure, Griffith found that

$$C = \sqrt{\frac{2E\gamma}{\pi}}$$

where E is the Young's modulus of the material and γ is the surface energy density of the material. Assuming $E = 62$ GPa and $\gamma = 1$ J/m^2 gives excellent agreement of Griffith's predicted fracture stress with experimental results for glass.

Irwin's modification



The plastic zone around crack tip in a ductile material.

Griffith's work was largely ignored by the engineering community until the early 1950s. The reasons for this appear to be (a) in the actual structural materials the level of energy needed to cause fracture is orders of magnitude higher than the corresponding surface energy, and (b) in structural materials there are always some inelastic deformations around the crack front that would make the assumption of linear elastic medium with infinite stresses at the crack tip highly unrealistic. **F. Erdogan (2000)**

Griffith's theory provides excellent agreement with experimental data for brittle materials such as glass. For ductile materials such as steel, though the relation $\sigma_y\sqrt{a} = C$ still holds, the surface energy (γ) predicted by Griffith's theory is usually unrealistically high. A group working under G. R. Irwin at the U.S. Naval Research Laboratory (NRL) during World War II realized that plasticity must play a significant role in the fracture of ductile materials.

In ductile materials (and even in materials that appear to be brittle), a plastic zone develops at the tip of the crack. As the applied load increases, the plastic zone increases in size until the crack grows and the material behind the crack tip unloads. The plastic loading and unloading cycle near the crack tip leads to the dissipation of energy as heat.

Hence, a dissipative term has to be added to the energy balance relation devised by Griffith for brittle materials. In physical terms, additional energy is needed for crack growth in ductile materials when compared to brittle materials.

Irwin's strategy was to partition the energy into two parts:

- the stored elastic strain energy which is released as a crack grows. This is the thermodynamic driving force for fracture.
- the dissipated energy which includes plastic dissipation and the surface energy (and any other dissipative forces that may be at work). The dissipated energy provides the thermodynamic resistance to fracture. Then the total energy dissipated is

  $$G = 2\gamma + G_p$$

where γ is the surface energy and G_p is the plastic dissipation (and dissipation from other sources) per unit area of crack growth.

The modified version of Griffith's energy criterion can then be written as

$$\sigma_f \sqrt{a} = \sqrt{\frac{E\,G}{\pi}}.$$

For brittle materials such as glass, the surface energy term dominates and $G \approx 2\gamma = 2\ J/m^2$. For ductile materials such as steel, the plastic dissipation term dominates and $G \approx G_p = 1000\ J/m^2$. For polymers close to the glass transition temperature, we have intermediate values of $G \approx 2 - 1000\ J/m^2$.

Stress intensity factor

Another significant achievement of Irwin and his colleagues was to find a method of calculating the amount of energy available for fracture in terms of the asymptotic stress and displacement fields around a crack front in a linear elastic solid. This asymptotic expression for the stress field around a crack tip is

$$\sigma_{ij} \approx \left(\frac{K}{\sqrt{2\pi r}}\right)\ f_{ij}(\theta)$$

where σ_{ij} are the Cauchy stresses, r is the distance from the crack tip, θ is the angle with respect to the plane of the crack, and f_{ij} are functions that are independent of the crack geometry and loading conditions. Irwin called the quantity K the *stress intensity factor*.

Since the quantity f_{ij} is dimensionless, the stress intensity factor can be expressed in units of $\mathrm{MPa}\text{-}\sqrt{\mathrm{m}}$.

When a rigid line inclusion is considered, a similar asymptotic expression for the stress fields is obtained.

Strain energy release

Irwin was the first to observe that if the size of the plastic zone around a crack is small compared to the size of the crack, the energy required to grow the crack will not be critically dependent on the state of stress at the crack tip. In other words, a purely elastic solution may be used to calculate the amount of energy available for fracture.

The energy release rate for crack growth or *strain energy release rate* may then be calculated as the change in elastic strain energy per unit area of crack growth, i.e.,

$$G := -\left[\frac{\partial U}{\partial a}\right]_P = -\left[\frac{\partial U}{\partial a}\right]_u$$

where U is the elastic energy of the system and a is the crack length. Either the load P or the displacement u can be kept fixed while evaluating the above expressions.

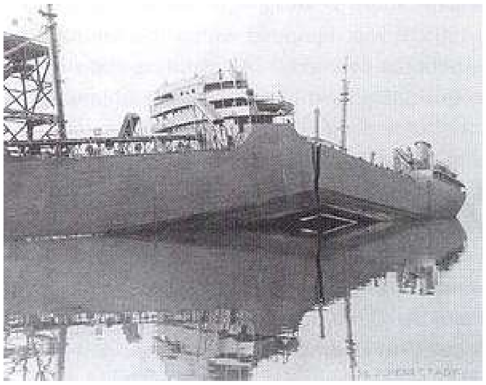
Irwin showed that for a mode I crack (opening mode) the strain energy release rate and the stress intensity factor are related by:

$$G = G_I = \begin{cases} \dfrac{K_I^2}{E} & \text{plane stress} \\ \dfrac{(1-\nu^2)K_I^2}{E} & \text{plane strain} \end{cases}$$

where E is the Young's modulus, v is Poisson's ratio, and K_I is the stress intensity factor in mode I. Irwin also showed that the strain energy release rate of a planar crack in a linear elastic body can be expressed in terms of the mode I, mode II (sliding mode), and mode III (tearing mode) stress intensity factors for the most general loading conditions.

Next, Irwin adopted the additional assumption that the size and shape of the energy dissipation zone remains approximately constant during brittle fracture. This assumption suggests that the energy needed to create a unit fracture surface is a constant that depends only on the material. This new material property was given the name *fracture toughness* and designated G_{Ic}. Today, it is the critical stress intensity factor K_{Ic} which is accepted as the defining property in linear elastic fracture mechanics.

Limitations



The S.S. *Schenectady* split apart by brittle fracture while in harbor (1944)

But a problem arose for the NRL researchers because naval materials, e.g., ship-plate steel, are not perfectly elastic but undergo significant plastic deformation at the tip of a crack. One basic assumption in Irwin's linear elastic fracture mechanics is that the size of the plastic zone is small compared to the crack length. However, this assumption is quite restrictive for certain types of failure in structural steels though such steels can be prone to brittle fracture, which has led to a number of catastrophic failures.

Linear-elastic fracture mechanics is of limited practical use for structural steels for another more practical reason. Fracture toughness testing is very expensive and engineers believe that sufficient information for selection of steels can be obtained from the simpler and cheaper Charpy impact test.

Nonlinear elasticity and plasticity

Vertical stabilizer, which separated from American Airlines Flight 587, leading to a fatal crash

Most engineering materials show some nonlinear elastic and inelastic behavior under operating conditions that involve large loads. In such materials the assumptions of linear elastic fracture mechanics may not hold, that is,

- the plastic zone at a crack tip may have a size of the same order of magnitude as the crack size
- the size and shape of the plastic zone may change as the applied load is increased and also as the crack length increases.

Therefore a more general theory of crack growth is needed for elastic-plastic materials that can account for:

- the local conditions for initial crack growth which include the nucleation, growth, and coalescence of voids or decohesion at a crack tip.
- a global energy balance criterion for further crack growth and unstable fracture.

R-curve

An early attempt in the direction of elastic-plastic fracture mechanics was Irwin's **crack extension resistance curve** or **R-curve**. This curve acknowledges the fact that the resistance to fracture increases with growing crack size in elastic-plastic materials. The R-curve is a plot of the total energy dissipation rate as a function of the crack size and can be used to examine the processes of slow stable crack growth and unstable fracture. However, the R-curve was not widely used in applications until the early 1970s. The main reasons appear to be that the R-curve depends on the geometry of the specimen and the crack driving force may be difficult to calculate.

J-integral

In the mid-1960s James R. Rice (then at Brown University) and G. P. Cherepanov independently developed a new toughness measure to describe the case where there is sufficient crack-tip deformation that the part no longer obeys the linear-elastic approximation. Rice's analysis, which assumes non-linear elastic (or monotonic deformation-theory plastic) deformation ahead of the crack tip, is designated the J integral. This analysis is limited to situations where plastic deformation at the crack tip does not extend to the furthest edge of the loaded part. It also demands that the assumed non-linear elastic behavior of the material is a reasonable approximation in shape and magnitude to the real material's load response. The elastic-plastic failure parameter is designated J_{Ic} and is conventionally converted to K_{Ic} using Equation (3.1) of the Appendix to this. Also note that the J integral approach reduces to the Griffith theory for linear-elastic behavior.

Fully plastic failure

If the material is so tough that the yielded region ahead of the crack extends to the far edge of the specimen before fracture, the crack is no longer an effective stress concentrator. Instead, the presence of the crack merely serves to reduce the load-bearing area. In this regime the failure stress is conventionally assumed to be the average of the yield and ultimate strengths of the material.

Engineering applications

The following information is needed for a fracture mechanics prediction of failure:

- Applied load
- Residual stress
- Size and shape of the part
- Size, shape, location, and orientation of the crack

Usually not all of this information is available and conservative assumptions have to be made.

Occasionally post-mortem fracture-mechanics analyses are carried out. In the absence of an extreme overload, the causes are either insufficient toughness (K_{Ic}) or an excessively large crack that was not detected during routine inspection.

Short summary

Arising from the manufacturing process, interior and surface flaws are found in all metal structures. Not all such flaws are unstable under service conditions. Fracture mechanics is the analysis of flaws to discover those that are safe (that is, do not grow) and those that are liable to propagate as cracks and so cause failure of the flawed structure. Ensuring safe operation of structure despite these inherent flaws is achieved through damage tolerance analysis. Fracture mechanics as a subject for critical study has barely been around for a century and thus is relatively new. There is a high demand for engineers with fracture mechanics expertise—particularly in this day and age where engineering failure is considered 'shocking' amongst the general public.

Appendix: mathematical relations

Griffith's criterion

For the simple case of a thin rectangular plate with a crack perpendicular to the load Griffith’s theory becomes:

$$G = \frac{\pi \sigma^2 a}{E} \qquad (1.1)$$

where *G* is the strain energy release rate, σ is the applied stress, *a* is half the crack length, and *E* is the Young’s modulus. The strain energy release rate can otherwise be understood as: *the rate at which energy is absorbed by growth of the crack*.

However, we also have that:

$$G_c = \frac{\pi \sigma_f^2 a}{E} \qquad (1.2)$$

If $G \geq G_c$, this is the criterion for which the crack will begin to propagate.

Irwin's modifications

Eventually a modification of Griffith's solids theory emerged from this work; a term called stress intensity replaced strain energy release rate and a term called fracture toughness replaced surface weakness energy. Both of these terms are simply related to the energy terms that Griffith used:

$$K_I = \sigma \sqrt{\pi a} \qquad (2.1)$$

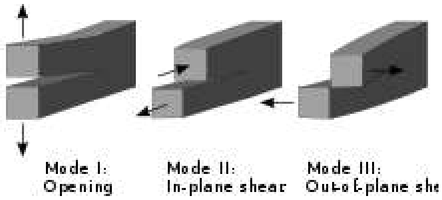
and

$$K_c = \sqrt{E G_c} \text{(for plane stress)} \qquad (2.2)$$

$$K_c = \sqrt{\frac{E G_c}{1 - \nu^2}} \text{(for plane strain)} \qquad (2.3)$$

where K_I is the stress intensity, K_c the fracture toughness, and ν is Poisson's ratio. It is important to recognize the fact that fracture parameter K_c has different values when measured under plane stress and plane strain

Fracture occurs when $K_I \geq K_c$. For the special case of plane strain deformation, K_c becomes K_{Ic} and is considered a material property. The subscript I arises because of the different ways of loading a material to enable a crack to propagate. It refers to so-called "mode I" loading as opposed to mode II or III:



The three fracture modes.

There are three ways of applying a force to enable a crack to propagate:

- **Mode I crack** – Opening mode (a tensile stress normal to the plane of the crack)

- **Mode II crack** – Sliding mode (a shear stress acting parallel to the plane of the crack and perpendicular to the crack front)
- **Mode III crack** – Tearing mode (a shear stress acting parallel to the plane of the crack and parallel to the crack front)

We must note that the expression for K_I in equation 2.1 will be different for geometries other than the center-cracked infinite plate, as discussed in stress intensity. Consequently, it is necessary to introduce a dimensionless correction factor, Y, in order to characterize the geometry. We thus have:

$$K_I = Y\sigma\sqrt{\pi a} \qquad (2.4)$$

where Y is a function of the crack length and width of sheet given by:

$$Y\left(\frac{a}{W}\right) = \sqrt{\sec\left(\frac{\pi a}{W}\right)} \qquad (2.5)$$

for a sheet of finite width W containing a through-thickness crack of length $2a$, or

$$Y\left(\frac{a}{W}\right) = 1.12 - \frac{0.41}{\sqrt{\pi}}\frac{a}{W} + \frac{18.7}{\sqrt{\pi}}\left(\frac{a}{W}\right)^2 - \cdots \qquad (2.6)$$

for a sheet of finite width W containing a through-thickness edge crack of length a

Elasticity and plasticity

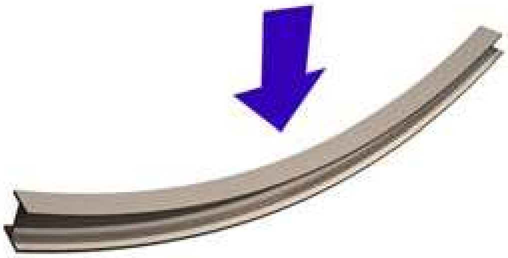
Since engineers became accustomed to using K_{Ic} to characterise fracture toughness, a relation has been used to reduce J_{Ic} to it:

$K_{Ic} = \sqrt{E^* J_{Ic}}$ where $E^* = E$ for plane stress and $E^* = \dfrac{E}{1-\nu^2}$ for plane strain (3.1)

The remainder of the mathematics employed in this approach is interesting, but is probably better summarised in external pages due to its complex nature.

Chapter- 11

Bending



Bending of an I-beam

In engineering mechanics, **bending** (also known as **flexure**) characterizes the behavior of a slender structural element subjected to an external load applied perpendicularly to a longitudinal axis of the element. The structural element is assumed to be such that at least one of its dimensions is a small fraction, typically 1/10 or less, of the other two. When the length is considerably longer than the width and the thickness, the element is called a beam. A closet rod sagging under the weight of clothes on clothes hangers is an example of a beam experiencing bending. On the other hand, a shell is a structure of any geometric form where the length and the width are of the same order of magnitude but the thickness of the structure (known as the 'wall') is considerably smaller. A large diameter, but thin-walled, short tube supported at its ends and loaded laterally is an example of a shell experiencing bending.

In the absence of a qualifier, the term *bending* is ambiguous because bending can occur locally in all objects. To make the usage of the term more precise, engineers refer to the *bending of rods*, the *bending of beams*, the *bending of plates*, the *bending of shells* and so on.

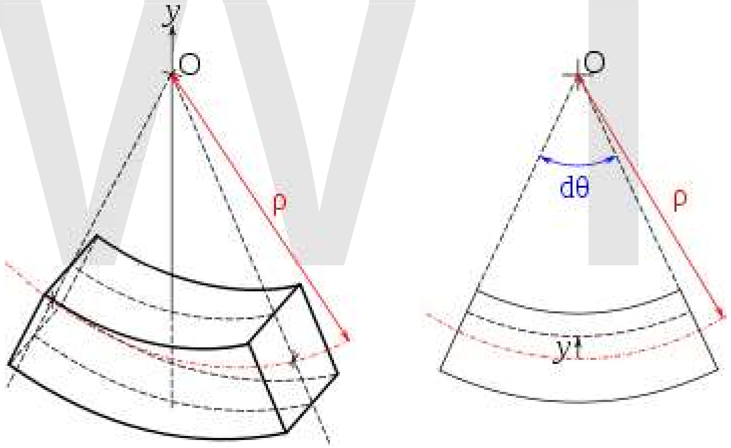
Quasistatic bending of beams

A beam deforms and stresses develop inside it when a transverse load is applied on it. In the quasistatic case, the amount of bending deflection and the stresses that develop are assumed not to change over time. In a horizontal beam supported at the ends and loaded downwards in the middle, the material at the over-side of the beam is compressed while the material at the underside is stretched. There are two forms of internal stresses caused by lateral loads:

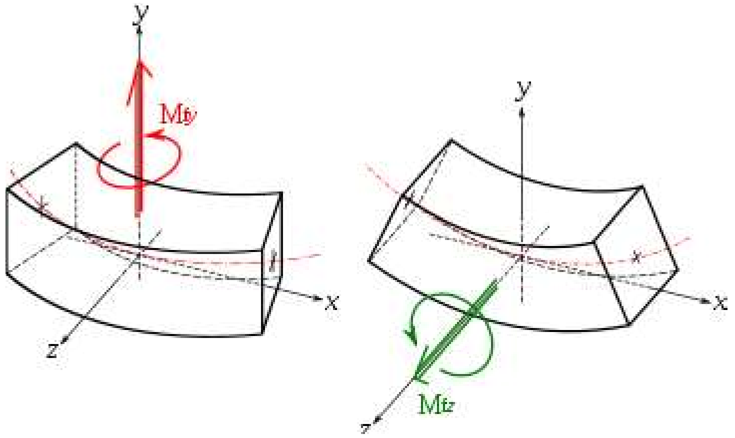
- Shear stress parallel to the lateral loading plus complementary shear stress on planes perpendicular to the load direction;
- Direct compressive stress in the upper region of the beam, and direct tensile stress in the lower region of the beam.

These last two forces form a couple or moment as they are equal in magnitude and opposite in direction. This bending moment resists the sagging deformation characteristic of a beam experiencing bending. The stress distribution in a beam can be predicted quite accurately even when some simplifying assumptions are used.

Euler-Bernoulli bending theory



Element of a bent beam: the fibers form concentric arcs, the top fibers are compressed and bottom fibers stretched.



Bending moments in a beam

In the Euler-Bernoulli theory of slender beams, a major assumption is that 'plane sections remain plane'. In other words, any deformation due to shear across the section is not accounted for (no shear deformation). Also, this linear distribution is only applicable if the maximum stress is less than the yield stress of the material. For stresses that exceed yield, refer to article plastic bending. At yield, the maximum stress experienced in the section (at the furthest points from the neutral axis of the beam) is defined as the flexural strength.

The Euler-Bernoulli equation for the quasistatic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load $q(x)$ is

$$EI\ \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x)$$

where E is the Young's modulus, I is the area moment of inertia of the cross-section, and $w(x)$ is the deflection of the neutral axis of the beam.

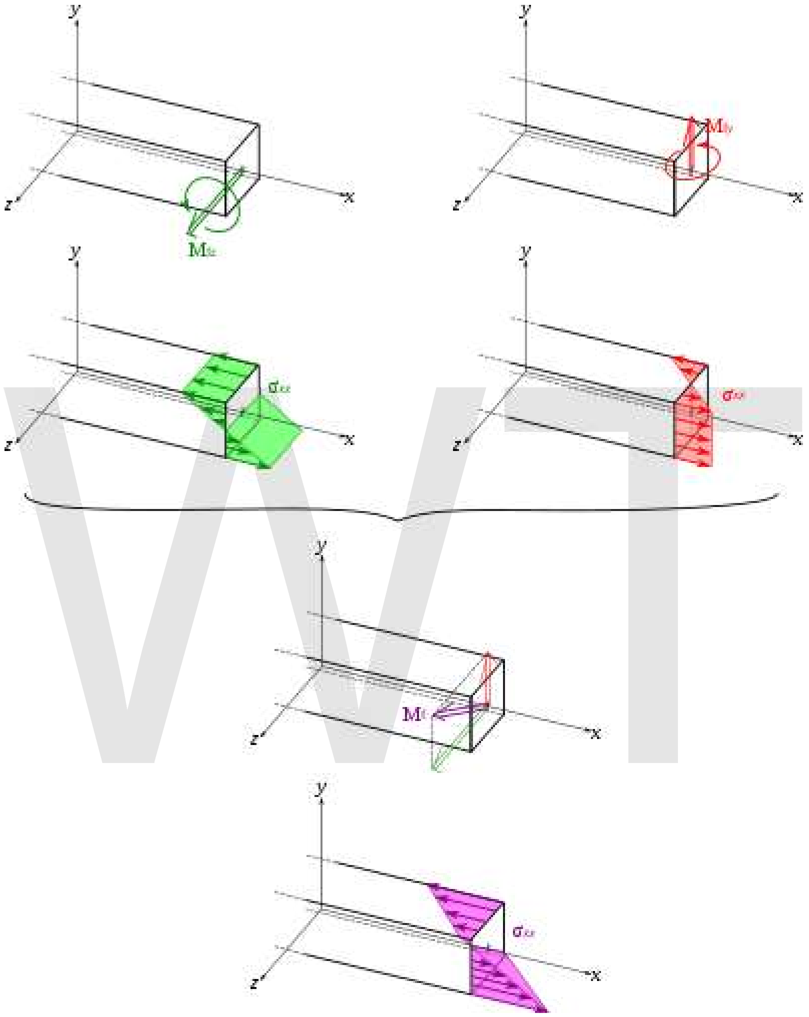
After a solution for the displacement of the beam has been obtained, the bending moment (M) and shear force (Q) in the beam can be calculated using the relations

$$M(x) = -EI\ \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\ ; \quad Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x}$$

Simple beam bending is often analyzed with the Euler-Bernoulli beam equation. The conditions for using simple bending theory are:

1. The beam is subject to pure bending. This means that the shear force is zero, and that no torsional or axial loads are present.
2. The material is isotropic and homogeneous.
3. The material obeys Hooke's law (it is linearly elastic and will not deform plastically).

4. The beam is initially straight with a cross section that is constant throughout the beam length.
5. The beam has an axis of symmetry in the plane of bending.
6. The proportions of the beam are such that it would fail by bending rather than by crushing, wrinkling or sideways buckling.
7. Cross-sections of the beam remain plane during bending.



Deflection of a beam deflected symmetrically and principle of superposition

Compressive and tensile forces develop in the direction of the beam axis under bending loads. These forces induce stresses on the beam. The maximum compressive stress is found at the uppermost edge of the beam while the maximum tensile stress is located at the lower edge of the beam. Since the stresses between these two opposing maxima vary linearly, there therefore exists a point on the linear path between them where there is no bending stress. The locus of these points is the neutral axis. Because of this area with no stress and the adjacent areas with low stress, using uniform cross section beams in bending is not a particularly efficient means of supporting a load as it does not use the full capacity of the beam until it is on the brink of collapse. Wide-flange beams (I-beams) and truss girders effectively address this inefficiency as they minimize the amount of material in this under-stressed region.

The classic formula for determining the bending stress in a beam under simple bending is:

$$\sigma = \frac{My}{I_x}$$

where

- σ is the bending stress
- M - the moment about the neutral axis
- y - the perpendicular distance to the neutral axis
- I_x - the second moment of area about the neutral axis x

Extensions of Euler-Bernoulli beam bending theory

Plastic bending

The equation $\sigma = \frac{My}{I_x}$ is valid only when the stress at the extreme fiber (i.e. the portion of the beam farthest from the neutral axis) is below the yield stress of the material from which it is constructed. At higher loadings the stress distribution becomes non-linear, and ductile materials will eventually enter a *plastic hinge* state where the magnitude of the stress is equal to the yield stress everywhere in the beam, with a discontinuity at the neutral axis where the stress changes from tensile to compressive. This plastic hinge state is typically used as a limit state in the design of steel structures.

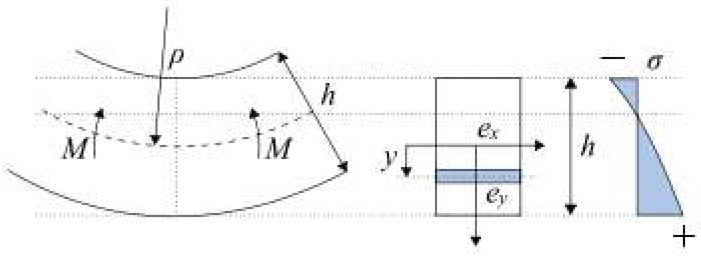
Complex or asymmetrical bending

The equation above is only valid if the cross-section is symmetrical. For homogeneous beams with asymmetrical sections, the axial stress in the beam is given by

$$\sigma_x(y,z) = -\frac{(M_z\ I_y + M_y\ I_{yz})}{I_y\ I_z - I_{yz}^2}y + \frac{(M_y\ I_z + M_z\ I_{yz})}{I_y\ I_z - I_{yz}^2}z$$

where y,z are the coordinates of a point on the cross section at which the stress is to be determined as shown to the right, M_y and M_z are the bending moments about the y and z centroid axes, I_y and I_z are the second moments of area (distinct from moments of inertia) about the y and z axes, and I_{yz} is the product of moments of area. Using this equation it is possible to calculate the bending stress at any point on the beam cross section regardless of moment orientation or cross-sectional shape. Note that M_y,M_z,I_y,I_z,I_{yz} do not change from one point to another on the cross section.

Large bending deformation



For large deformations of the body, the stress in the cross-section is calculated using an extended version of this formula. First the following assumptions must be made:

1. Assumption of flat sections - before and after deformation the considered section of body remains flat (i.e. is not swirled).
2. Shear and normal stresses in this section that are perpendicular to the normal vector of cross section have no influence on normal stresses that are parallel to this section.

Large bending considerations should be implemented when the bending radius ρ is smaller than ten section heights h:

$\rho < 10h$

With those assumptions the stress in large bending is calculated as:

$$\sigma = \frac{F}{A} + \frac{M}{\rho A} + \frac{M}{I_x{}'} y \frac{\rho}{\rho + y}$$

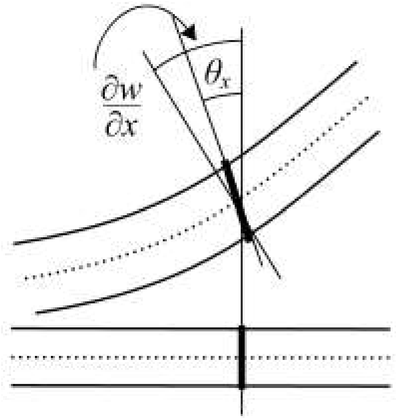
where

F is the normal force
A is the section area
M is the bending moment
ρ is the local bending radius (the radius of bending at the current section)
I_x' is the area moment of inertia along the **x** axis, at the y place
y is the position along **y** axis on the section area in which the stress σ is calculated

When bending radius ρ approaches infinity and y is near zero, the original formula is back:

$$\sigma = \frac{F}{A} \pm \frac{My}{I}.$$

Timoshenko bending theory



Deformation of a Timoshenko beam. The normal rotates by an amount θ which is not equal to *dw* / *dx*.

In 1921, Timoshenko improved upon the Euler-Bernoulli theory of beams by adding the effect of shear into the beam equation. The kinematic assumptions of the Timoshenko theory are

- normals to the axis of the beam remain straight after deformation
- there is no change in beam thickness after deformation

However, normals to the axis are not required to remain perpendicular to the axis after deformation.

The equation for the quasistatic bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - \frac{EI}{kAG}\,\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}$$

where *I* is the area moment of inertia of the cross-section, *A* is the cross-sectional area, *G* is the shear modulus, and *k* is a **shear correction factor**. For materials with Poisson's ratios (ν) close to 0.3, the shear correction factor for a rectangular cross-section is approximately

$$k = \frac{5+5\nu}{6+5\nu}$$

The rotation ($\varphi(x)$) of the normal is described by the equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{q(x)}{kAG}$$

The bending moment (*M*) and the shear force (*Q*) are given by

$$M(x) = -EI\,\frac{\mathrm{d}\varphi}{\mathrm{d}x}\,;\quad Q(x) = kAG\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \varphi\right) = -EI\,\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{\mathrm{d}M}{\mathrm{d}x}$$

Dynamic bending of beams

The dynamic bending of beams, also known as flexural vibrations of beams, was first investigated by Daniel Bernoulli in the late 18th century. Bernoulli's equation of motion of a vibrating beam tended to overestimate the natural frequencies of beams and was improved marginally by Rayleigh in 1877 by the addition of a mid-plane rotation. In 1921 Stephen Timoshenko improved the theory further by incorporating the effect of shear on the dynamic response of bending beams. This allowed the theory to be used for problems involving high frequencies of vibration where the dynamic Euler-Bernoulli theory is inadequate. The Euler-Bernoulli and Timoshenko theories for the dynamic bending of beams continue to be used widely by engineers.

Euler-Bernoulli theory

The Euler-Bernoulli equation for the dynamic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load *q*(*x*,*t*) is

$$EI\,\frac{\partial^4 w}{\partial x^4} + m\,\frac{\partial^2 w}{\partial t^2} = q(x,t)$$

where *E* is the Young's modulus, *I* is the area moment of inertia of the cross-section, *w*(*x*,*t*) is the deflection of the neutral axis of the beam, and *m* is mass per unit length of the beam.

Free vibrations

For the situation where there is no transverse load on the beam, the bending equation takes the form

$$EI\,\frac{\partial^4 w}{\partial x^4} + m\,\frac{\partial^2 w}{\partial t^2} = 0$$

Free, harmonic vibrations of the beam can then be expressed as

$$w(x,t) = \mathrm{Re}[\hat{w}(x)\ e^{-i\omega t}] \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial t^2} = -\omega^2\ w(x,t)$$

and the bending equation can be written as

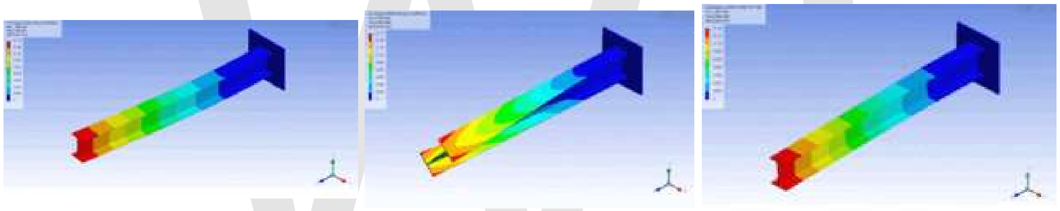
$$EI\ \frac{\mathrm{d}^4\hat{w}}{\mathrm{d}x^4} - m\omega^2\hat{w} = 0$$

The general solution of the above equation is

$$\hat{w} = A_1\cosh(\beta x) + A_2\sinh(\beta x) + A_3\cos(\beta x) + A_4\sin(\beta x)$$

where A_1,A_2,A_3,A_4 are constants and $\beta := \left(\frac{m}{EI}\,\omega^2\right)^{1/4}$

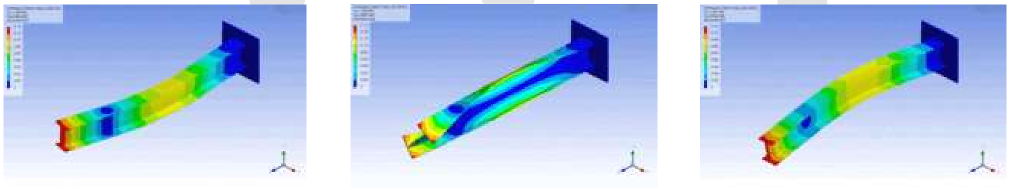
The mode shapes of a cantilevered I-beam



1st lateral bending

1st torsional

1st vertical bending



2nd lateral bending

2nd torsional

2nd vertical bending

Timoshenko-Rayleigh theory

In 1877, Rayleigh proposed an improvement to the dynamic Euler-Bernoulli beam theory by including the effect of rotational inertia of the cross-section of the beam. Timoshenko improved upon that theory in 1922 by adding the effect of shear into the beam equation. Shear deformations of the normal to the mid-surface of the beam are allowed in the Timoshenko-Rayleigh theory.

The equation for the bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\ \frac{\partial^4 w}{\partial x^4}+m\ \frac{\partial^2 w}{\partial t^2}-\left(J+\frac{EIm}{kAG}\right)\ \frac{\partial^4 w}{\partial x^2\ \partial t^2}+\frac{Jm}{kAG}\ \frac{\partial^4 w}{\partial t^4}=q(x,t)+\frac{J}{kAG}\ \frac{\partial^2 q}{\partial t^2}-\frac{EI}{kAG}\ \frac{\partial^2 q}{\partial x^2}$$

where $J=\frac{mI}{A}$ is the polar moment of inertia of the cross-section, $m = \rho A$ is the mass per unit length of the beam, ρ is the density of the beam, A is the cross-sectional area, G is the shear modulus, and k is a **shear correction factor**. For materials with Poisson's ratios (ν) close to 0.3, the shear correction factor are approximately

$$\begin{aligned} k &= \tfrac{5+5\nu}{6+5\nu} \quad \text{rectangular cross-section} \\ &= \tfrac{6+12\nu+6\nu^2}{7+12\nu+4\nu^2} \quad \text{circular cross-section} \end{aligned}$$

Free vibrations

For free, harmonic vibrations the Timoshenko-Rayleigh equations take the form

$$EI\ \frac{\mathrm{d}^4\hat{w}}{\mathrm{d}x^4}+m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}x^2}+m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)\ \hat{w}=0$$

This equation can be solved by noting that all the derivatives of w must have the same form to cancel out and hence as solution of the form e^{kx} may be expected. This observation leads to the characteristic equation

$$\alpha\, k^4+\beta\, k^2+\gamma=0\,;\ \ \alpha:=EI\,,\ \ \beta:=m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\ ,\ \ \gamma:=m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)$$

The solutions of this quartic equation are

$$k_1=+\sqrt{z_+}\ ,\quad k_2=-\sqrt{z_+}\ ,\quad k_3=+\sqrt{z_-}\ ,\quad k_4=-\sqrt{z_-}$$

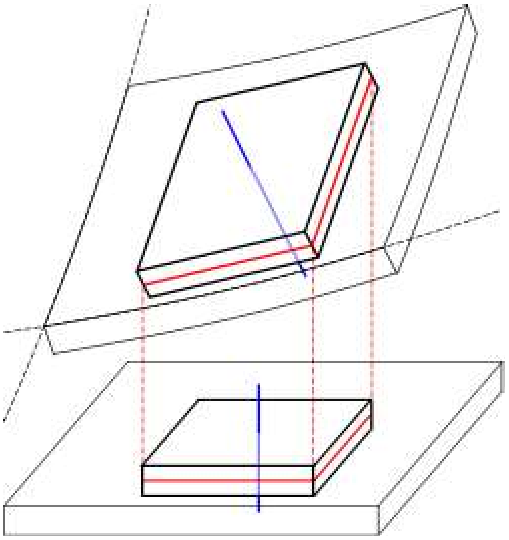
where

$$z_+:=\frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\ ,\quad z_-:=\frac{-\beta-\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$$

The general solution of the Timoshenko-Rayleigh beam equation for free vibrations can then be written as

$$\hat{w}=A_1\ e^{k_1 x}+A_2\ e^{-k_1 x}+A_3\ e^{k_3 x}+A_4\ e^{-k_3 x}$$

Quasistatic bending of plates



Deformation of a thin plate highlighting the displacement, the mid-surface (red) and the normal to the mid-surface (blue)

The defining feature of beams is that one of the dimensions is much *larger* than the other two. A structure is called a plate when it is flat and one of its dimensions is much *smaller* than the other two. There are several theories that attempt to describe the deformation and stress in a plate under applied loads two of which have been used widely. These are

- the Kirchhoff-Love theory of plates (also called classical plate theory)
- the Mindlin-Reissner plate theory (also called the first-order shear theory of plates)

Kirchhoff-Love theory of plates

The assumptions of Kirchhoff-Love theory are

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

These assumptions imply that

$$u_\alpha(\mathbf{x}) = -x_3\ \frac{\partial w^0}{\partial x_\alpha} = -x_3\ w^0_{,\alpha}\ ;\quad \alpha = 1,2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

where $\mathbf{u}$ is the displacement of a point in the plate and w^0 is the displacement of the mid-surface.

The strain-displacement relations are

$$\begin{aligned}\varepsilon_{\alpha\beta} &= -x_3\ w^0_{,\alpha\beta}\\ \varepsilon_{\alpha 3} &= 0\\ \varepsilon_{33} &= 0\end{aligned}$$

The equilibrium equations are

$$M_{\alpha\beta,\alpha\beta} + q(x) = 0\ ; \quad M_{\alpha\beta} := \int_{-h}^{h} x_3\ \sigma_{\alpha\beta}\ dx_3$$

where $q(x)$ is an applied load normal to the surface of the plate.

In terms of displacements, the equilibrium equations for an isotropic, linear elastic plate in the absence of external load can be written as

$$w^0_{,1111} + 2\ w^0_{,1212} + w^0_{,2222} = 0$$

In direct tensor notation,

$$\nabla^2\nabla^2 w = 0$$

Mindlin-Reissner theory of plates

The special assumption of this theory is that normals to the mid-surface remain straight and inextensible but not necessarily normal to the mid-surface after deformation. The displacements of the plate are given by

$$\begin{aligned}u_\alpha(\mathbf{x}) &= -x_3\ \varphi_\alpha\ ; \quad \alpha = 1,2\\ u_3(\mathbf{x}) &= w^0(x_1, x_2)\end{aligned}$$

where φ_α are the rotations of the normal.

The strain-displacement relations that result from these assumptions are

$$\begin{aligned}\varepsilon_{\alpha\beta} &= -x_3\ \varphi_{\alpha,\beta}\\ \varepsilon_{\alpha 3} &= \frac{1}{2}\,\kappa\left(w^0_{,\alpha} - \varphi_\alpha\right)\\ \varepsilon_{33} &= 0\end{aligned}$$

where κ is a shear correction factor.

The equilibrium equations are

$$M_{\alpha\beta,\beta} - Q_\alpha = 0$$
$$Q_{\alpha,\alpha} + q = 0$$

where

$$Q_\alpha := \kappa \int_{-h}^{h} \sigma_{\alpha 3}\, dx_3$$

Dynamic bending of plates

Dynamics of thin Kirchhoff plates

The dynamic theory of plates determines the propagation of waves in the plates, and the study of standing waves and vibration modes. The equations that govern the dynamic bending of Kirchhoff plates are

$$M_{\alpha\beta,\alpha\beta} - q(x,t) = J_1\, \ddot{w}^0 - J_3\, \ddot{w}^0_{,\alpha\alpha}$$

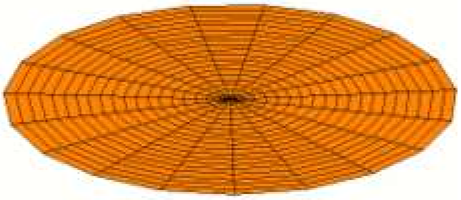
where, for a plate with density ρ = ρ(*x*),

$$J_1 := \int_{-h}^{h} \rho\, dx_3 \;;\quad J_3 := \int_{-h}^{h} x_3^2\, \rho\, dx_3$$

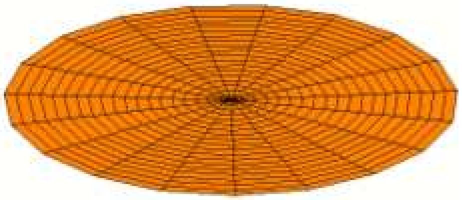
and

$$\ddot{w}^0 = \frac{\partial^2 w^0}{\partial t^2} \;;\quad \ddot{w}^0_{,\alpha\beta} = \frac{\partial^2 \ddot{w}^0}{\partial x_\alpha \partial x_\beta}$$

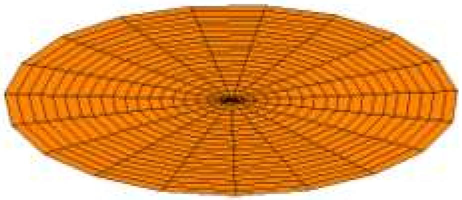
The figures below show some vibrational modes of a circular plate.



mode $k = 0$, $p = 1$



mode $k = 0$, $p = 2$



mode $k = 1$, $p = 2$

Chapter- 12

Failure Theory (Material)

Failure theory is the science of predicting the conditions under which solid materials lose their strength under the action of external loads. The failure of a material is usually classified into brittle failure (fracture) or ductile failure (yield). Depending on the conditions (such as temperature, state of stress, loading rate) most materials can fail in a brittle or ductile manner or both. However, for most practical situations, a material may be classified as either brittle or ductile. Though failure theory has been in development for over 200 years, its level of acceptability is yet to reach that of continuum mechanics.

In mathematical terms, failure theory is expressed in the form of various failure criteria which are valid for specific materials. Failure criteria are functions in stress or strain space which separate "failed" states from "unfailed" states. A precise physical definition of a "failed" state is not easily quantified and several working definitions are in use in the engineering community. Quite often, phenomenological failure criteria of the same form are used to predict brittle failure and ductile yield.

Material failure

In materials science, **material failure** is the loss of load carrying capacity of a material unit. This definition *per se* introduces the fact that material failure can be examined in different scales, from microscopic, to macroscopic. In structural problems, where the structural response may be beyond the initiation of nonlinear material behaviour, material failure is of profound importance for the determination of the integrity of the structure. On the other hand, due to the lack of globally accepted fracture criteria, the determination of the structure's damage, due to material failure, is still under intensive research.

Types of material failure

Material failure can be distinguished in two broader categories depending on the scale in which the material is examined:

Microscopic failure

Microscopic material failure is defined in terms of crack propagation and initiation. Such methodologies are useful for gaining insight in the cracking of specimens and simple structures under well defined global load distributions. Microscopic failure considers the initiation and propagation of a crack. Failure criteria in this case are related to microscopic fracture. Some of the most popular failure models in this area are the micromechanical failure models, which combine the advantages of continuum mechanics and classical fracture mechanics. Such models are based on the concept that during plastic deformation, microvoids nucleate and grow until a local plastic neck or fracture of the intervoid matrix occurs, which causes the coalescence of neighbouring voids. Such a model, proposed by Gurson and extended by Tvergaard and Needleman, is known as GTN. Another approach, proposed by Rousselier, is based on continuum damage mechanics (CDM) and thermodynamics. Both models form a modification of the von Mises yield potential by introducing a scalar damage quantity, which represents the void volume fraction of cavities, the porosity f.

Macroscopic failure

Macroscopic material failure is defined in terms of load carrying capacity or energy storage capacity, equivalently. Li presents a classification of macroscopic failure criteria in four categories:

- Stress or strain failure
- Energy type failure (S-criterion, T-criterion)
- Damage failure
- Empirical failure.

Five general levels are considered, at which the meaning of deformation and failure is interpreted differently: the structural element scale, the macroscopic scale where macroscopic stress and strain are defined, the mesoscale which is represented by a typical void, the microscale and the atomic scale. The material behaviour at one level is considered as a collective of its behaviour at a sublevel. An efficient deformation and failure model should be consistent at every level.

Brittle material failure criteria

Failure of brittle materials can be determined using several approaches:

- Phenomenological failure criteria
- Linear elastic fracture mechanics
- elastic-plastic fracture mechanics
- Energy-based methods
- Cohesive zone methods

Phenomenological failure criteria

The failure criteria that were developed for brittle solids were the maximum stress/strain criteria. The **maximum stress criterion** assumes that a material fails when the maximum principal stress σ_1 in a material element exceeds the uniaxial tensile strength of the material. Alternatively, the material will fail if the minimum principal stress σ_3 is less than the uniaxial compressive strength of the material. If the uniaxial tensile strength of the material is σ_t and the uniaxial compressive strength is σ_c, then the safe region for the material is assumed to be

$$\sigma_c < \sigma_3 < \sigma_1 < \sigma_t$$

Note that the convention that tension is positive has been used in the above expression.

The **maximum strain criterion** has a similar form except that the principal strains are compared with experimentall determined uniaxial strains at failure, i.e.,

$$\varepsilon_c < \varepsilon_3 < \varepsilon_1 < \varepsilon_t$$

The maximum principal stress and strain criteria continue to be widely used in spite of severe shortcomings.

Numerous other phenomenological failure criteria can be found in the engineering literature. The degree of success of these criteria in predicting failure has been limited. For brittle materials, some popular failure criteria are

- criteria based on invariants of the Cauchy stress tensor
- the Tresca or maximum shear stress failure criterion
- the von Mises or maximum elastic distortional energy criterion
- the Mohr-Coulomb failure criterion for cohesive-frictional solids
- the Drucker-Prager failure criterion for pressure-dependent solids
- the Bresler-Pister failure criterion for concrete
- the Willam-Warnke failure criterion for concrete
- the Hankinson criterion, an empirical failure criterion that is used for orthotropic materials such as wood.
- the Hill yield criteria for anisotropic solids
- the Tsai-Wu failure criterion for anisotropic composites
- the Johnson–Holmquist damage model for high-rate deformations of isotropic solids
- the Hoek-Brown failure criterion for rock masses

Linear elastic fracture mechanics

The approach taken in linear elastic fracture mechanics is to estimate the amount of energy needed to grow a preexisting crack in a brittle material. The earliest fracture mechanics approach for unstable crack growth is Griffiths' theory. When applied to the mode I opening of a crack, Griffiths' theory predicts that the critical stress (σ) needed to propagate the crack is given by

$$\sigma = \sqrt{\frac{2E\gamma}{\pi\ a}}$$

where E is the Young's modulus of the material, γ is the surface energy per unit area of the crack, and $2a$ is the crack length. The quantity $\sigma\sqrt{\pi a}$ is postulated as a material parameter called the '*fracture toughness*. The mode I fracture toughness is defined as

$$K_{Ic} = \sigma_c\sqrt{\pi a}$$

and is determined experimentally. Similar quantities K_{IIc} and K_{IIIc} can be determined for mode II and model III loading conditions.

The state of stress around cracks of various shapes can be expressed in terms of their stress intensity factors. Linear elastic fracture mechanics predicts that a crack will extend when the stress intensity factor at the crack tip is greater than the fracture toughness of the material. Therefore the critical applied stress can also be determined once the stress intensity factor at a crack tip is known.

Energy-based methods

The linear elastic fracture mechanics method is difficult to apply for anisotropic materials (such as composites) or for situations where the loading or the geometry are complex. The **strain energy release rate** approach has proved quite useful for such situations. The strain energy release rate for a mode I crack which runs through the thickness of a plate is defined as

$$G_I := \frac{P}{2t}\,\frac{du}{da}$$

where P is the applied load, t is the thickness of the plate, u is the displacement at the point of application of the load due to crack growth, and $2a$ is the length of the crack. The crack is expected to propagate when the strain energy release rate exceeds a critical value G_{Ic} - called the **critical strain energy release rate**.

The fracture toughness and the critical stain energy release rate are related by

$$G_{Ic} = \frac{1}{E}\,K_{Ic}^2$$

where E is the Young's modulus. If an initial crack size is known, then a critical stress can be determined using the strain energy release rate criterion.

Ductile material failure criteria

Criteria used to predict the failure of ductile materials are usually called yield criteria. Commonly used failure criteria for ductile materials are:

- the Tresca or maximum shear stress criterion.
- the von Mises yield criterion or distortional strain energy density criterion.
- the Gurson yield criterion for pressure-dependent metals.
- the Hosford yield criterion for metals.
- the Hill yield criteria.
- various criteria based on the invariants of the Cauchy stress tensor.

The yield surface of a ductile material usually changes as the material experiences increased deformation. Models for the evolution of the yield surface with increasing strain, temperature, and strain rate are used in conjunction with the above failure criteria for isotropic hardening, kinematic hardening, and viscoplasticity. Some such models are:

- the Johnson-Cook model
- the Steinberg-Guinan model
- the Zerilli-Armstrong model
- the Mechanical threshold stress model
- the Preston-Tonks-Wallace model

There is another important aspect to ductile materials - the prediction of the ultimate failure strength of a ductile material. Several models for predicting the ultimate strength have been used by the engineering community with varying levels of success. For metals, such failure criteria are usually expressed in terms of a combination of porosity and strain to failure or in terms of a damage parameter.

Chapter- 13

Hooke's Law

Hooke's law models the properties of springs for small changes in length

In mechanics, and physics, **Hooke's law** of elasticity is an approximation that states that the extension of a spring is in direct proportion with the load applied to it. Many materials obey this law as long as the load does not exceed the material's elastic limit. Materials for which Hooke's law is a useful approximation are known as linear-elastic or "Hookean" materials. Hooke's law in simple terms says that strain is directly proportional to stress.

Mathematically, Hooke's law states that

$$\mathbf{F} = -k\mathbf{x},$$

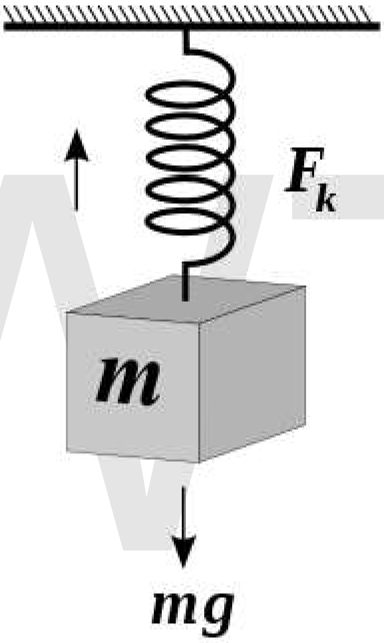
where

$\boldsymbol{x}$ is the displacement of the spring's end from its equilibrium position (a distance, in SI units: meters);
$\boldsymbol{F}$ is the restoring force exerted by the spring on that end (in SI units: N or $kg \cdot m \cdot s^{-2}$); and
k is a constant called the *rate* or *spring constant* (in SI units: $N \cdot m^{-1}$ or $kg \cdot s^{-2}$).

When this holds, the behavior is said to be *linear*. If shown on a graph, the line should show a direct variation. There is a negative sign on the right hand side of the equation because the restoring force always acts in the opposite direction of the displacement (for example, when a spring is stretched to the left, it pulls back to the right).

Hooke's law is named after the 17th century British physicist Robert Hooke. He first stated this law in 1660 as a Latin anagram, whose solution he published in 1678 as *Ut tensio, sic vis*, meaning, "As the extension, so the force".

General application to elastic materials



Hooke's law describes how far the spring will stretch under a specific force

Objects that quickly regain their original shape after being deformed by a force, with the molecules or atoms of their material returning to the initial state of stable equilibrium, often obey Hooke's law.

We may view a rod of any elastic material as a linear spring. The rod has length L and cross-sectional area A. Its extension (strain) is linearly proportional to its tensile stress σ, by a constant factor, the inverse of its modulus of elasticity, E, hence,

$$\sigma = E\varepsilon$$

or

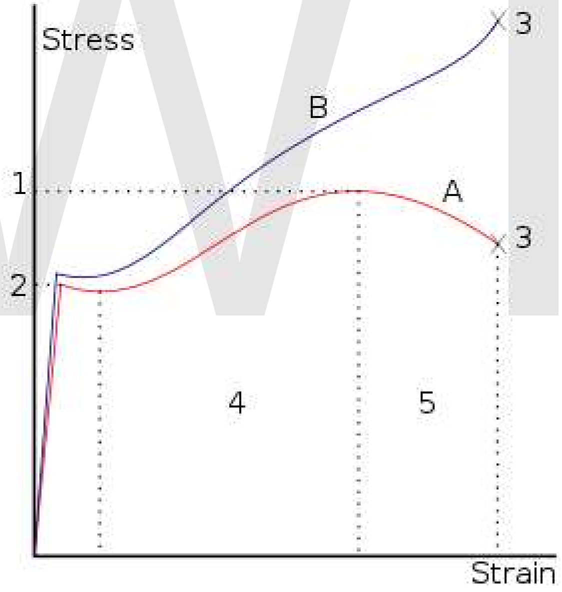
$$\Delta L = \frac{F}{EA} L = \frac{\sigma}{E} L.$$

Hooke's law only holds for some materials under certain loading conditions. Steel exhibits linear-elastic behavior in most engineering applications; Hooke's law is valid for it throughout its **elastic range** (i.e., for stresses below the yield strength). For some other materials, such as aluminium, Hooke's law is only valid for a portion of the elastic range. For these materials a proportional limit stress is defined, below which the errors associated with the linear approximation are negligible.

Rubber is generally regarded as a "non-hookean" material because its elasticity is stress dependent and sensitive to temperature and loading rate.

Applications of the law include spring operated weighing machines, stress analysis and modelling of materials.

The spring equation



Stress–strain curve for low-carbon steel. Hooke's law is only valid for the portion of the curve between the origin and the yield point(2).
1. Ultimate strength
2. Yield strength - corresponds to yield point
3. Rupture
4. Strain hardening region
5. Necking region
A: (F/A_0)
B: True stress (F/A)

The most commonly encountered form of Hooke's law is probably the *spring equation*, which relates the force exerted by a spring to the distance it is stretched by a *spring constant*, *k*, measured in force per length.

$$F = -kx$$

The negative sign indicates that the force exerted by the spring is in direct opposition to the direction of displacement. It is called a "restoring force", as it tends to restore the system to equilibrium. The potential energy stored in a spring is given by

$$PE = \frac{1}{2}kx^2$$

which comes from adding up the energy it takes to incrementally compress the spring. That is, the integral of force over distance. (Note that potential energy of a spring is always non-negative.)

This potential can be visualized as a parabola on the *U*-*x* plane. As the spring is stretched in the positive x-direction, the potential energy increases (the same thing happens as the spring is compressed). The corresponding point on the potential energy curve is higher than that corresponding to the equilibrium position ($x = 0$). The tendency for the spring is to therefore decrease its potential energy by returning to its equilibrium (unstretched) position, just as a ball rolls downhill to decrease its gravitational potential energy.

If a mass *m* is attached to the end of such a spring, the system becomes a harmonic oscillator. It will oscillate with a natural frequency given either as an angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

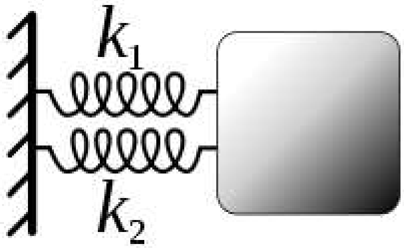
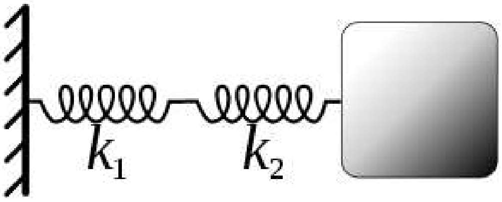
or as a natural frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

This idealized description of spring mechanics works as long as the mass of the spring is very small compared to the mass *m*, there is no significant friction on the system, and the spring is not overextended beyond its natural range (which can deform it permanently).

Multiple springs

When two springs are attached to a mass and compressed, the following table compares values of the springs.

Comparison	In Parallel	In Series
		
Equivalent spring constant	$k_{eq} = k_1 + k_2$	$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$
Compressed distance	$x_1 = x_2$	$\frac{x_1}{x_2} = \frac{k_2}{k_1}$
Energy stored	$\frac{E_1}{E_2} = \frac{k_1}{k_2}$	$\frac{E_1}{E_2} = \frac{k_2}{k_1}$

Derivation

Equivalent Spring Constant (Series)

$$\begin{aligned} F_b &= -k_2x_2 + k_2x_1 \\ &= -k_2x_2 + k_2\left(\frac{k_2}{k_1+k_2}x_2\right) \\ &= -k_2x_2\left(\frac{k_1+k_2}{k_1+k_2}\right) + \frac{k_2^2}{k_1+k_2}x_2 \\ &= x_2\frac{-k_1k_2 - k_2^2 + k_2^2}{k_1+k_2} \end{aligned}$$

Equivalent Spring Constant (Parallel)

$$\begin{aligned} F_b &= F_1 + F_2 \\ &= -k_1x - k_2x \end{aligned}$$

Compressed Distance

$$|F_1| = |F_2|$$

$$k_1 x_1 = k_2 (x_2 - x_1) .$$

Energy Stored

Tensor expression of Hooke's Law

When working with a three-dimensional stress state, a 4th order tensor $\mathbf{c}$($c_{ijk\ell}$) containing 81 elastic coefficients must be defined to link the stress tensor $\boldsymbol{\sigma}$(σ_{ij}) and the strain tensor $\boldsymbol{\epsilon}$($\epsilon_{k\ell}$).

$$\boldsymbol{\sigma} = \mathbf{c} : \boldsymbol{\epsilon} \ .$$

Expressed in terms of components with respect to an orthonormal basis, the generalized form of Hooke's law is written as (using the summation convention)

$$\sigma_{ij} = c_{ijk\ell} \ \epsilon_{k\ell}$$

The tensor $\mathbf{c}$ is called the **stiffness tensor** or the **elasticity tensor**. Due to the symmetry of the stress tensor, strain tensor, and stiffness tensor, only 21 elastic coefficients are independent. As stress is measured in units of pressure and strain is dimensionless, the entries of $c_{ijk\ell}$ are also in units of pressure.

The expression for generalized Hooke's law can be inverted to get a relation for the strain in terms of stress:

$$\boldsymbol{\epsilon} = \mathbf{s} : \boldsymbol{\sigma} \qquad \text{or} \qquad \epsilon_{ij} = s_{ijk\ell} \ \sigma_{k\ell} \ .$$

The tensor $\mathbf{s}$ is called the **compliance tensor**.

Generalization for the case of large deformations is provided by models of neo-Hookean solids and Mooney-Rivlin solids.

Isotropic materials

Isotropic materials are characterized by properties which are independent of direction in space. Physical equations involving isotropic materials must therefore be independent of the coordinate system chosen to represent them. The strain tensor is a symmetric tensor. Since the trace of any tensor is independent of any coordinate system, the most complete coordinate-free decomposition of a symmetric tensor is to represent it as the sum of a constant tensor and a traceless symmetric tensor.:Ch. 10 Thus:

$$\varepsilon_{ij} = \left(\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + \left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)$$

where δ_{ij} is the Kronecker delta. In direct tensor notation

$$\boldsymbol{\varepsilon} = \text{vol}(\boldsymbol{\varepsilon}) + \text{dev}(\boldsymbol{\varepsilon}) \ ; \quad \text{vol}(\boldsymbol{\varepsilon}) := \tfrac{1}{3}\ \text{tr}(\boldsymbol{\varepsilon})\ \mathbf{I} \ ; \quad \text{dev}(\boldsymbol{\varepsilon}) := \boldsymbol{\varepsilon} - \text{vol}(\boldsymbol{\varepsilon})$$

where $\mathbf{I}$is the second-order identity tensor. The first term on the right is the constant tensor, also known as the **volumetric strain tensor**, and the second term is the traceless symmetric tensor, also known as the **deviatoric strain tensor** or shear tensor.

The most general form of Hooke's law for isotropic materials may now be written as a linear combination of these two tensors:

$$\sigma_{ij} = 3K\left(\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + 2G\left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) \quad ; \quad \boldsymbol{\sigma} = 3K\ \text{vol}(\boldsymbol{\varepsilon}) + 2G\ \text{dev}(\boldsymbol{\varepsilon})$$

where *K* is the bulk modulus and *G* is the shear modulus.

Using the relationships between the elastic moduli, these equations may also be expressed in various other ways. A common form of Hooke's law for isotropic materials, expressed in direct tensor notation, is

$$\boldsymbol{\sigma} = \lambda\ \text{tr}(\boldsymbol{\varepsilon})\ \mathbf{I} + 2\mu\ \boldsymbol{\varepsilon} = \mathsf{c} : \boldsymbol{\varepsilon} \ ; \quad \mathsf{c} = \lambda\ \mathbf{I}\otimes\mathbf{I} + 2\mu\ \mathsf{I}$$

where $\lambda := K - 2/3G$ and $\mu := G$ are the Lamé constants, $\mathbf{I}$is the second-order identity tensor, and $\mathsf{I}$is the symmetric part of the fourth-order identity tensor. In terms of components with respect to a Cartesian basis,

$$\sigma_{ij} = \lambda\ \varepsilon_{kk}\ \delta_{ij} + 2\mu\ \varepsilon_{ij} = c_{ijk\ell}\ \varepsilon_{k\ell} \ ; \quad c_{ijk\ell} = \lambda\ \delta_{ij}\ \delta_{k\ell} + \mu\ (\delta_{ik}\ \delta_{j\ell} + \delta_{i\ell}\ \delta_{jk})$$

The inverse relationship is

$$\boldsymbol{\varepsilon} = \frac{1}{2\mu}\ \boldsymbol{\sigma} - \frac{\lambda}{2\mu(3\lambda+2\mu)}\ \text{tr}(\boldsymbol{\sigma})\ \mathbf{I} = \frac{1}{2G}\ \boldsymbol{\sigma} + \left(\frac{1}{9K} - \frac{1}{6G}\right)\ \text{tr}(\boldsymbol{\sigma})\ \mathbf{I}$$

Therefore the compliance tensor in the relation $\boldsymbol{\varepsilon} = \mathsf{s} : \boldsymbol{\sigma}$is

$$\mathsf{s} = -\frac{\lambda}{2\mu(3\lambda+2\mu)}\ \mathbf{I}\otimes\mathbf{I} + \frac{1}{2\mu}\ \mathsf{I} = \left(\frac{1}{9K} - \frac{1}{6G}\right)\ \mathbf{I}\otimes\mathbf{I} + \frac{1}{2G}\ \mathsf{I}$$

In terms of Young's modulus and Poisson's ratio, Hooke's law for isotropic materials can then be expressed as

$$\boldsymbol{\varepsilon} = \frac{1}{E}\ \boldsymbol{\sigma} - \frac{\nu}{E}\left[\text{tr}(\boldsymbol{\sigma})\ \mathbf{I} - \boldsymbol{\sigma}\right]$$

This is the form in which the strain is expressed in terms of the stress tensor in engineering. The expression in expanded form is

$$
\begin{aligned}
\varepsilon_{11} &= \tfrac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22}+\sigma_{33})\right] \\
\varepsilon_{22} &= \tfrac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11}+\sigma_{33})\right] \\
\varepsilon_{33} &= \tfrac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11}+\sigma_{22})\right] \\
\varepsilon_{12} &= \tfrac{1}{2G}\,\sigma_{12}\ ; \quad \varepsilon_{13} = \tfrac{1}{2G}\,\sigma_{13}\ ; \quad \varepsilon_{23} = \tfrac{1}{2G}\,\sigma_{23}
\end{aligned}
$$

where *E* is the modulus of elasticity and ν is Poisson's ratio.

In matrix form, Hooke's law for isotropic materials can be written as

$$
\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix}
=
\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix}
= \frac{1}{E}
\begin{bmatrix}
1 & -\nu & -\nu & 0 & 0 & 0\\
-\nu & 1 & -\nu & 0 & 0 & 0\\
-\nu & -\nu & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 2(1+\nu) & 0 & 0\\
0 & 0 & 0 & 0 & 2(1+\nu) & 0\\
0 & 0 & 0 & 0 & 0 & 2(1+\nu)
\end{bmatrix}
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}
$$

where $\gamma_{ij} := 2\varepsilon_{ij}$ is the **engineering shear strain**. The inverse relation may be written as

$$
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}
= \frac{E}{(1+\nu)(1-2\nu)}
\begin{bmatrix}
1-\nu & \nu & \nu & 0 & 0 & 0\\
\nu & 1-\nu & \nu & 0 & 0 & 0\\
\nu & \nu & 1-\nu & 0 & 0 & 0\\
0 & 0 & 0 & (1-2\nu)/2 & 0 & 0\\
0 & 0 & 0 & 0 & (1-2\nu)/2 & 0\\
0 & 0 & 0 & 0 & 0 & (1-2\nu)/2
\end{bmatrix}
\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix}
$$

which expression can be simplified thanks to the Lamé constants:

$$
\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}
=
\begin{bmatrix}
2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0\\
\lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0\\
\lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0\\
0 & 0 & 0 & \mu & 0 & 0\\
0 & 0 & 0 & 0 & \mu & 0\\
0 & 0 & 0 & 0 & 0 & \mu
\end{bmatrix}
\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31}\\ 2\varepsilon_{12}\end{bmatrix}
$$

Plane stress Hooke's law

Under plane stress conditions $\sigma_{33} = \sigma_{31} = \sigma_{23} = 0$. In that case Hooke's law takes the form

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}$$

The inverse relation is usually written in the reduced form

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ 2\varepsilon_{12}\end{bmatrix}$$

Anisotropic materials

The symmetry of the Cauchy stress tensor ($\sigma_{ij} = \sigma_{ji}$) and the generalized Hooke's laws ($\sigma_{ij} = c_{ijk\ell}\ \epsilon_{k\ell}$) implies that $c_{ijk\ell} = c_{jik\ell}$. Similarly, the symmetry of the infinitesimal strain tensor implies that $c_{ijk\ell} = c_{ij\ell k}$. These symmetries are called the **minor symmetries** of the **stiffness tensor** (c).

If in addition, since the displacement gradient and the Cauchy stress are work conjugate, the stress-strain relation can be derived from a strain energy density functional (*U*), then

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \quad \implies \quad c_{ijk\ell} = \frac{\partial^2 U}{\partial \epsilon_{ij}\partial \epsilon_{k\ell}}\ .$$

The arbitrariness of the order of differentiation implies that $c_{ijk\ell} = c_{k\ell ij}$. These are called the **major symmetries** of the stiffness tensor. The major and minor symmetries indicate that the stiffness tensor has only 21 independent components.

Matrix representation (stiffness tensor)

It is often useful to express the anisotropic form of Hooke's law in matrix notation, also called Voigt notation. To do this we take advantage of the symmetry of the stress and strain tensors and express them as six-dimensional vectors in an orthonormal coordinate system ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} ; \quad [\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} \equiv \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

Then the stiffness tensor (C) can be expressed as

$$[\mathsf{C}] = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1131} & c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & c_{2223} & c_{2231} & c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & c_{3323} & c_{3331} & c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & c_{2323} & c_{2331} & c_{2312} \\ c_{3111} & c_{3122} & c_{3133} & c_{3123} & c_{3131} & c_{3112} \\ c_{1211} & c_{1222} & c_{1233} & c_{1223} & c_{1231} & c_{1212} \end{bmatrix} \equiv \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

and Hooke's law is written as

$$[\boldsymbol{\sigma}] = [\mathsf{C}][\boldsymbol{\epsilon}] \qquad \text{or} \qquad \sigma_i = C_{ij}\epsilon_j \ .$$

Similarly the compliance tensor (S) can be written as

$$[\mathsf{S}] = \begin{bmatrix} s_{1111} & s_{1122} & s_{1133} & 2s_{1123} & 2s_{1131} & 2s_{1112} \\ s_{2211} & s_{2222} & s_{2233} & 2s_{2223} & 2s_{2231} & 2s_{2212} \\ s_{3311} & s_{3322} & s_{3333} & 2s_{3323} & 2s_{3331} & 2s_{3312} \\ 2s_{2311} & 2s_{2322} & 2s_{2333} & 4s_{2323} & 4s_{2331} & 4s_{2312} \\ 2s_{3111} & 2s_{3122} & 2s_{3133} & 4s_{3123} & 4s_{3131} & 4s_{3112} \\ 2s_{1211} & 2s_{1222} & 2s_{1233} & 4s_{1223} & 4s_{1231} & 4s_{1212} \end{bmatrix} \equiv \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix}$$

Change of coordinate system

If a linear elastic material is rotated from a reference configuration to another, then the material is symmetric with respect to the rotation if the components of the stiffness tensor in the rotated configuration are related to the components in the reference configuration by the relation

$$c_{pqrs} = l_{pi}\ l_{qj}\ l_{rk}\ l_{s\ell}\ c_{ijk\ell}$$

where l_{ab} are the components of an orthogonal rotation matrix $[L]$. The same relation also holds for inversions.

In matrix notation, if the transformed basis (rotated or inverted) is related to the reference basis by

$$[\mathbf{e}_i'] = [L][\mathbf{e}_i]$$

then

$$C_{ij}\ \epsilon_i\ \epsilon_j = C_{ij}'\ \epsilon_i'\ \epsilon_j'\ .$$

In addition, if the material is symmetric with respect to the transformation $[L]$ then

$$C_{ij} = C_{ij}' \quad \implies \quad C_{ij}\ (\epsilon_i\ \epsilon_j - \epsilon_i'\ \epsilon_j') = 0\ .$$

Orthotropic materials

Orthotropic materials have three orthogonal planes of symmetry. If the basis vectors ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) are normals to the planes of symmetry then the coordinate transformation relations imply that

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

The inverse of this relation is commonly written as

$$\begin{bmatrix}\epsilon_{\rm xx}\\ \epsilon_{\rm yy}\\ \epsilon_{\rm zz}\\ 2\epsilon_{\rm yz}\\ 2\epsilon_{\rm zx}\\ 2\epsilon_{\rm xy}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_{\rm x}} & -\frac{\nu_{\rm xy}}{E_{\rm x}} & -\frac{\nu_{\rm xz}}{E_{\rm x}} & 0 & 0 & 0\\ -\frac{\nu_{\rm yx}}{E_{\rm y}} & \frac{1}{E_{\rm y}} & -\frac{\nu_{\rm yz}}{E_{\rm y}} & 0 & 0 & 0\\ -\frac{\nu_{\rm zx}}{E_{\rm z}} & -\frac{\nu_{\rm zy}}{E_{\rm z}} & \frac{1}{E_{\rm z}} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G_{\rm yz}} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{\rm zx}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{\rm xy}}\end{bmatrix} \begin{bmatrix}\sigma_{\rm xx}\\ \sigma_{\rm yy}\\ \sigma_{\rm zz}\\ \sigma_{\rm yz}\\ \sigma_{\rm zx}\\ \sigma_{\rm xy}\end{bmatrix}$$

where

$E_{\rm i}$ is the Young's modulus along axis *i*
$G_{\rm ij}$ is the shear modulus in direction *j* on the plane whose normal is in direction *i*
$\nu_{\rm ij}$ is the Poisson's ratio that corresponds to a contraction in direction *j* when an extension is applied in direction *i*.

Transversely isotropic materials

A transversely isotropic material is symmetric with respect to a rotation about an axis of symmetry. For such a material, if $\mathbf{e}_3$is the axis of symmetry, Hooke's law can be expressed as

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_3\\\sigma_4\\\sigma_5\\\sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix} \begin{bmatrix}\epsilon_1\\\epsilon_2\\\epsilon_3\\\epsilon_4\\\epsilon_5\\\epsilon_6\end{bmatrix}$$

More frequently, the $x \equiv \mathbf{e}_1$axis is taken to be the axis of symmetry and the inverse Hooke's law is written as

$$\begin{bmatrix}\epsilon_{\rm xx}\\\epsilon_{\rm yy}\\\epsilon_{\rm zz}\\2\epsilon_{\rm yz}\\2\epsilon_{\rm zx}\\2\epsilon_{\rm xy}\end{bmatrix} = \begin{bmatrix} \frac{1}{E_{\rm x}} & -\frac{\nu_{\rm xy}}{E_{\rm x}} & -\frac{\nu_{\rm xy}}{E_{\rm x}} & 0 & 0 & 0 \\ -\frac{\nu_{\rm yx}}{E_{\rm y}} & \frac{1}{E_{\rm y}} & -\frac{\nu_{\rm yz}}{E_{\rm y}} & 0 & 0 & 0 \\ -\frac{\nu_{\rm yx}}{E_{\rm y}} & -\frac{\nu_{\rm zy}}{E_{\rm y}} & \frac{1}{E_{\rm y}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2(1+\nu_{\rm yz})}{E_{\rm y}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{\rm xy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{\rm xy}} \end{bmatrix} \begin{bmatrix}\sigma_{\rm xx}\\\sigma_{\rm yy}\\\sigma_{\rm zz}\\\sigma_{\rm yz}\\\sigma_{\rm zx}\\\sigma_{\rm xy}\end{bmatrix}$$

Thermodynamic basis of Hooke's law

Linear deformations of elastic materials can be approximated as adiabatic. Under these conditions and for quasistatic processes the first law of thermodynamics for a deformed body can be expressed as

$$\delta W = \delta U$$

where δU is the increase in internal energy and δW is the work done by external forces. The work can be split into two terms

$$\delta W = \delta W_s + \delta W_b$$

where δW_s is the work done by surface forces while δW_b is the work done by body forces. If $\delta\mathbf{u}$is a variation of the displacement field $\mathbf{u}$in the body, then the two external work terms can be expressed as

$$\delta W_s = \int_{\partial\Omega} \mathbf{t}\cdot\delta\mathbf{u}~\mathrm{dS}~;~~\delta W_b = \int_{\Omega} \mathbf{b}\cdot\delta\mathbf{u}~\mathrm{dV}$$

where $\mathbf{t}$ is the surface traction vector, $\mathbf{b}$ is the body force vector, Ω represents the body and $\partial\Omega$ represents its surface. Using the relation between the Cauchy stress and the surface traction, $\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}$ (where $\mathbf{n}$ is the unit outward normal to $\partial\Omega$), we have

$$\delta W = \delta U = \int_{\partial\Omega} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}~\mathrm{dS} + \int_{\Omega} \mathbf{b}\cdot\delta\mathbf{u}~\mathrm{dV}$$

Converting the surface integral into a volume integral via the divergence theorem gives

$$\delta U = \int_{\Omega} [\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u}) + \mathbf{b}\cdot\delta\mathbf{u}]~\mathrm{dV}~.$$

Using the symmetry of the Cauchy stress and the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\cdot\mathbf{b}) = (\boldsymbol{\nabla}\cdot\boldsymbol{A})\cdot\mathbf{b} + \tfrac{1}{2}[\boldsymbol{A}^T : \boldsymbol{\nabla}\mathbf{b} + \boldsymbol{A} : (\boldsymbol{\nabla}\mathbf{b})^T]$$

we have

$$\delta U = \int_{\Omega} [\boldsymbol{\sigma} : \tfrac{1}{2}\{\boldsymbol{\nabla}\delta\mathbf{u} + (\boldsymbol{\nabla}\delta\mathbf{u})^T\} + \{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{b}\}\cdot\delta\mathbf{u}]~\mathrm{dV}~.$$

From the definition of strain and from the equations of equilibrium we have

$$\delta\boldsymbol{\epsilon} = \tfrac{1}{2}[\boldsymbol{\nabla}\delta\mathbf{u} + (\boldsymbol{\nabla}\delta\mathbf{u})^T]~;~~\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}~.$$

Hence we can write

$$\delta U = \int_{\Omega} \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}~\mathrm{dV}$$

and therefore the variation in the internal energy density is given by

$$\delta U_0 = \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}~.$$

An elastic material is defined as one in which the total internal energy is equal to the potential energy of the internal forces (also called the **elastic strain energy**). Therefore the internal energy density is a function of the strains, $U_0 = U_0(\boldsymbol{\epsilon})$ and the variation of the internal energy can be expressed as

$$\delta U_0 = \frac{\partial U_0}{\partial \boldsymbol{\epsilon}} : \delta \boldsymbol{\epsilon} \ .$$

Since the variation of strain is arbitrary, the stress-strain relation of an elastic material is given by

$$\boldsymbol{\sigma} = \frac{\partial U_0}{\partial \boldsymbol{\epsilon}} \ .$$

For a linear elastic material, the quantity $\partial U_0/\partial \boldsymbol{\epsilon}$is a linear function of $\boldsymbol{\epsilon}$, and can therefore be expressed as

$$\boldsymbol{\sigma} = \mathsf{c} : \boldsymbol{\epsilon}$$

where $\mathsf{c}$is a fourth-order tensor of material constants, also called the **stiffness tensor**.