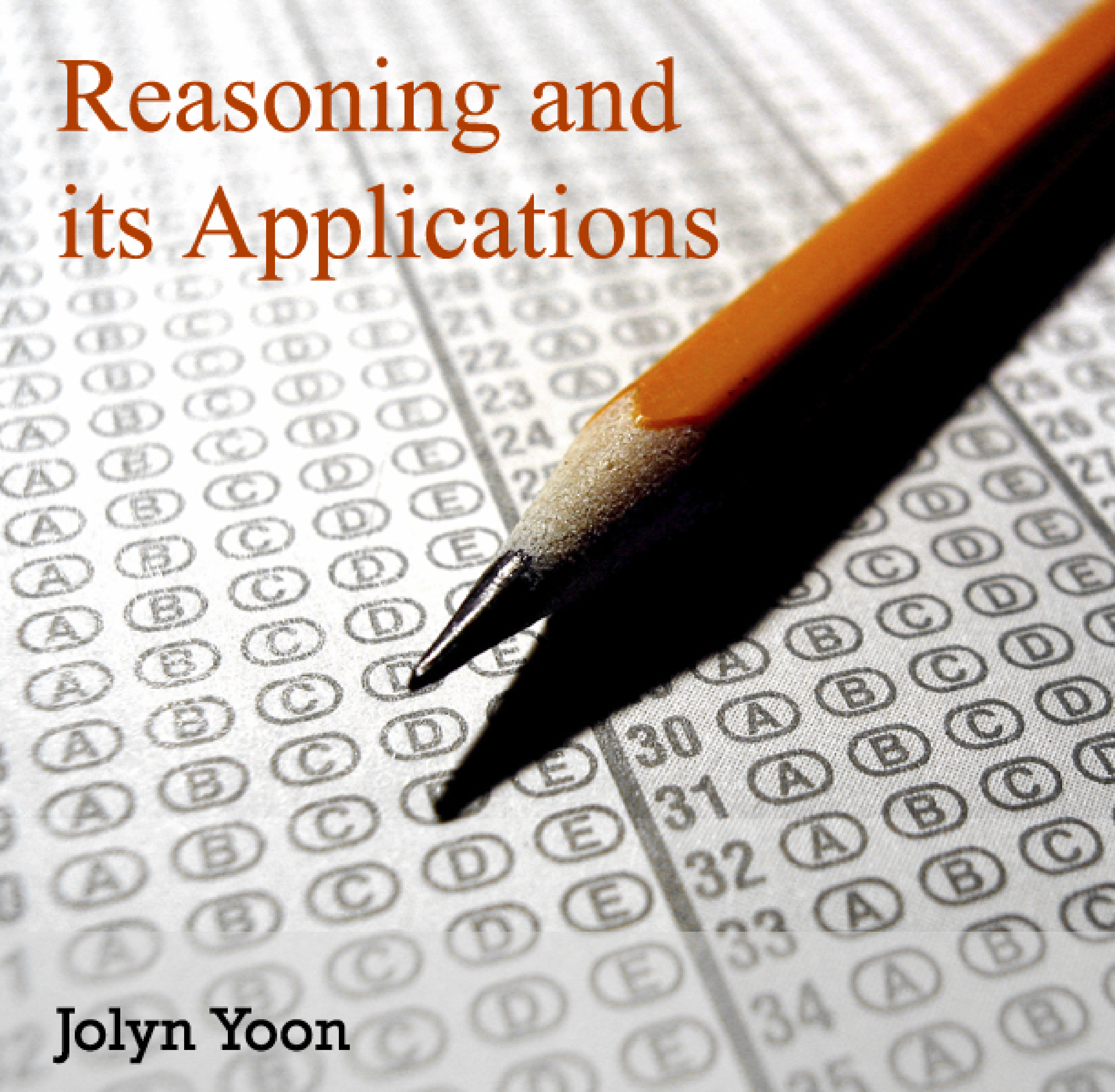
Reasoning and its Applications



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First Edition, 2012

ISBN 978-81-323-2690-8

Published by:
Orange Apple
4735/22 Prakashdeep Bldg,
Ansari Road, Darya Ganj,
Delhi - 110002
Email: info@wtbooks.com

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Chapter- 1

Introduction to Reasoning

Reasoning is the cognitive process of looking for reasons, beliefs, conclusions, actions or feelings.

Different forms of such reflection on reasoning occur in different fields. In philosophy, the study of reasoning typically focuses on what makes reasoning efficient or inefficient, appropriate or inappropriate, good or bad. Philosophers do this by either examining the form or structure of the reasoning within arguments, or by considering the broader methods used to reach particular goals of reasoning. Psychologists and cognitive scientists, in contrast, tend to study how people reason, which cognitive and neural processes are engaged, how cultural factors affect the inferences people draw. The properties of logic which may be used to reason are studied in mathematical logic. The field of automated reasoning studies how reasoning may be modelled computationally. Lawyers also study reasoning.

History of reasoning

It is likely that humans have used reasoning to work out what they should believe or do for a very long time. However, some researchers have tried to determine when, in the history of human development, humans began using formal techniques of reasoning.

Babylonian reasoning

In Mesopotamia, Esagil-kin-apli's medical Diagnostic Handbook written in the 11th century BC was based on a logical set of axioms and assumptions, including the modern view that through the examination and inspection of the symptoms of a patient, it is possible to determine the patient's disease, its aetiology and future development, and the chances of the patient's recovery.

During the 8th and 7th centuries BC, Babylonian astronomers began employing an internal logic within their predictive planetary systems, which was an important contribution to logic and the philosophy of science. Babylonian thought had a considerable influence on early Greek thought.

Greek reasoning

The works of Homer, written in the 8th century BC, contain mythic stories that use gods to explain the formation of the world. However, only two centuries later, late in the 6th century BC, Xenophanes of Colophon began to question the Homeric accounts of the creation of nature and the gods. He wrote:

- "Homer and Hesiod attribute all things to the gods that among men are shame and a disgrace" (frag. 11).
- "God is one, greatest among gods and among men, in no way like men in form and thought" (frag. 23).
- "If oxen and horses and lions had hands or could paint and make things with their hands like men, then they would paint the forms of gods and make their bodies each according to their own shapes, horses like horses, oxen like oxen" (frag. 15).

According to David Furley, "the basis of [Xenophanes'] criticism appears to have been that he saw an inconsistency between the concept of god as something different from man, and the stories told about the gods, which made them behave as men do." In the same period, other Greek thinkers began to develop theories about the nature of the world that suggest that they believed that there were regularities in nature and that humans could use reasoning to develop a consistent story about the nature of the world. Thales of Miletus, c. 624 BC – c. 546 BC, proposed that all is water. Anaximenes of Miletus, c. 585 BC – c. 525 BC, claimed that air is the source of everything.

Aristotle is, so far as we know, the first writer to give an extended, systematic treatment of the methods of human reasoning. He identified two major methods of reasoning, analysis and synthesis. In the first, we try to understand an object by looking at its component parts. In the second, we try to understand a class of objects by looking at the common properties of each object in that class.

Aristotle developed what is known as syllogistic logic, which makes it possible to analyse reasoning in a way that ignores the content of the argument and focuses on the form or structure of the argument. In the Prior Analytics, Aristotle begins by pointing out that:

"[If] no pleasure is a good, neither will any good be a pleasure."

He then argues that this argument is an example of a rule of reasoning of the following form:

Premise: "Aristotle is Greek" and "All Greeks are human"
Conclusion: "Aristotle is human"

Aristotle points out that by understanding the reasoning involved in this type of argument, we can know that whatever the As and Bs are, we can reach the same conclusion about the relationship between them. This is a simple and straightforward argument, but it is a sign of an amazing leap in understanding and research into reason and was the beginning of the development of formal logic.

Aristotle's system of logic was responsible for the introduction of hypothetical syllogism, temporal modal logic, and inductive logic.

Indian reasoning

Two of the six Indian schools of thought deal with logic: Nyaya and Vaisheshika. The Nyaya Sutras of Aksapada Gautama constitute the core texts of the Nyaya school, one of the six orthodox schools of Hindu philosophy. This realist school developed a rigid five-member schema of inference involving an initial premise, a reason, an example, an application and a conclusion. The idealist Buddhist philosophy became the chief opponent to the Naiyayikas. Nagarjuna, the founder of the Madhyamika "Middle Way" developed an analysis known as the "catuskoti" or tetralemma. This four-cornered argumentation systematically examined and rejected the affirmation of a proposition, its denial, the joint affirmation and denial, and finally, the rejection of its affirmation and denial. But it was with Dignaga and his successor Dharmakirti that Buddhist logic reached its height. Their analysis centred on the definition of necessary logical entailment, "vyapti", also known as invariable concomitance or pervasion. To this end a doctrine known as "apoha" or differentiation was developed. This involved what might be called inclusion and exclusion of defining properties. The difficulties involved in this enterprise, in part, stimulated the neo-scholastic school of Navya-Nyāya, which developed a formal analysis of inference in the 16th century.

Chinese reasoning

In China, a contemporary of Confucius, Mozi, "Master Mo", is credited with founding the Mohist school, whose canons dealt with issues relating to valid inference and the conditions of correct conclusions. In particular, one of the schools that grew out of Mohism, the Logicians, are credited by some scholars for their early investigation of formal logic. However, due to the harsh rule of Legalism in the subsequent Qin Dynasty, this line of investigation disappeared in China until the introduction of Indian philosophy by Buddhists.

Islamic reasoning

For a time after prophet Muhammad's death, Islamic law placed importance on formulating standards of argument, which gave rise to a novel approach to logic in Kalam, but this approach was later influenced by ideas from Greek philosophy and Hellenistic philosophy with the rise of the Mu'tazili philosophers, who highly valued Aristotle's Organon. The works of Hellenistic-influenced Islamic philosophers were crucial in the reception of Aristotelian logic in medieval Europe, along with the commentaries on the Organon by Averroes. The works of al-Farabi, Avicenna, al-Ghazali and other Muslim logicians who often criticized and corrected Aristotelian logic and introduced their own forms of logic, also played a central role in the subsequent development of medieval European logic.

Islamic logic not only included the study of formal patterns of inference and their validity but also elements of the philosophy of language and elements of epistemology and metaphysics. Due to disputes with Arabic grammarians, Islamic philosophers were very interested in working out the relationship between logic and language, and they devoted much discussion to the question of the subject matter and aims of logic in relation to reasoning and speech. In the area of formal logical analysis, they elaborated upon the theory of terms, propositions and syllogisms. They considered the syllogism to be the form to which all rational argumentation could be reduced, and they regarded syllogistic theory as the focal point of logic. Even poetics was considered as a syllogistic art in some fashion by many major Islamic logicians.

Important developments made by Muslim logicians included the development of "Avicenna's logic" as a replacement of Aristotelian logic. Other important developments in Islamic philosophy include the development of a strict citation practice, the isnad or "backing", and the development of a scientific method of open inquiry to disprove claims, the ijtihad, which could be generally applied to many types of questions.

Reasoning methods and argumentation

One approach to the study of reasoning is to identify various forms of reasoning that may be used to support or justify conclusions. The main division between forms of reasoning that is made in philosophy is between deductive reasoning and inductive reasoning. Formal logic has been described as "the science of deduction". The study of inductive reasoning is generally carried out within the field known as informal logic or critical thinking.

Behavioral experiments on human reasoning

Experimental cognitive psychologists carry out research on reasoning behaviour. Such research may focus, for example, on how people perform on tests of reasoning such as intelligence or IQ tests, or on how well people's reasoning matches ideals set by logic. Experiments examine how people make inferences from conditionals e.g., If A then B and how they make inferences about alternatives, e.g., A or else B. They test whether people can make valid deductions about spatial and temporal relations, e.g., A is to the left of B, or A happens after B, and about quantified assertions, e.g., All the A are B. Experiments investigate how people make inferences about factual situations, hypothetical possibilities, probabilities, and counterfactual situations.

Developmental studies of children's reasoning

Developmental psychologists investigate the development of reasoning from birth to adulthood. Piaget's theory of cognitive development was the first complete theory of reasoning development. Subsequently, several alternative theories were proposed, including the neo-Piagetian theories of cognitive development.

Neuroscience of reasoning

The biological functioning of the brain is studied by neurophysiologists and neuropsychologists. Research in this area includes research into the structure and function of normally functioning brains, and of damaged or otherwise unusual brains. In addition to carrying out research into reasoning, some psychologists, for example, clinical psychologists and psychotherapists work to alter people's reasoning habits when they are unhelpful..

Legal reasoning

Legal reasoning is used when reflecting on the nature of existing laws or when reaching decisions about the relationship between laws and particular court cases.

Thorne McCarty did pioneering early work in the mechanisation of legal reasoning for taxation using Micro Planner. More recent work on the formalisation and mechanisation of legal reasoning can be found in the proceedings of the International Conferences on Artificial Intelligence and Law (most recently at Stanford in June 2007).

Chapter- 2

Deductive Reasoning and Inductive Reasoning

Deductive reasoning

Deductive reasoning, also called **deductive logic**, is reasoning which constructs or evaluates deductive arguments. Deductive arguments are attempts to show that a conclusion necessarily follows from a set of premises. A deductive argument is valid if the conclusion does follow necessarily from the premises, i.e., if the conclusion must be true provided that the premises are true. A deductive argument is sound if it is valid and its premises are true. Deductive arguments are valid or invalid, sound or unsound, but are never false nor true. Deductive Reasoning is a method of gaining knowledge. It was advanced by the French Philosopher and mathematician René Descartes: (1596–1650).

An example of a deductive argument:

1. All men are mortal
2. Socrates is a man
3. Therefore, Socrates is mortal

The first premise states that all objects classified as 'men' have the attribute 'mortal'. The second premise states that 'Socrates' is classified as a man- a member of the set 'men'. The conclusion states that 'Socrates' must be mortal because he inherits this attribute from his classification as a man. Deductive reasoning is sometimes contrasted with inductive reasoning.

Deductive logic

Deductive arguments are generally evaluated in terms of their validity and soundness. An argument is valid if it is impossible both for its premises to be true and its conclusion to be false. An argument can be valid even though the premises are false.

This is an example of a valid argument. The first premise is false, yet the argument is still valid.

1. Everyone who eats steak is a quarterback.
2. John eats steak.
3. Therefore, John is a quarterback.

This argument is valid but not sound. For a deductive argument to be considered sound the argument must not only be valid, but the premises must be true as well.

A theory of deductive reasoning known as categorical or term logic was developed by Aristotle, but was superseded by propositional (sentential) logic and predicate logic.

Inductive reasoning can be contrasted with deductive reasoning. In cases of inductive reasoning, it is possible for the conclusion to be false even though the premises are true and the argument's form is cogent.

Hume's Skepticism

Philosopher David Hume presented grounds to doubt deduction by questioning induction. Hume's problem of induction starts by suggesting that the use of even the simplest forms of induction simply cannot be justified by inductive reasoning itself. Moreover, induction cannot be justified by deduction either. Therefore, induction cannot be justified rationally. Consequentially, if induction is not yet justified, then deduction seems to be left to rationally justify itself – an objectionable conclusion to Hume.

Hume did not provide a strictly rational solution per se. He simply explained that we cannot help but induce, but that it is lucky that we do so. Certainly we must appeal to first principles of some kind, including laws of thought.

Inductive reasoning

Inductive reasoning, also known as **induction** or **inductive logic**, or **educated guess** in colloquial English, is a kind of reasoning that draws generalized conclusions from a finite collection of specific observations. The premises of an inductive logical argument indicate some degree of support (inductive probability) for the conclusion but do not entail it; that is, they suggest truth but do not ensure it.

Induction is employed, for example, in the following argument:

All of the ice we have examined so far is cold. (Specific observations)
Therefore, all ice is cold. (Generalized conclusion)

Inductive reasoning allows for the possibility that the conclusion is false, even where all of the premises are true. For example:

All of the swans we have seen are white.
All swans are white. (Only if we disregard Black Swans)

Note that this definition of inductive reasoning excludes mathematical induction, which is considered to be a form of deductive reasoning.

Strong and weak induction

The words 'strong' and 'weak' are sometimes used to praise or demean the goodness of an inductive argument. The idea is that you say "this is an example of strong induction" when you would decide to believe the conclusion if presented with the premises. Alternatively, you say "that is weak induction" when your particular world view does not allow you to see that the conclusions are likely given the premises.

Strong induction

The equation, "the gravitational force between two objects equals the gravitational constant times the product of the masses divided by the distance between them squared," has allowed us to describe the rate of fall of all objects we have observed.
Therefore:
The gravitational force between two objects equals the gravitational constant times the product of the masses divided by the distance between them squared.

The conclusion of this argument is not absolutely certain, even given the premise. At speeds we normally experience, Newtonian mechanics holds quite well. But at speeds approaching that of light, the Newtonian system is not accurate and the conclusion in that case would be false. However, since, in most cases that we experience, the premise as stated would usually lead to the conclusion given, we are logical in calling this argument an instance of strong induction.

Weak induction

Consider this example:

I always hang pictures on nails.
Therefore:
All pictures hang from nails.

Here, the link between the premise and the conclusion is very weak. Not only is it possible for the conclusion to be false given the premise, it is even fairly likely that the conclusion is false. Not all pictures are hung from nails; moreover, not all pictures are hung. Thus we say that this argument is an instance of weak induction.

Problem of induction

The **problem of induction** is the philosophical question of whether inductive reasoning leads to knowledge. That is, what is the justification for either:

1. generalizing about the properties of a class of objects based on some number of observations of particular instances of that class (for example, the inference that "all swans we have seen are white, and therefore all swans are white," before the discovery of black swans) or
2. presupposing that a sequence of events in the future will occur as it always has in the past (for example, that the laws of physics will hold as they have always been observed to hold). Hume called this the Principle of Uniformity of Nature.

The problem calls into question all empirical claims made in everyday life or through the scientific method. Although the problem arguably dates back to the Pyrrhonism of ancient philosophy, David Hume introduced it in the mid-18th century, with the most notable response provided by Karl Popper two centuries later. A more recent, probability-based extension is the "no-free-lunch theorem for supervised learning" of Wolpert and Macready.

Formulation of the problem

Induction as a method cannot be justified using induction, nor deduction, or therefore at all by pure reason

In inductive reasoning, one makes a series of observations and infers a new claim based on them. For instance, from a series of observations that a woman walks her dog by the market at 8am on Monday, it seems valid to infer that next Monday she will do the same, or that, in general, the woman walks her dog by the market every Monday. That next Monday the woman walks by the market merely adds to the series of observations, it does not prove she will walk by the market every Monday. First of all, it is not certain, regardless of the number of observations, that the woman always walks by the market at 8am on Monday. In fact, Hume would even argue that we cannot claim it is "more probable", since this still requires the assumption that the past predicts the future. Second, the observations themselves do not establish the validity of inductive reasoning, except inductively.

Ancient and early modern origins

Pyrrhonian skeptic Sextus Empiricus first questioned the validity of inductive reasoning, positing that a universal rule could not be established from an incomplete set of particular instances. He wrote :

When they propose to establish the universal from the particulars by means of induction, they will effect this by a review of either all or some of the particulars. But if they review some, the induction will be insecure, since some of the particulars omitted in the induction may contravene the universal; while if they are to review all, they will be toiling at the impossible, since the particulars are infinite and indefinite.

The focus upon the gap between the premises and conclusion present in the above passage appears different from Hume's focus upon the circular reasoning of induction. However, Weintraub claims in The Philosophical Quarterly that although Sextus' approach to the problem appears different, Hume's approach was actually an application of another argument raised by Sextus :

Those who claim for themselves to judge the truth are bound to possess a criterion of truth. This criterion, then, either is without a judge's approval or has been approved. But if it is without approval, whence comes it that it is truthworthy? For no matter of dispute is to be trusted without judging. And, if it has been approved, that which approves it, in turn, either has been approved or has not been approved, and so on ad infinitum.

Although the criterion argument applies to both deduction and induction, Weintraub believes that Sextus' argument "is precisely the strategy Hume invokes against induction: it cannot be justified, because the purported justification, being inductive, is circular." She concludes that "Hume's most important legacy is the supposition that the justification of induction is not analogous to that of deduction." She ends with a discussion of Hume's implicit sanction of the validity of deduction, which Hume describes as intuitive in a manner analogous to modern foundationalism.

Medieval writers such as al-Ghazali and William of Ockham connected the problem with God's absolute power, asking how we can be certain that the world will continue behaving as expected when God could at any moment miraculously cause the opposite. Duns Scotus however argued that inductive inference from a finite number of particulars to a universal generalization was justified by "a proposition reposing in the soul, 'Whatever occurs in a great many instances by a cause that is not free, is the natural effect of that cause.'" Some 17th century Jesuits argued that although God could create the end of the world at any moment, it was necessarily a rare event and hence our confidence that it would not happen very soon was largely justified.

David Hume

David Hume described the problem in An Enquiry concerning Human Understanding, §4, based on his epistemological framework. Here, "reason" refers to deductive reasoning and "induction" refers to inductive reasoning.

First, Hume ponders the discovery of causal relations, which form the basis for what he refers to as "matters of fact." He argues that causal relations are found not by reason, but by induction. This is because for any cause, multiple effects are conceivable, and the actual effect cannot be determined by reasoning about the cause; instead, one must observe occurrences of the causal relation to discover that it holds. For example, when one thinks of "a billiard ball moving in a straight line toward another," one can conceive that the first ball bounces back with the second ball remaining at rest, the first ball stops and the second ball moves, or the first ball jumps over the second, etc. There is no reason to conclude any of these possibilities over the others. Only through previous observation can it be predicted, inductively, what will actually happen with the balls. In general, it is not necessary that causal relation in the future resemble causal relations in the past, as it is always conceivable otherwise; for Hume, this is because the negation of the claim does not lead to a contradiction.

Next, Hume ponders the justification of induction. If all matters of fact are based on causal relations, and all causal relations are found by induction, then induction must be shown to be valid somehow. He uses the fact that induction assumes a valid connection between the proposition "I have found that such an object has always been attended with such an effect" and the proposition "I foresee that other objects which are in appearance similar will be attended with similar effects." One connects these two propositions not by reason, but by induction. This claim is supported by the same reasoning as that for causal relations above, and by the observation that even rationally inexperienced or inferior people can infer, for example, that touching fire causes pain. Hume challenges other philosophers to come up with a (deductive) reason for the connection. If he is right, then the justification of induction can be only inductive. But this begs the question; as induction is based on an assumption of the connection, it cannot itself explain the connection.

In this way, the problem of induction is not only concerned with the uncertainty of conclusions derived by induction, but doubts the very principle through which those uncertain conclusions are derived.

Nelson Goodman's New Problem of Induction

Nelson Goodman presented a different description of the problem of induction in the chapter of [[Fact, Fiction, and Forecast]] entitled "The New Riddle of Induction" (1954). Goodman proposed a new predicate, "grue". Something is grue if and only if it has been observed to be green before a certain time or blue after that time. The "new" problem of induction is, since all emeralds we have ever seen are both green and grue, why do we suppose that after time t we will find green but not grue emeralds? The standard scientific response is to invoke Occam's razor.

Goodman, however, points out that the predicate "grue" only appears more complex than the predicate "green" because we have defined grue in terms of blue and green. If we had always been brought up to think in terms of "grue" and "bleen" (where bleen is blue before time t, or green thereafter), we would intuitively consider "green" to be a crazy and complicated predicate. Goodman believed that which scientific hypotheses we favour depend on which predicates are "entrenched" in our language.

W.V.O. Quine offers the most practicable solution to this problem by making the metaphysical claim that only predicates that identify a "natural kind" (i.e. a real property of real things) can be legitimately used in a scientific hypothesis.

Interpretations and proposed explanations

Hume

Although induction is not made by reason, Hume observes that we nonetheless perform it and improve from it. He proposes a descriptive explanation for the nature of induction in §5 of the Enquiry, titled "Skeptical solution of these doubts". It is by custom or habit that one draws the inductive connection described above, and "without the influence of custom we would be entirely ignorant of every matter of fact beyond what is immediately present to the memory and senses." The result of custom is belief, which is instinctual and much stronger than imagination alone.

Rather than unproductive radical skepticism about everything, Hume said that he was actually advocating a practical skepticism based on common sense, wherein the inevitability of induction is accepted. Someone who insists on reason for certainty might, for instance, starve to death, as they would not infer the benefits of food based on previous observations of nutrition.

Colin Howson

Colin Howson interpreted Hume to say that an inductive inference must be backed, not only by observations, but also by an independent "inductive assumption." Howson combined this idea with Frank P. Ramsey's view on probabilistic reasoning to conclude that "there is a genuine logic of induction which exhibits inductive reasoning as logically quite sound given suitable premisses, but does not justify those premisses." In other words, there is logic to induction, but it relies on premises. In this sense, the strength of inductive reasoning is comparable to the strength (or lack thereof) of deductive reasoning.

David Stove and Donald Williams

David Stove's principal positive argument for induction was presented in the Rationality of Induction and was developed from an argument put forward by one of Stove's heroes, the late Donald Cary Williams (formerly Professor at Harvard) in his book The Ground of Induction. Stove argued that it is a statistical truth that the great majority of the possible subsets of specified size (as long as this size is not too small) are similar to the larger population to which they belong. For example, the majority of the subsets which contain 3000 ravens which you can form from the raven population are similar to the population itself (and this applies no matter how large the raven population is, as long as it is not infinite). Consequently, Stove argued that if you find yourself with such a subset then the chances are that this subset is one of the ones that are similar to the population, and so you are justified in concluding that it is likely that this subset 'matches' the population reasonably closely. The situation would be analogous to drawing a ball out of a barrel of balls, 99% of which are red. In such a case you have a 99% chance of drawing a red ball. Similarly, when getting a sample of ravens the probability is very high that the sample is one of the matching or 'representative' ones. So as long as you have no reason to think that your sample is one unrepresentative you are justified in thinking that probably (although not certainly) that it is.

Karl Popper

Karl Popper, a philosopher of science, sought to solve the problem of induction . He argued that science does not use induction, and induction is in fact a myth . Instead, knowledge is created by conjecture and criticism. The main role of observations and experiments in science, he argued, is in our attempts to criticize and refute existing theories .

According to Popper, the problem of induction as usually conceived is asking the wrong question: it is asking how we can justify our theories given they cannot be justified by induction. Popper argued that we do not need justification at all, and seeking justification "begs for an authoritarian answer". Instead, Popper said, we should be looking to find and correct errors. Popper regarded theories that have survived criticism as better corroborated in proportion to the amount and stringency of the criticism, but, in sharp contrast to the inductivist theories of knowledge, emphatically as less likely to be true. Popper held that seeking for theories with a high probability of being true was a false goal that is in conflict with the search for knowledge. Science should seek for theories that are most probably false on the one hand (which is the same as saying that they are highly falsifiable and so there are lots of ways that they could turn out to be wrong), but still all actual attempts to falsify them have failed so far (that they are highly corroborated).

Wesley C. Salmon criticises Popper on the grounds that predictions need to be made both for practical purposes and in order to test theories. That means Popperians need to make a selection from the number of unfalsified theories available to them, which is generally more than one. Popperians would wish to choose well-corroborated theories, in their sense of corroboration, but face a dilemma: either they are making the essentially inductive claim that a theory's having survived criticism in the past means it will be a reliable predictor in the future; or Popperian corroboration is no indicator of predictive power at all, so there is no rational motivation for their preferred selection principle.

David Miller has criticized this kind of critism of Salmon and others, because it makes inductivist assumptions. Popper does not say that corroboration is an indicator of predictive power. The predictive power is in the theory itself, not in its corroboration. The rational motivation for choosing a well-corrobated theory is that it is simply easier to falsify: Well-corroborated means that at least one kind of experiment (already conducted at least once) could (but did not) falsify the one theory, while the same kind of experiment, regardless of its outcome, would not falsify the other. So it is rational to chose the well-corroborated theory; it may not be more likely to be true, but at least it is easier to get rid of if not.

Other consequences

The most commonly suggested consequence of Hume's Problem of Induction is some kind of wide spread skepticism or fallibilism . This is especially so because Hume's skepticism did not end with an attack on induction. If his argument is successful, it means that induction cannot be used to justify truth claims, and deduction may be lost as well. In other words, since deduction can no longer be justified by induction, and it would be circular to use deduction to justify itself, it seems that it too is unjustifiable by pure reason. This argument seems to suggest that there are necessarily some initial premises (e.g. laws of thought) for any belief system that cannot be justified by reason alone, but rather must be granted before discourse or thought can begin .

Types of inductive reasoning

Generalization

A generalization (more accurately, an inductive generalization) proceeds from a premise about a sample to a conclusion about the population.

The proportion Q of the sample has attribute A.
Therefore:
The proportion Q of the population has attribute A.
Example

There are 20 balls in an urn, either black or white. To estimate their respective numbers you draw a sample of 4 balls and find that 3 are black, one is white. A good inductive generalisation would be: there are 15 black and 5 white balls in the urn.

How great the support is which the premises provide for the conclusion is dependent on (a) the number of individuals in the sample group compared to the number in the population; and (b) the degree to which the sample is representative of the population (which may be achieved by taking a random sample). The hasty generalization and biased sample are fallacies related to generalisation.

Statistical syllogism

A statistical syllogism proceeds from a generalization to a conclusion about an individual.

A proportion Q of population P has attribute A.
An individual X is a member of P.
Therefore:
There is a probability which corresponds to Q that X has A.

The proportion in the first premise would be something like "3/5ths of", "all", "few", etc. Two dicto simpliciter fallacies can occur in statistical syllogisms: "accident" and "converse accident".

Simple induction

Simple induction proceeds from a premise about a sample group to a conclusion about another individual.

Proportion Q of the known instances of population P has attribute A.
Individual I is another member of P.
Therefore:
There is a probability corresponding to Q that I has A.

This is a combination of a generalization and a statistical syllogism, where the conclusion of the generalization is also the first premise of the statistical syllogism.

Argument from analogy

Some philosophers believe that an argument from analogy is a kind of inductive reasoning.

An argument from analogy has the following form:

I has attributes A, B, and C
J has attributes A and B
So, J has attribute C

An analogy relies on the inference that the attributes known to be shared (the similarities) imply that C is also a shared property. The support which the premises provide for the conclusion is dependent upon the relevance and number of the similarities between I and J. The fallacy related to this process is false analogy. As with other forms of inductive argument, even the best reasoning in an argument from analogy can only make the conclusion probable given the truth of the premises, not certain.

Analogical reasoning is very frequent in common sense, science, philosophy and the humanities, but sometimes it is accepted only as an auxiliary method. A refined approach is case-based reasoning.

Causal inference

A causal inference draws a conclusion about a causal connection based on the conditions of the occurrence of an effect. Premises about the correlation of two things can indicate a causal relationship between them, but additional factors must be confirmed to establish the exact form of the causal relationship.

Prediction

A prediction draws a conclusion about a future individual from a past sample.

Proportion Q of observed members of group G have had attribute A.
Therefore:
There is a probability corresponding to Q that other members of group G will have attribute A when next observed.

Bayesian inference

Of the candidate systems for an inductive logic, the most influential is Bayesianism. As a logic of induction rather than a theory of belief, Bayesianism does not determine which beliefs are a priori rational, but rather determines how we should rationally change the beliefs we have when presented with evidence. We begin by committing to a (really any) hypothesis, and when faced with evidence, we adjust the strength of our belief in that hypothesis in a precise manner using Bayesian logic.

Chapter- 3

Abductive Reasoning

Abduction is a kind of logical inference described by Charles Sanders Peirce as "guessing". The term refers to the process of arriving at an explanatory hypothesis. Peirce said that to abduce a hypothetical explanation a from an observed surprising circumstance b is to surmise that a may be true because then b would be a matter of course. Thus, to abduce a from b involves determining that a is sufficient (or nearly sufficient), but not necessary, for b.

For example, the lawn is wet. But if it rained last night, then it would be unsurprising that the lawn is wet. Therefore, by abductive reasoning, it rained last night. (But note that Peirce did not remain convinced that a single logical form covers all abduction.)

Peirce argues that good abductive reasoning from P to Q involves not simply a determination that, e.g., Q is sufficient for P, but also that Q is among the most economical explanations for P. Simplification and economy are what call for the 'leap' of abduction.

There has been renewed interest in the subject of abduction in the fields of computer science and artificial intelligence research.

Deduction, induction, and abduction

Deduction

    allows deriving b from a only where b is a formal consequence of a. In other words, deduction is the process of deriving the consequences of what is assumed. Given the truth of the assumptions, a valid deduction guarantees the truth of the conclusion. For example, given that all bachelors are unmarried males, and given that this person is a bachelor, it can be deduced that this person is an unmarried male.

Induction

    allows inferring b from a, where b does not follow necessarily from a. a might give us very good reason to accept b, but it does not ensure that b. For example, if all of the swans that we have observed so far are white, we may induce that all swans are white. We have good reason to believe the conclusion from the premise, but the truth of the conclusion is not guaranteed. (Indeed, it turns out that some swans are black.)

Abduction

allows inferring a as an explanation of b. Because of this, abduction allows the precondition a to be abduced from the consequence b. Deduction and abduction thus differ in the direction in which a rule like "a entails b" is used for inference. As such abduction is formally equivalent to the logical fallacy affirming the consequent or Post hoc ergo propter hoc, because there are multiple possible explanations for b. For example, after glancing up and seeing the eight ball moving towards us we may abduce that it was struck by the cue ball. The cue ball's strike would account for the eight ball's movement. It serves as a hypothesis that explains our observation. There are in fact infinitely many possible explanations for the eight ball's movement, and so our abduction does not leave us certain that the cue ball did in fact strike the eight ball, but our abduction is still useful and can serve to orient us in our surroundings. This process of abduction is an instance of the scientific method. There are infinite possible explanations for any of the physical processes we observe, but we are inclined to abduce a single explanation (or a few explanations) for them in the hopes that we can better orient ourselves in our surroundings and eliminate some of the possibilities.

Formalizations of abduction

Logic-based abduction

In logic, explanation is done from a logical theory T representing a domain and a set of observations O. Abduction is the process of deriving a set of explanations of O according to T and picking out one of those explanations. For E to be an explanation of O according to T, it should satisfy two conditions:

- O follows from E and T;
- E is consistent with T.

In formal logic, O and E are assumed to be sets of literals. The two conditions for E being an explanation of O according to theory T are formalized as:

$T \cup E \models O$;
$T \cup E$ is consistent.

Among the possible explanations E satisfying these two conditions, some other condition of minimality is usually imposed to avoid irrelevant facts (not contributing to the entailment of O) being included in the explanations. Abduction is then the process that picks out some member of E. Criteria for picking out a member representing "the best" explanation include the simplicity, the prior probability, or the explanatory power of the explanation.

A proof theoretical abduction method for first order classical logic based on the sequent calculus and a dual one, based on semantic tableaux (analytic tableaux) have been proposed (Cialdea Mayer & Pirri 1993). The methods are sound and complete and work for full first order logic, without requiring any preliminary reduction of formulae into normal forms. These methods have also been extended to modal logic.

Abductive logic programming is a computational framework that extends normal logic programming with abduction. It separates the theory T into two components, one of which is a normal logic program, used to generate E by means of backward reasoning, the other of which is a set of integrity constraints, used to filter the set of candidate explanations.

Set-cover abduction

A different formalization of abduction is based on inverting the function that calculates the visible effects of the hypotheses. Formally, we are given a set of hypotheses H and a set of manifestations M; they are related by the domain knowledge, represented by a function e that takes as an argument a set of hypotheses and gives as a result the corresponding set of manifestations. In other words, for every subset of the hypotheses $H' \subseteq H$, their effects are known to be e(H').

Abduction is performed by finding a set $H' \subseteq H$ such that $M \subseteq e(H')$. In other words, abduction is performed by finding a set of hypotheses H' such that their effects e(H') include all observations M.

A common assumption is that the effects of the hypotheses are independent, that is, for every $H' \subseteq H$, it holds that $e(H') = \bigcup_{h \in H'} e(\{h\})$. If this condition is met, abduction can be seen as a form of set covering.

Abductive validation

Abductive validation is the process of validating a given hypothesis through abductive reasoning. This can also be called reasoning through successive approximation. Under this principle, an explanation is valid if it is the best possible explanation of a set of known data. The best possible explanation is often defined in terms of simplicity and elegance. Abductive validation is common practice in hypothesis formation in science; moreover, Peirce argues it is a ubiquitous aspect of thought:

Looking out my window this lovely spring morning I see an azalea in full bloom. No, no! I do not see that; though that is the only way I can describe what I see. That is a proposition, a sentence, a fact; but what I perceive is not proposition, sentence, fact, but only an image, which I make intelligible in part by means of a statement of fact. This statement is abstract; but what I see is concrete. I perform an abduction when I so much as express in a sentence anything I see. The truth is that the whole fabric of our knowledge is one matted felt of pure hypothesis confirmed and refined by induction. Not the smallest advance can be made in knowledge beyond the stage of vacant staring, without making an abduction at every step.

It was Peirce's own maxim that "Facts cannot be explained by a hypothesis more extraordinary than these facts themselves; and of various hypotheses the least extraordinary must be adopted." After obtaining results from an inference procedure, we may be left with multiple assumptions, some of which may be contradictory. Abductive validation is a method for identifying the assumptions that will lead to your goal.

Probabilistic abduction

Probabilistic abductive reasoning is a form of abductive validation, and is used extensively in areas where conclusions about possible hypotheses need to be derived, such as for making diagnoses from medical tests. For example, a pharmaceutical company that develops a test for a particular infectious disease will typically determine the reliability of the test by letting a group of infected and a group of non-infected people undergo the test. Assume the statements x: "Positive test", $\overline{x}$: "Negative test", y: "Infected", and $\overline{y}$: "Not infected". The result of these trials will then determine the reliability of the test in terms of its sensitivity p(x | y) and false positive rate $p(x|\overline{y})$. The interpretations of the conditionals are: p(x | y): "The probability of positive test given infection", and $p(x|\overline{y})$: "The probability of positive test in the absence of infection". The problem with applying these conditionals in a practical setting is that they are expressed in the opposite direction to what the practitioner needs. The conditionals needed for making the diagnosis are: p(y | x): "The probability of infection given positive test", and $p(y|\overline{x})$: "The probability of infection given negative test". The probability of infection could then have been conditionally deduced as $p(y\|x) = p(x)p(y|x) + p(\overline{x})p(y|\overline{x})$, where "$\|$" denotes conditional deduction. Unfortunately the required conditionals are usually not directly available to the medical practitioner, but they can be obtained if the base rate of the infection in the population is known.

The required conditionals can be correctly derived by inverting the available conditionals using Bayes rule. The inverted conditionals are obtained as follows:

$$\begin{cases} p(x|y) = \frac{p(x \wedge y)}{p(y)} \\ p(y|x) = \frac{p(x \wedge y)}{p(x)} \end{cases} \Rightarrow \quad p(y|x) = \frac{p(y)p(x|y)}{p(x)}.$$

The term p(y) on the right hand side of the equation expresses the base rate of the infection in the population. Similarly, the term p(x) expresses the default likelihood of positive test on a random person in the population. In the expressions below a(y) and $a(\overline{y}) = 1 - a(y)$denote the base rates of y and its complement $\overline{y}$respectively, so that e.g.

$p(x) = a(y)p(x|y) + a(\overline{y})p(x|\overline{y})$. The full expression for the required conditionals p(y | x) and $p(y|\overline{x})$ are then: $\begin{cases} p(y|x) = \frac{a(y)p(x|y)}{a(y)p(x|y)+a(\overline{y})p(x|\overline{y})} \\ p(y|\overline{x}) = \frac{a(y)p(\overline{x}|y)}{a(y)p(\overline{x}|y)+a(\overline{y})p(\overline{x}|\overline{y})} \end{cases}$

The full expression for the conditionally abduced probability of infection in a tested person, expressed as $p(y\overline{\|}x)$, given the outcome of the test, the base rate of the infection, as well as the test's sensitivity and false positive rate, is then given by:

$$p(y\overline{\|}x) = p(x)\left(\frac{a(y)p(x|y)}{a(y)p(x|y) + a(\overline{y})p(x|\overline{y})}\right) + p(\overline{x})\left(\frac{a(y)p(\overline{x}|y)}{a(y)p(\overline{x}|y) + a(\overline{y})p(\overline{x}|\overline{y})}\right).$$

Probabilistic abduction can thus be described as a method for inverting conditionals in order to apply probabilistic deduction.

A medical test result is typically considered positive or negative, so when applying the above equation it can be assumed that either p(x) = 1 (positive) or $p(\overline{x}) = 1$(negative). In case the patient tests positive, the above equation can be simplified to $p(y\overline{\|}x) = p(y|x)$which will give the correct likelihood that the patient actually is infected.

The Base rate fallacy in medicine, or the Prosecutor's fallacy in legal reasoning, consists of making the erroneous assumption that p(y | x) = p(x | y). While this reasoning error often can produce a relatively good approximation of the correct hypothesis probability value, it can lead to a completely wrong result and wrong conclusion in case the base rate is very low and the reliability of the test is not perfect. An extreme example of the base rate fallacy is to conclude that a male person is pregnant just because he tests positive in a pregnancy test. Obviously, the base rate of male pregnancy is zero, and assuming that the test is not perfect, it would be correct to conclude that the male person is not pregnant.

The expression for probabilistic abduction can be generalised to multinomial cases, i.e., with a state space X of multiple x_i and a state space Y of multiple states y_j.

Subjective logic abduction

Subjective logic generalises probabilistic logic by including parameters for uncertainty in the input arguments. Abduction in subjective logic is thus similar to probabilistic abduction described above. The input arguments in subjective logic are composite functions called subjective opinions which can be binomial when the opinion applies to a single proposition or multinomial when it applies to a set of propositions. A multinomial opinion thus applies to a frame X(i.e. a state space of exhaustive and mutually disjoint propositions x_i), and is denoted by the composite function $\omega_X = (\vec{b}, u, \vec{a})$, where $\vec{b}$ is a vector of belief masses over the propositions of X, $u$is the uncertainty mass, and $\vec{a}$is a vector of base rate values over the propositions of X. These components satisfy $u + \sum \vec{b}(x_i) = 1$ and $\sum \vec{a}(x_i) = 1$ as well as $\vec{b}(x_i), u, \vec{a}(x_i) \in [0,1]$.

Assume the frames X and Y, the sets of conditional opinions $\omega_{X|Y}$ and $\omega_{X|\overline{Y}}$, the opinion ω_X on X, and the base rate function a_Y on Y. Based on these parameters, subjective logic provides a method for deriving the set of inverted conditionals $\omega_{Y|X}$ and $\omega_{Y|\overline{X}}$. Using these inverted conditionals, subjective logic also provides a method for deduction. Abduction in subjective logic consists of inverting the conditionals and then applying deduction.

The symbolic notation for conditional abduction is "$\overline{\|}$", and the operator itself is denoted as $\overline{\circledcirc}$. The expression for subjective logic abduction is then:

$$\omega_{Y\overline{\|}X} = \omega_X \ \overline{\circledcirc} \ (\omega_{X|Y}, \omega_{X|\overline{Y}}, a_Y).$$

The advantage of using subjective logic abduction compared to probabilistic abduction is that uncertainty about the probability values of the input arguments can be explicitly expressed and taken into account during the analysis. It is thus possible to perform abductive analysis in the presence of missing or incomplete input evidence, which normally results in degrees of uncertainty in the output conclusions.

History of the concept

The philosopher Charles Sanders Peirce (1839–1914) introduced abduction into modern logic. Over the years he called such inference hypothesis, abduction, presumption, and retroduction. He considered it a topic in logic as a normative field in philosophy, not in purely formal or mathematical logic, and eventually as a topic also in economics of research.

As two stages of the development, extension, etc., of a hypothesis in scientific inquiry, abduction and induction are often collapsed into one overarching concept — the hypothesis. That is why, in the scientific method pioneered by Galileo and Bacon, the abductive stage of hypothesis formation is conceptualized simply as induction. In the twentieth century this collapse was reinforced by Karl Popper's explication of the hypothetico-deductive model, where the hypothesis is considered to be just "a guess" (in the spirit of Peirce). However, when the formation of a hypothesis is considered the result of a process it becomes clear that this "guess" has already been tried and made more robust in thought as a necessary stage of its acquiring the status of hypothesis. Indeed many abductions are rejected or heavily modified by subsequent abductions before they ever reach this stage.

Before 1900, Peirce treated abduction as the use of a known rule to explain an observation, e.g., it is a known rule that if it rains the grass is wet; so, to explain the fact that the grass is wet; one infers that it has rained. This remains the common use of the term "abduction" in the social sciences and in artificial intelligence.

Peirce consistently characterized it as the kind of inference that originates a hypothesis by concluding in an explanation, though an unassured one, for some very curious or surprising (anomalous) observation stated in a premise. As early as 1865 he wrote that all conceptions of cause and force are reached through hypothetical inference; in the 1900s he wrote that all explanatory content of theories is reached through abduction. In other respects Peirce revised his view of abduction over the years.

In later years his view came to be:

- Abduction is guessing. It is "very little hampered" by rules of logic. Oftenest even a well-prepared mind guesses wrong. But the success of our guesses far exceeds that of random luck and seems born of attunement to nature by instinct (some speak of intuition in such contexts).
- Abduction guesses a new or outside idea so as to account in a plausible, instinctive, economical way for a surprising or very complicated phenomenon. That is its proximate aim.
- Its longer aim is to economize inquiry itself. Its rationale is inductive: it works often enough, is the only source of new ideas, and has no substitute in expediting the discovery of new truths. Its rationale especially involves its role in coordination with other modes of inference in inquiry. It is inference to explanatory hypotheses for selection of those best worth trying.
- Pragmatism is the logic of abduction. Upon the generation of an explanation (which he came to regard as instinctively guided), the pragmatic maxim gives the necessary and sufficient logical rule to abduction in general. The hypothesis, being insecure, needs to have conceivable implications for informed practice, so as to be testable and, through its trials, to expedite and economize inquiry. The economy of research is what calls for abduction and governs its art.

Writing in 1910, Peirce admits that "in almost everything I printed before the beginning of this century I more or less mixed up hypothesis and induction" and he traces the confusion of these two types of reasoning to logicians' too "narrow and formalistic a conception of inference, as necessarily having formulated judgments from its premises."

He started out in the 1860s treating hypothetical inference in a number of ways which he eventually peeled away as inessential or, in some cases, mistaken:

- as inferring the occurrence of a character (a characteristic) from the combined occurrence of multiple characters which it necessarily involves (but by 1878 he no longer regarded that as common to all hypothetical inference.)
- as aiming for a more or less probable hypothesis (in 1867 and 1883 but not in 1878; anyway by 1900 the justification is not probability but the lack of alternatives to guessing and the fact that guessing is fruitful. ; by 1903 he speaks of the "likely" in the sense of nearing the truth in an "indefinite sense" ; by 1908 he discusses plausibility as instinctive appeal.)
- as induction from characters (but as early as 1900 he characterized abduction as guessing)

- as citing a known rule in a premise rather than hypothesizing a rule in the conclusion (but by 1903 he allowed either approach)
- as basically a transformation of a deductive categorical syllogism (but in 1903 he offered a variation on modus ponens instead , and by 1911 he was unconvinced that any one form covers all hypothetical inference).

In 1867, in "The Natural Classification of Arguments", hypothetical inference always deals with a cluster of characters (call them P′, P″, P‴, etc.) known to occur at least whenever a certain character (M) occurs. (Note that categorical syllogisms have elements traditionally called middles, predicates, and subjects. For example: All men [middle] are mortal [predicate]; Socrates [subject] is a man [middle]; ergo Socrates [subject] is mortal [predicate]". Below, 'M' stands for a middle; 'P' for a predicate; 'S' for a subject. Note also that Peirce held that all deduction can be put into the form of the categorical syllogism Barbara (AAA).)

[Deduction].	Induction.	Hypothesis.
[Any] M is P [Any] S is M ∴[Any] S is P.	S′, S″, S‴, &c. are taken at random as M's; S′, S″, S‴, &c. are P: ∴Any M is probably P.	Any M is, for instance, P′, P″, P‴, &c.; S is P′, P″, P‴, &c.: ∴S is probably M.

In 1878, in "Deduction, Induction, and Hypothesis" , there is no longer a need for multiple characters or predicates in order for an inference to be hypothetical, although it is still helpful. Moreover Peirce no longer poses hypothetical inference as concluding in a probable hypothesis. In the forms themselves, it is understood but not explicit that induction involves random selection and that hypothetical inference involves response to a "very curious circumstance". The forms instead emphasize the modes of inference as rearrangements of one another's propositions (without the bracketed hints shown below).

Deduction.	Induction.	Hypothesis.
Rule: All the beans from this bag are white. Case: These beans are from this bag. ∴Result: These beans are white.	Case: These beans are [randomly selected] from this bag. Result: These beans are white. ∴Rule: All the beans from this bag are white.	Rule: All the beans from this bag are white. Result: These beans [oddly] are white. ∴Case: These beans are from this bag.

Peirce long treated abduction in terms of induction from characters or traits (weighed, not counted like objects), explicitly so in his influential 1883 "A Theory of Probable Inference", in which he returns to involving probability in the hypothetical conclusion. Like "Deduction, Induction, and Hypothesis" in 1878, it was widely read, unlike his later amendments of his conception of abduction. Today abduction remains most commonly understood as induction from characters and extension of a known rule to cover unexplained circumstances.

In 1902 Peirce wrote that he now regarded the syllogistical forms and the doctrine of extension and comprehension (i.e., objects and characters as referenced by terms), as being less fundamental than he had earlier thought. In 1903 he offered the following form for abduction:

The surprising fact, C, is observed;
But if A were true, C would be a matter of course,
Hence, there is reason to suspect that A is true.

The hypothesis is framed, but not asserted, in a premise, then asserted as rationally suspectable in the conclusion. Thus, as in the earlier categorical syllogistic form, the conclusion is formulated from some premise(s). But all the same the hypothesis consists more clearly than ever in a new or outside idea beyond what is known or observed. Induction in a sense goes beyond observations already reported in the premises, but it merely amplifies ideas already known to represent occurrences, or tests an idea supplied by hypothesis; either way it requires previous abductions in order to get such ideas in the first place. Induction seeks facts to test a hypothesis; abduction seeks a hypothesis to account for facts.

Note that the hypothesis ("A") could be of a rule. It need not even be a rule strictly necessitating the surprising observation ("C"), which needs to follow only as a "matter of course"; or the "course" itself could amount to some known rule, merely alluded to, and also not necessarily a rule of strict necessity. In the same year, Peirce wrote that reaching a hypothesis may involve placing a surprising observation under either a newly hypothesized rule or a hypothesized combination of a known rule with a peculiar state of facts, so that the phenomenon would be not surprising but instead either necessarily implied or at least likely.

Peirce did not remain quite convinced about any such form as the categorical syllogistic form or the 1903 form. In 1911, he wrote, "I do not, at present, feel quite convinced that any logical form can be assigned that will cover all "Retroductions". For what I mean by a Retroduction is simply a conjecture which arises in the mind."

Peirce came over the years to divide (philosophical) logic into three departments:

1. Stechiology or speculative grammar, on the conditions for meaningfulness of signs and on classification of signs (semblances, symptoms, symbols, etc.) and of their combinations.
2. Logical critic, the critique of arguments in their distinct modes (deduction, induction, abduction).
3. Methodeutic or speculative rhetoric, on methods of inquiry in its coordinated modes.

Peirce had, from the start, seen the modes of inference as being coordinated together in scientific inquiry and, by the 1900s, held that hypothetical inference in particular is inadequately treated at the level of critique of arguments. To increase the assurance of a hypothetical conclusion, one needs to deduce implications about evidence to be found, predictions which induction can test through observation so as to evaluate the hypothesis. For Peirce that is the outline of the scientific method of inquiry and is studied in methodology of inquiry, a methodology including pragmatism or, as he later called it, pragmaticism, the clarification of ideas in terms of their conceivable implications regarding informed practice.

- At the critical level Peirce held that the hypothesis should economize explanation for plausibility in terms of the feasible and natural. In 1908 Peirce described this plausibility in some detail . It involves not likeliness based on observations (which is instead the inductive evaluation of a hypothesis), but instead optimal simplicity in the sense of the "facile and natural", as by Galileo's natural light of reason and as distinct from "logical simplicity" (Peirce does not dismiss logical simplicity entirely but sees it in a subordinate role; taken to its logical extreme it would favor adding no explanation to the observation at all). Even a well-prepared mind guesses oftener wrong than right, but our guesses succeed better than random luck at reaching the truth or at least advancing the inquiry, and that indicates to Peirce that they are based in instinctive attunement to nature, an affinity between the mind's processes and the processes of the real, which would account for why appealingly "natural" guesses are the ones that oftenest (or least seldom) succeed; to which Peirce added the argument that such guesses are to be preferred since, without "a natural bent like nature's", people would have no hope of understanding nature. In 1910 Peirce made a three-way distinction between probability, verisimilitude, and plausibility, and defined plausibility with a normative "ought": "By plausibility, I mean the degree to which a theory ought to recommend itself to our belief independently of any kind of evidence other than our instinct urging us to regard it favorably." The phrase "inference to the best explanation" (not used by Peirce but often applied to hypothetical inference) is not always understood as referring to the most simple and natural. However, in other senses of "best", such as "standing up best to tests", it is hard to know which is the best explanation to form, since one has not tested it yet.
- At the methodeutical level Peirce held that a hypothesis is judged and selected for testing because it offers, via its trial, to economize and expedite the inquiry process itself toward new truths, first of all by being testable and also by further economies , in terms of cost, value, and relationships among guesses (hypotheses). Here, considerations such as probability, absent from the treatment of abduction at the critical level, come into play. For examples:
    - Cost: A simple but low-odds guess, if low in cost to test for falsity, may belong first in line for testing, to get it out of the way. If surprisingly it stands up to tests, that is worth knowing early in the inquiry, which otherwise might have stayed long on a wrong though seemingly likelier track.

- Value: A guess's objective probability recommonds it for trial, while subjective likelihood can be misleading.
- Interrelationships: Guesses can be chosen for trial strategically, for which Peirce gave as example the game of Twenty Questions.

In 1901 Peirce wrote, "There would be no logic in imposing rules, and saying that they ought to be followed, until it is made out that the purpose of hypothesis requires them." In 1903 Peirce called pragmatism "the logic of abduction" and said that the pragmatic maxim gives the necessary and sufficient logical rule to abduction in general. The pragmatic maxim is: "Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have. Then, our conception of these effects is the whole of our conception of the object." It is a method for fruitful clarification of conceptions by equating the meaning of a conception with the conceivable practical implications of its object's conceived effects. Peirce held that that is precisely tailored to abduction's purpose in inquiry, the forming of an idea that could conceivably shape informed conduct. In various writings in the 1900s he said that the conduct of abduction (or retroduction) is governed by considerations of economy, belonging in particular to the economics of research. He regarded economics as a normative science whose analytic portion might be part of logical methodeutic.

Norwood Russell Hanson, a philosopher of science, wanted to grasp a logic explaining how scientific discoveries take place. He used Peirce's notion of abduction for this.

Further development of the concept can be found in Peter Lipton's Inference to the Best Explanation (Lipton, 1991).

Applications

Applications in artificial intelligence include fault diagnosis, belief revision, and automated planning. The most direct application of abduction is that of automatically detecting faults in systems: given a theory relating faults with their effects and a set of observed effects, abduction can be used to derive sets of faults that are likely to be the cause of the problem.

Abduction can also be used to model automated planning. Given a logical theory relating action occurrences with their effects (for example, a formula of the event calculus), the problem of finding a plan for reaching a state can be modeled as the problem of abducting a set of literals implying that the final state is the goal state.

In intelligence analysis, Analysis of Competing Hypotheses and Bayesian networks, probabilistic abductive reasoning is used extensively. Similarly in medical diagnosis and legal reasoning, the same methods are being used, although there have been many examples of errors, especially caused by the base rate fallacy and the prosecutor's fallacy.

Belief revision, the process of adapting beliefs in view of new information, is another field in which abduction has been applied. The main problem of belief revision is that the new information may be inconsistent with the corpus of beliefs, while the result of the incorporation cannot be inconsistent. This process can be done by the use of abduction: once an explanation for the observation has been found, integrating it does not generate inconsistency. This use of abduction is not straightforward, as adding propositional formulae to other propositional formulae can only make inconsistencies worse. Instead, abduction is done at the level of the ordering of preference of the possible worlds. Preference models use fuzzy logic or utility models.

In the philosophy of science, abduction has been the key inference method to support scientific realism, and much of the debate about scientific realism is focused on whether abduction is an acceptable method of inference.

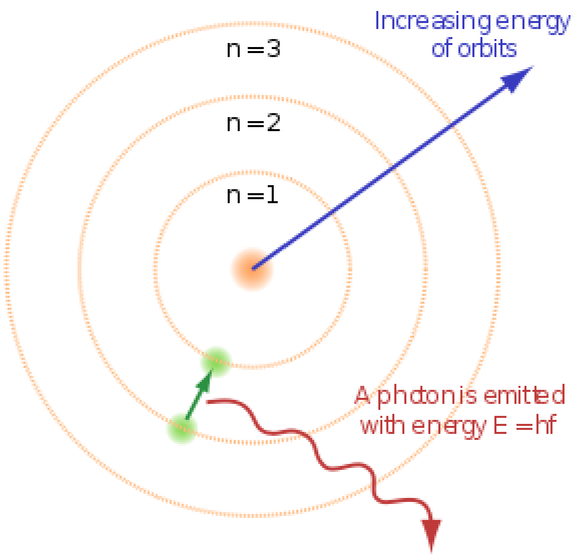
In historical linguistics, abduction during language acquisition is often taken to be an essential part of processes of language change such as reanalysis and analogy.

In anthropology, Alfred Gell in his influential book Art and Agency defined abduction, (after Eco) as "a case of synthetic inference 'where we find some very curious circumstances, which would be explained by the supposition that it was a case of some general rule, and thereupon adopt that supposition". Gell criticizes existing 'anthropological' studies of art, for being too preoccupied with aesthetic value and not preoccupied enough with the central anthropological concern of uncovering 'social relationships' specifically the social contexts in which artworks are produced, circulated, and received. Abduction is used as the basis of one gets from art to agency in the sense of a theory of how works of art can inspire a sensus communis, or the commonly-held views that a characteristic of a given society because they are shared by everyone in that society. The question Gell asks in the book is, 'how does initially to 'speak' to people?' He answers by saying that "No reasonable person could suppose that art-like relations between people and things do not involve at least some form of semiosis." However, he rejects any intimation that semiosis can be thought of as a language because then he would have to admit to some pre-established existence of the sensus communis that he wants to claim only emerges afterward out of art. Abduction is the answer to this conundrum because the tentative nature of the abduction concept (Pierce likened it to guessing) means that not only can it operate outside of any pre-existing framework, but moreover, it can actually intimate the existence of a framework. As Gell reasons in his analysis, the physical existence of the artwork prompts the viewer to perform an abduction that imbues the artwork with intentionality. A statue of a goddess, for example, in some senses actually becomes the goddess in the mind of the beholder; and represents not only the form of the deity but also her intentions (which are adduced from the feeling of her very presence). Therefore through abduction, Gell claims that art can have the kind of agency that plants the seeds that grow into cultural myths. The power of agency is the power to motivate actions and inspire ultimately the shared understanding that characterizes any given society.

Chapter- 4

Analogy

Analogy is a cognitive process of transferring information or meaning from a particular subject (the analogue or source) to another particular subject (the target), and a linguistic expression corresponding to such a process. In a narrower sense, analogy is an inference or an argument from one particular to another particular, as opposed to deduction, induction, and abduction, where at least one of the premises or the conclusion is general. The word analogy can also refer to the relation between the source and the target themselves, which is often, though not necessarily, a similarity, as in the biological notion of analogy.



Niels Bohr's model of the atom, though inaccurate, made an analogy between the atom and the solar system.

Analogy plays a significant role in problem solving, decision making, perception, memory, creativity, emotion, explanation and communication. It lies behind basic tasks such as the identification of places, objects and people, for example, in face perception and facial recognition systems. It has been argued that analogy is "the core of cognition". Specific analogical language comprises exemplification, comparisons, metaphors, similes, allegories, and parables, but not metonymy. Phrases like and so on, and the like, as if, and the very word like also rely on an analogical understanding by the receiver of a message including them. Analogy is important not only in ordinary language and common sense (where proverbs and idioms give many examples of its application) but also in science, philosophy and the humanities. The concepts of association, comparison, correspondence, mathematical and morphological homology, homomorphism, iconicity, isomorphism, metaphor, resemblance, and similarity are closely related to analogy. In cognitive linguistics, the notion of conceptual metaphor may be equivalent to that of analogy.

Analogy has been studied and discussed since classical antiquity by philosophers, scientists and lawyers. The last few decades have shown a renewed interest in analogy, most notable in cognitive science.

Usage of the terms source and target

With respect to the terms source and target there are two distinct traditions of usage:

- The logical and mathematical tradition speaks of an arrow, homomorphism, mapping, or morphism from what is typically the more complex domain or source to what is typically the less complex codomain or target, using all of these words in the sense of mathematical category theory.

- The tradition that appears to be more common in cognitive psychology, literary theory, and specializations within philosophy outside of logic, speaks of a mapping from what is typically the more familiar area of experience, the source, to what is typically the more problematic area of experience, the target.

Models and theories

Identity of relation

In ancient Greek the word αναλογια (analogia) originally meant proportionality, in the mathematical sense, and it was indeed sometimes translated to Latin as proportio. From there analogy was understood as **identity of relation** between any two ordered pairs, whether of mathematical nature or not. Kant's Critique of Judgment held to this notion. Kant argued that there can be exactly the same relation between two completely different objects. The same notion of analogy was used in the US-based SAT tests, that included "analogy questions" in the form "A is to B as C is to what?" For example, "Hand is to palm as foot is to ____?" These questions were usually given in the Aristotelian format:

HAND : PALM : : FOOT : ____

While most competent English speakers will immediately give the right answer to the analogy question (sole), it is more difficult to identify and describe the exact relation that holds both between hand and palm, and between foot and sole. This relation is not apparent in some lexical definitions of palm and sole, where the former is defined as the inner surface of the hand, and the latter as the underside of the foot. Analogy and abstraction are different cognitive processes, and analogy is often an easier one.

Recently a computer algorithm has achieved human-level performance on multiple-choice analogy questions from the SAT test. The algorithm measures the similarity of relations between pairs of words (e.g., the similarity between the pairs HAND:PALM and FOOT:SOLE) by statistical analysis of a large collection of text. It answers SAT questions by selecting the choice with the highest relational similarity.

Shared abstraction

In several cultures, the sun is the source of an analogy to God

Greek philosophers such as Plato and Aristotle actually used a wider notion of analogy. They saw analogy as a **shared abstraction**. Analogous objects did not share necessarily a relation, but also an idea, a pattern, a regularity, an attribute, an effect or a function. These authors also accepted that comparisons, metaphors and "images" (allegories) could be used as arguments, and sometimes they called them analogies. Analogies should also make those abstractions easier to understand and give confidence to the ones using them.

The Middle Ages saw an increased use and theorization of analogy. Roman lawyers had already used analogical reasoning and the Greek word analogia. Medieval lawyers distinguished analogia legis and analogia iuris (see below). In Islamic logic, analogical reasoning was used for the process of Qiyas in Islamic sharia law and fiqh jurisprudence. In Christian theology, analogical arguments were accepted in order to explain the attributes of God. Aquinas made a distinction between equivocal, univocal and analogical terms, the latter being those like healthy that have different but related meanings. Not only a person can be "healthy", but also the food that is good for health. Thomas Cajetan wrote an influential treatise on analogy. In all of these cases, the wide Platonic and Aristotelian notion of analogy was preserved. James Francis Ross in Portraying Analogy (1982), the first substantive examination of the topic since Cajetan's De Nominum Analogia, demonstrated that analogy is a systematic and universal feature of natural languages, with identifiable and law-like characteristics which explain how the meanings of words in a sentence are interdependent.

Special case of induction

On the contrary, Ibn Taymiyya, Francis Bacon and later John Stuart Mill argued that analogy is simply **a special case of induction**. In their view analogy is an inductive inference from common known attributes to another probable common attribute, which is known only about the source of the analogy, in the following form:

Premises
 a is C, D, E, F, G
 b is C, D, E, F
Conclusion
 b is probably G.
Alternative conclusion
 every C, D, E, F is probably G.

This view does not accept analogy as an autonomous mode of thought or inference, reducing it to induction. However, autonomous analogical arguments are still useful in science, philosophy and the humanities (see below), which makes this reduction philosophically uninteresting. Moreover, induction tries to achieve general conclusions, while analogy looks for particular ones.

Hidden deduction

The opposite move could also be tried, **reducing analogy to deduction**. It is argued that every analogical argument is partially superfluous and can be rendered as a deduction stating as a premise a (previously hidden) universal proposition which applied both to the source and the target. In this view, instead of an argument with the form:

Premises
 a is analogous to b.
 b is F.
Conclusion
 a is plausibly F.

We should have:

Hidden universal premise

all Gs are plausibly Fs.
Hidden singular premise
a is G.
Conclusion
a is plausibly F.

This would mean that premises referring the source and the analogical relation are themselves superfluous. However, it is not always possible to find a plausibly true universal premise to replace the analogical premises. And analogy is not only an argument, but also a distinct cognitive process.

Shared structure

Contemporary cognitive scientists use a wide notion of analogy, extensionally close to that of Plato and Aristotle, but framed by Gentner's (1983) **structure mapping theory**. The same idea of mapping between source and target is used by conceptual metaphor and conceptual blending theorists. Structure mapping theory concerns both psychology and computer science. According to this view, analogy depends on the mapping or alignment of the elements of source and target. The mapping takes place not only between objects, but also between relations of objects and between relations of relations. The whole mapping yields the assignment of a predicate or a relation to the target. Structure mapping theory has been applied and has found considerable confirmation in psychology. It has had reasonable success in computer science and artificial intelligence (see below). Some studies extended the approach to specific subjects, such as metaphor and similarity.

Keith Holyoak and Paul Thagard (1997) developed their **multiconstraint theory** within structure mapping theory. They defend that the "coherence" of an analogy depends on structural consistency, semantic similarity and purpose. Structural consistency is maximal when the analogy is an isomorphism, although lower levels are admitted. Similarity demands that the mapping connects similar elements and relations of source and target, at any level of abstraction. It is maximal when there are identical relations and when connected elements have many identical attributes. An analogy achieves its purpose insofar as it helps solve the problem at hand. The multiconstraint theory faces some difficulties when there are multiple sources, but these can be overcome. Hummel and Holyoak (2005) recast the multiconstraint theory within a neural network architecture. A problem for the multiconstraint theory arises from its concept of similarity, which, in this respect, is not obviously different from analogy itself. Computer applications demand that there are some identical attributes or relations at some level of abstraction. Human analogy does not, or at least not apparently.

Mark T. Keane and Brayshaw (1988) developed their Incremental Analogy Machine (IAM) to include working memory constraints as well as structural, semantic and pragmatic constraints, so that a subset of the base analog is selected and mapping from base to target occurs in a serial manner. Empirical evidence shows that human analogical mapping performance is influenced by information presentation order.

High-level perception

Douglas Hofstadter and his team challenged the shared structure theory and mostly its applications in computer science. They argue that there is no line between perception, including high-level perception, and analogical thought. In fact, analogy occurs not only after, but also before and at the same time as high-level perception. In high-level perception, humans make representations by selecting relevant information from low-level stimuli. Perception is necessary for analogy, but analogy is also necessary for high-level perception. Chalmers et al. conclude that analogy **is** high-level perception. Forbus et al. (1998) claim that this is only a metaphor. It has been argued (Morrison and Dietrich 1995) that Hofstadter's and Gentner's groups do not defend opposite views, but are instead dealing with different aspects of analogy.

Analogy and Complexity

Antoine Cornuéjols has presented analogy as a principle of economy and computational complexity.

Reasoning by analogy is a process of, from a given pair (x,f(x)), extrapolating the function f. In the standard modeling, analogical reasoning involves two "objects": the source and the target. The target is supposed to be incomplete and in need for a complete description using the source. The target has an existing part S_t and a missing part R_t. We assume that we can isolate a situation of the source S_s, which corresponds to a situation of target S_t, and the result of the source R_s, which correspond to the result of the target R_t. With B_s, the relation between S_s and R_s, we want B_t, the relation between S_t and R_t.

If the source and target are completely known:

Using Kolmogorov complexity K(x), defined as the size of the smallest description of x and Solomonoff's approach to induction, Rissanen (89), Wallace & Boulton (68) proposed the principle of Minimum description length. This principle leads to minimize the complexity K(target| Source) of producing the target from the source.

This is unattractive in Artificial Intelligence, as it requires a computation over abstract Turing machines. Suppose that M_s and M_t are local theories of the source and the target, available to the observer. The best analogy between a source case a and target case is the analogy that minimizes:

$$K(M_s) + K(S_s|M_s) + K(B_s|M_s) + K(M_t|M_s) + K(S_t|M_t) + K(B_t|M_t) \ (1).$$

If the target is completely unknown:

All models and descriptions M_s, M_t, B_s, S_s, and S_t leading to the minimization of:

$$K(M_s) + K(S_s|M_s) + K(B_s|M_s) + K(M_t|M_s) + K(S_t|M_t) \ (2)$$

are also those who allow to obtain the relationship B_t, and thus the most satisfactory R_t for formula (1).

The analogical hypothesis, which solves an analogy between a source case and a target case, has two parts:

- Analogy, like induction, is a principle of economy. The best analogy between two cases is the one which minimizes the amount of information necessary for the derivation of the source from the target (1). Its most fundamental measure is the computational complexity theory.
- When solving or completing a target case with a source case, the parameters which minimize (2) are postulated to minimize (1), and thus, produce the best response.

However, a cognitive agent may simply reduce the amount of information necessary for the interpretation of the source and the target, without taking into account the cost of data replication. So, it may prefer to the minimization of (2) the minimization of the following simplified formula:

$$K(M_s) + K(B_s|M_s) + K(M_t|M_s) \quad (3).$$

Applications and types

In language

Rhetoric

- An analogy can be a spoken or textual comparison between two words (or sets of words) to highlight some form of semantic similarity between them. Such analogies can be used to strengthen political and philosophical arguments, even when the semantic similarity is weak or non-existent (if crafted carefully for the audience). Analogies are sometimes used to persuade those that cannot detect the flawed or non-existent arguments.

Linguistics

- An analogy can be the linguistic process that reduces word forms perceived as irregular by remaking them in the shape of more common forms that are governed by rules. For example, the English verb help once had the preterite holp and the past participle holpen. These obsolete forms have been discarded and replaced by helped by the power of analogy (or by widened application of the productive Verb-ed rule.) This is called leveling. However, irregular forms can sometimes be created by analogy; one example is the American English past tense form of dive: dove, formed on analogy with words such as drive: drove.
- Neologisms can also be formed by analogy with existing words. A good example is software, formed by analogy with hardware; other analogous neologisms such

as firmware and vaporware have followed. Another example is the humorous term underwhelm, formed by analogy with overwhelm.

- Analogy is often presented as an alternative mechanism to generative rules for explaining productive formation of structures such as words. Others argue that in fact they are the same mechanism, that rules are analogies that have become entrenched as standard parts of the linguistic system, whereas clearer cases of analogy have simply not (yet) done so (e.g. Langacker 1987.445–447). This view has obvious resonances with the current views of analogy in cognitive science which are discussed above.

Mathematics

Some types of analogies can have a precise mathematical formulation through the concept of isomorphism. In detail, this means that given two mathematical structures of the same type, an analogy between them can be thought of as a bijection between them which preserves some or all of the relevant structure. For example, $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic as vector spaces, but the complex numbers, $\mathbb{C}$, have more structure than $\mathbb{R}^2$ does – $\mathbb{C}$ is a field as well as a vector space.

Category theory takes the idea of mathematical analogy much further with the concept of functors. Given two categories C and D a functor F from C to D can be thought of as an analogy between C and D, because F has to map objects of C to objects of D and arrows of C to arrows of D in such a way that the compositional structure of the two categories is preserved. This is similar to the structure mapping theory of analogy of Dedre Gentner, in that it formalizes the idea of analogy as a function which satisfies certain conditions.

Chapter- 5

Fallacy and Formal Fallacy

Fallacy

In logic and rhetoric, a **fallacy** is incorrect reasoning in argumentation resulting in a misconception. By accident or design, fallacies may exploit emotional triggers in the listener or interlocutor (e.g. appeal to emotion), or take advantage of social relationships between people (e.g. argument from authority). Fallacious arguments are often structured using rhetorical patterns that obscure the logical argument, making fallacies more difficult to diagnose. Also, the components of the fallacy may be spread out over separate arguments.

Fallacies

The taxonomy of material fallacies is based on that of Aristotle's Organon (Sophistici elenchi). This taxonomy is as follows:

Fallacy of Accident or Sweeping Generalization

- Fallacy of Accident or Sweeping Generalization: a generalization that disregards exceptions.
    - Example

  Argument: Cutting people is a crime. Surgeons cut people. Therefore, surgeons are criminals.
  Problem: Cutting people is only sometimes a crime.
  Argument: It is illegal for a stranger to enter someone's home uninvited. Firefighters enter people's homes uninvited, therefore firefighters are breaking the law.
  Problem: The exception does not break nor define the rule; a dicto simpliciter ad dictum secundum quid (where an accountable exception is ignored).

Converse Fallacy of Accident or Hasty Generalization

- Converse Fallacy of Accident or Hasty Generalization: argues from a special case to a general rule.

- Example

Argument: Every person I've met speaks English, so it must be true that all people speak English.
Problem: Those one has met are a subset of the entire set.

- Also called reverse accident, destroying the exception, a dicto secundum quid ad dictum simpliciter

Irrelevant Conclusion

- Irrelevant Conclusion: diverts attention away from a fact in dispute rather than address it directly.
  - Example

Argument: Billy believes that war is justifiable, therefore it must be justifiable.
Problem: Billy can be wrong. (In particular this is an appeal to authority.)

- Special cases:
  - purely personal considerations (argumentum ad hominem),
  - popular sentiment (argumentum ad populum—appeal to the majority; appeal to loyalty.),
  - fear (argumentum ad baculum),
  - conventional propriety (argumentum ad verecundiam—appeal to authority)
  - to arouse pity for getting one's conclusion accepted (argumentum ad misericordiam)
  - proving the proposition under dispute without any certain proof (argumentum ad ignorantiam)
  - assuming a perceived defect in the origin of a claim discredits the claim itself (genetic fallacy)
- Also called Ignoratio Elenchi, a "red herring"

Affirming the Consequent

- Affirming the Consequent: draws a conclusion from premises that do not support that conclusion.
  - Example:

Argument: If people have the flu, they cough. Billy is coughing. Therefore, Billy has the flu.
Problem: Other things, such as asthma, can cause someone to cough.
Argument: If it rains, the ground gets wet. The ground is wet, therefore it rained.
Problem: There are other ways by which the ground could get wet (e.g. dew).

Denying the antecedent

- Denying the antecedent: draws a conclusion from premises that do not support that conclusion.
    - Example

  Argument: If it is raining outside, it must be cloudy. It is not raining outside. Therefore, it is not cloudy.
  Problem: There does not have to be rain in order for it to be cloudy.

Begging the question

- Begging the question: demonstrates a conclusion by means of premises that assume that conclusion.
    - Example

  Argument: Billy always tells the truth, I know this because he told me so.
  Problem: Billy may be lying.

    - Also called Petitio Principii, Circulus in Probando, arguing in a circle, assuming the answer. Begging the question does not preclude the possibility that the statement in question is correct, but is insufficient proof in and of itself.

Fallacy of False Cause

- Fallacy of False Cause or Non Sequitur: incorrectly assumes one thing is the cause of another. Non Sequitur is Latin for "It does not follow."
    - Example

  Argument: Taxes fund necessary services such as police, courts, and roads; this demonstrates the necessity of taxation.
  Problem: The fact that taxes currently fund certain services does not prove that taxation is the only means, or the best means, of funding those services. Although, in all fairness, it is a deductive fallacy to claim that the logical possibility of something (funding public services without taxes) implies its practicality, probability or even existence.

    - Special cases
        - post hoc ergo propter hoc: believing that temporal succession implies a causal relation.
            - Example

  Argument: After Billy was vaccinated he developed autism, therefore the vaccine caused his autism.

Problem: This does not provide any evidence that the vaccine was the cause. The characteristics of autism may generally become noticeable at the age just following the typical age children receive vaccinations.

- cum hoc ergo propter hoc: believing that correlation implies a causal relation.
  - Example

Argument: More cows die in India in the summer months. More ice cream is consumed in summer months. Therefore, the consumption of ice cream in the summer months is killing Indian cows.
Problem: It is hotter in the summer, resulting in both the death of cows and the consumption of ice cream.
Also called causation versus correlation.

Fallacy of many questions

- Fallacy of many questions or loaded question: groups more than one question in the form of a single question.
  - Example

Argument: Have you stopped beating your wife?
Problem: A yes or no answer will still be an admission of guilt to beating your wife at some point.

- Also called Plurium Interrogationum and other terms

Straw man

- Straw man: A straw man argument is an informal fallacy based on misrepresentation of an opponent's position.
  - Example

Person A claims: Sunny days are good.
Argument Person B: If all days were sunny, we'd never have rain, and without rain, we'd have famine and death. Therefore, you are wrong.
Problem: B has falsely framed A's claim to imply that A says that only sunny days are good, and has argued against that assertion instead of the assertion A has made.

Verbal fallacies

Verbal fallacies are those in which a conclusion is obtained by improper or ambiguous use of words. They are generally classified as follows.

Equivocation

- Equivocation consists in employing the same word in two or more senses, e.g. in a syllogism, the middle term being used in one sense in the major and another in the minor premise, so that in fact there are four not three terms.

  Example Argument: All heavy things have a great mass; this is heavy fog; therefore this fog has a great mass.
  Problem: Heavy describes more than just weight. In the case of fog, it means that the fog is nearly opaque, not that it has a great mass. In fairness, a heavy fog does have significant mass, but not for the above reason.

Connotation fallacies

- Connotation fallacies occur when a dysphemistic word is substituted for the speaker's actual quote and used to discredit the argument. It is a form of attribution fallacy.

Argument by innuendo

- Argument by innuendo involves implicitly suggesting a conclusion without stating it outright. For example, a job reference that says a former employee "was never caught taking money from the cash box" implies that the employee was a thief, even though it does not make (or justify) a direct negative statement.

Amphiboly

- Amphiboly is the result of ambiguity of grammatical structure.

  Example: The position of the adverb "only" in a sentence starting with "He only said that" results in a sentence in which it is uncertain as to which of the other three words the speaker is intending to modify with the adverb.

Fallacy of Composition

- Fallacy of Composition "From Each to All". Arguing from some property of constituent parts, to the conclusion that the composite item has that property. This can be acceptable (i.e., not a fallacy) with certain arguments such as spatial arguments (e.g. "all the parts of the car are in the garage, therefore the car is in the garage").

  Example Argument: All the band members (constituent parts) are highly skilled, therefore the band (composite item) is highly skilled.
  Problem: The band members may be skilled musicians but lack the ability to function properly as a group.

Division

- Division, the converse of the preceding, arguing from a property of the whole, to each constituent part.

  Example Argument: "The university (the whole) is 700 years old, therefore, all the staff (each part) are 700 years old".
  Problem: Each and every person currently on staff is younger than 700 years. The university continues to exist even when, one by one, each and every person on the original staff leaves and is replaced by a younger person.
  Example Argument: "This cereal is part of a nutritious breakfast therefore the cereal is nutritious."
  Problem: Simply because the breakfast taken as a whole is nutritious does not necessarily mean that each part of that breakfast is nutritious.

Proof by verbosity

- Proof by verbosity, sometimes colloquially referred to as argumentum verbosium - a rhetorical technique that tries to persuade by overwhelming those considering an argument with such a volume of material that the argument sounds plausible, superficially appears to be well-researched, and it is so laborious to untangle and check supporting facts that the argument might be allowed to slide by unchallenged.

Accent

- Accent, which occurs only in speaking and consists of emphasizing the wrong word in a sentence. e.g., "He is a fairly good pianist," according to the emphasis on the words, may imply praise of a beginner's progress or insult of an expert pianist.

- "He is a fairly good pianist." This argument places emphasis on the fact that "He", as opposed to anyone else, is a good pianist.
- "He is a fairly good pianist." This is an assertion that "He" is a good pianist, as opposed to a poor one.
- "He is a fairly good pianist." This is an insult to his ability as a pianist and indicates that many others are better than he.
- "He is a fairly good pianist." This is isolating his ability as only being good in the field of musical instruments, namely, the piano, and excludes the idea that he is good at anything else. .

- "I killed my wife?" in response to a police officier asking if he killed his wife. In court, the police officier states his reply to his question was "I killed my wife"

Figure of Speech

- Figure of Speech, the confusion between the metaphorical and ordinary uses of a word or phrase.

  Example: The sailor was at home on the sea.
  Problem: The expression 'to be at home' does not literally mean that one's domicile is in that location.

Fallacy of Misplaced Concretion

- Fallacy of Misplaced Concretion, identified by Whitehead in his discussion of metaphysics, this refers to the reification of concepts which exist only in discussion.

Example 1

Timmy argues:

1. Billy is a good tennis player.
2. Therefore, Billy is 'good', that is to say a 'morally' good person.

Here the problem is that the word good has different meanings, which is to say that it is an ambiguous word. In the premise, Timmy says that Billy is good at some particular activity, in this case tennis. In the conclusion, Timmy states that Billy is a morally good person. These are clearly two different senses of the word "good". The premise might be true but the conclusion can still be false: Billy might be the best tennis player in the world but a rotten person morally. However, it is not legitimate to infer he is a bad person on the ground there has been a fallacious argument on the part of Timmy. Nothing concerning Billy's moral qualities is to be inferred from the premise. Appropriately, since it plays on an ambiguity, this sort of fallacy is called the fallacy of equivocation, that is, equating two incompatible terms or claims.

Example 2

One posits the argument:

1. Nothing is better than eternal happiness.
2. Eating a hamburger is better than nothing.
3. Therefore, eating a hamburger is better than eternal happiness.

This argument has the appearance of an inference that applies transitivity of the two-placed relation is better than, which in this critique we grant is a valid property. The argument is an example of syntactic ambiguity. In fact, the first premise semantically does not predicate an attribute of the subject, as would for instance the assertion

Nothing is better than eternal happiness.

In fact it is semantically equivalent to the following universal quantification:

Everything fails to be better than eternal happiness.

So instantiating this fact with eating a hamburger, it logically follows that

Eating a hamburger fails to be better than eternal happiness.

Note that the premise A hamburger is better than nothing does not provide anything to this argument. This fact really means something such as

Eating a hamburger is better than eating nothing at all.

Thus this is a fallacy of equivocation.

Deductive fallacy

In philosophy, the term **logical fallacy** properly refers to a formal fallacy: a flaw in the structure of a deductive argument which renders the argument invalid.

However, it is often used more generally in informal discourse to mean an argument which is problematic for any reason, and thus encompasses informal fallacies as well as formal fallacies.

The presence of a formal fallacy in a deductive argument does not imply anything about the argument's premises or its conclusion. Both may actually be true, or even more probable as a result of the argument (e.g., appeal to authority), but the deductive argument is still invalid because the conclusion does not follow from the premises in the manner described. By extension, an argument can contain a formal fallacy even if the argument is not a deductive one; for instance an inductive argument that incorrectly applies principles of probability or causality can be said to commit a formal fallacy.

Formalisms and frameworks used to understand fallacies

A different approach to understanding and classifying fallacies is provided by argumentation theory; see for instance the van Eemeren, Grootendorst reference below. In this approach, an argument is regarded as an interactive protocol between individuals which attempts to resolve a disagreement. The protocol is regulated by certain rules of interaction, and violations of these rules are fallacies. Many of the fallacies in the list below are best understood as being fallacies in this sense.

Other systems of classification

Of other classifications of fallacies in general the most famous are those of Francis Bacon and J. S. Mill. Bacon (Novum Organum, Aph. 33, 38 sqq.) divided fallacies into four Idola (Idols, i.e. False Appearances), which summarize the various kinds of mistakes to which the human intellect is prone. With these should be compared the Offendicula of Roger Bacon, contained in the Opus maius, pt. i. J. S. Mill discussed the subject in book v. of his Logic, and Jeremy Bentham's Book of Fallacies (1824) contains valuable remarks.

Formal fallacy

In philosophy, a **formal fallacy** is a pattern of reasoning that is always wrong. This is due to a flaw in the logical structure of the argument which renders the argument invalid. A formal fallacy is contrasted with an informal fallacy, which may have a valid logical form, but be false due to the characteristics of its premises, or its justification structure.

The term fallacy is often used more generally to mean an argument that is problematic for any reason, whether it is formal or informal.

The presence of a formal fallacy in a deductive argument does not imply anything about the argument's premises or its conclusion. Both may actually be true, or even more probable as a result of the argument (e.g. appeal to authority), but the deductive argument is still invalid because the conclusion does not follow from the premises in the manner described. By extension, an argument can contain a formal fallacy even if the argument is not a deductive one; for instance an inductive argument that incorrectly applies principles of probability or causality can be said to commit a formal fallacy.

Recognizing fallacies in everyday arguments may be difficult since arguments are often embedded in rhetorical patterns that obscure the logical connections between statements. Informal fallacies may also exploit the emotional , intellectual, or psychological weaknesses of the audience. Having the capability to recognize fallacies in arguments is one way to reduce the likelihood of such occurrences.

A different approach to understanding and classifying fallacies is provided by argumentation theory. In this approach, an argument is regarded as an interactive protocol between individuals which attempts to resolve their disagreements. The protocol is regulated by certain rules of interaction and violations of these rules are fallacies. Many of the fallacies in the list below are best understood as being fallacies in this sense.

Such fallacies are used in many forms of modern communications where the intention is to influence behavior and change beliefs - examples in the mass media today include but are not limited to propaganda, advertisements, politics, newspaper editorials and opinion-based news shows.

In contrast to informal fallacy

As modus ponens, the following argument contains no formal fallacies.

1. If **P** then **Q**
2. **P**
3. Therefore **Q**

If statements 1 and 2 are true, it will absolutely follow that statement 3 is true. However, it may still be the case that statement 1 or 2 is not true. For example:

1. If a scientist makes a statement about science, it is correct.
2. Albert Einstein states that all quantum mechanics is deterministic.
3. Therefore it's true that quantum mechanics is deterministic.

In this case, statement 1 is false. The particular informal fallacy being committed in this assertion is Argument from authority. By contrast, an argument with a formal fallacy could still contain all true premises:

1. If Bill Gates owns Fort Knox, then he is rich.
2. Bill Gates is rich.
3. Therefore, Bill Gates owns Fort Knox.

Though, 1 and 2 are true statements, 3 does not follow because the argument commits the formal fallacy of affirming the consequent.

An argument could contain both an informal fallacy and a formal fallacy yet have a correct conclusion, for example, again affirming the consequent

1. If a scientist makes a statement about science, it is correct.
2. It's true that quantum mechanics is deterministic.
3. Therefore a scientist has made a statement about it.

Chapter- 6

Informal Fallacy and Fuzzy Logic

Informal fallacy

An **informal fallacy** is an argument whose stated premises fail to support their proposed conclusion. The deviation in an informal fallacy often stems from a flaw in the path of reasoning that links the premises to the conclusion. In contrast to a formal fallacy, the error has to do with issues of ratiocination manifest in language used to state the propositions; the range of elements that can be symbolized by language is broader than that which the symbolism of formal logic can represent.

Deductive and inductive informal fallacies

Informal fallacies of deductive reasoning contain a fundamental disconnect between the premises and the conclusion that renders the argument invalid. This disconnect often stems from the presence of a hidden co-premise that, if presented, would validate the argument.

Inductive informal fallacies are slightly different from their deductive counterparts, as their merit rests in the inductive strength of the premise-conclusion link rather than in the presence of hidden premises. For instance, the fallacy of hasty generalization, can be roughly stated as:

p) A is an X
p) A is also a Y
c) therefore, all Xs are also Ys

If the populations X and Y are both too large to sample completely, then the statement is inductive. In such a case, a hasty generalization occurs when the number of Xs and Ys is insufficient to represent the respective populations. It is important to distinguish between a principle of reasoning (deductive or inductive) and the premise of an argument.

Fuzzy logic

Fuzzy logic is a form of multi-valued logic derived from fuzzy set theory to deal with reasoning that is approximate rather than accurate. In contrast with "crisp logic", where binary sets have binary logic, fuzzy logic variables may have a truth value that ranges between 0 and 1 and is not constrained to the two truth values of classic propositional logic. Furthermore, when linguistic variables are used, these degrees may be managed by specific functions.

Fuzzy logic emerged as a consequence of the 1965 proposal of fuzzy set theory by Lotfi Zadeh. Though fuzzy logic has been applied to many fields, from control theory to artificial intelligence, it still remains controversial among most statisticians, who prefer Bayesian logic, and some control engineers, who prefer traditional two-valued logic.

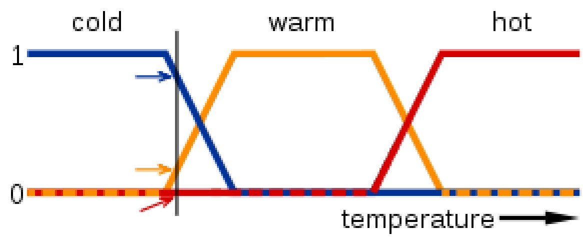
Degrees of truth

Fuzzy logic and probabilistic logic are mathematically similar – both have truth values ranging between 0 and 1 – but conceptually distinct, due to different interpretations. Fuzzy logic corresponds to "degrees of truth", while probabilistic logic corresponds to "probability, likelihood"; as these differ, fuzzy logic and probabilistic logic yield different models of the same real-world situations.

Both degrees of truth and probabilities range between 0 and 1 and hence may seem similar at first. For example, let a 100 ml glass contain 30 ml of water. Then we may consider two concepts: Empty and Full. The meaning of each of them can be represented by a certain fuzzy set. Then one might define the glass as being 0.7 empty and 0.3 full. Note that the concept of emptiness would be subjective and thus would depend on the observer or designer. Another designer might equally well design a set membership function where the glass would be considered full for all values down to 50 ml. It is essential to realize that fuzzy logic uses truth degrees as a mathematical model of the vagueness phenomenon while probability is a mathematical model of ignorance. The same could be achieved using probabilistic methods, by defining a binary variable "full" that depends on a continuous variable that describes how full the glass is. There is no consensus on which method should be preferred in a specific situation.

Applying truth values

A basic application might characterize subranges of a continuous variable. For instance, a temperature measurement for anti-lock brakes might have several separate membership functions defining particular temperature ranges needed to control the brakes properly. Each function maps the same temperature value to a truth value in the 0 to 1 range. These truth values can then be used to determine how the brakes should be controlled.



Fuzzy logic temperature

In this image, the meaning of the expressions cold, warm, and hot is represented by functions mapping a temperature scale. A point on that scale has three "truth values"—one for each of the three functions. The vertical line in the image represents a particular temperature that the three arrows (truth values) gauge. Since the red arrow points to zero, this temperature may be interpreted as "not hot". The orange arrow (pointing at 0.2) may describe it as "slightly warm" and the blue arrow (pointing at 0.8) "fairly cold".

Linguistic variables

While variables in mathematics usually take numerical values, in fuzzy logic applications, the non-numeric linguistic variables are often used to facilitate the expression of rules and facts.

A linguistic variable such as age may have a value such as young or its antonym old. However, the great utility of linguistic variables is that they can be modified via linguistic hedges applied to primary terms. The linguistic hedges can be associated with certain functions. For example, L. A. Zadeh proposed to take the square of the membership function. This model, however, does not work properly.

Example

Fuzzy set theory defines fuzzy operators on fuzzy sets. The problem in applying this is that the appropriate fuzzy operator may not be known. For this reason, fuzzy logic usually uses IF-THEN rules, or constructs that are equivalent, such as fuzzy associative matrices.

Rules are usually expressed in the form:
IF variable IS property THEN action

For example, a simple temperature regulator that uses a fan might look like this:

IF temperature IS very cold THEN stop fan
IF temperature IS cold THEN turn down fan
IF temperature IS normal THEN maintain level
IF temperature IS hot THEN speed up fan

There is no "ELSE" – all of the rules are evaluated, because the temperature might be "cold" and "normal" at the same time to different degrees.

The AND, OR, and NOT operators of boolean logic exist in fuzzy logic, usually defined as the minimum, maximum, and complement; when they are defined this way, they are called the Zadeh operators. So for the fuzzy variables x and y:

NOT x = (1 - truth(x))
x AND y = minimum(truth(x), truth(y))
x OR y = maximum(truth(x), truth(y))

There are also other operators, more linguistic in nature, called hedges that can be applied. These are generally adverbs such as "very", or "somewhat", which modify the meaning of a set using a mathematical formula.

Logical analysis

In mathematical logic, there are several formal systems of "fuzzy logic"; most of them belong among so-called t-norm fuzzy logics.

Propositional fuzzy logics

The most important propositional fuzzy logics are:

- Monoidal t-norm-based propositional fuzzy logic MTL is an axiomatization of logic where conjunction is defined by a left continuous t-norm, and implication is defined as the residuum of the t-norm. Its models correspond to MTL-algebras that are prelinear commutative bounded integral residuated lattices.
- Basic propositional fuzzy logic BL is an extension of MTL logic where conjunction is defined by a continuous t-norm, and implication is also defined as the residuum of the t-norm. Its models correspond to BL-algebras.
- Łukasiewicz fuzzy logic is the extension of basic fuzzy logic BL where standard conjunction is the Łukasiewicz t-norm. It has the axioms of basic fuzzy logic plus an axiom of double negation, and its models correspond to MV-algebras.
- Gödel fuzzy logic is the extension of basic fuzzy logic BL where conjunction is Gödel t-norm. It has the axioms of BL plus an axiom of idempotence of conjunction, and its models are called G-algebras.
- Product fuzzy logic is the extension of basic fuzzy logic BL where conjunction is product t-norm. It has the axioms of BL plus another axiom for cancellativity of conjunction, and its models are called product algebras.
- Fuzzy logic with evaluated syntax (sometimes also called Pavelka's logic), denoted by EVŁ, is a further generalization of mathematical fuzzy logic. While the above kinds of fuzzy logic have traditional syntax and many-valued semantics, in EVŁ is evaluated also syntax. This means that each formula has an evaluation. Axiomatization of EVŁ stems from Łukasziewicz fuzzy logic. A generalization of classical Gödel completeness theorem is provable in EVŁ.

Predicate fuzzy logics

These extend the above-mentioned fuzzy logics by adding universal and existential quantifiers in a manner similar to the way that predicate logic is created from propositional logic. The semantics of the universal (resp. existential) quantifier in t-norm fuzzy logics is the infimum (resp. supremum) of the truth degrees of the instances of the quantified subformula.

Decidability issues for fuzzy logic

The notions of a "decidable subset" and "recursively enumerable subset" are basic ones for classical mathematics and classical logic. Then, the question of a suitable extension of such concepts to fuzzy set theory arises. A first proposal in such a direction was made by E.S. Santos by the notions of fuzzy Turing machine, Markov normal fuzzy algorithm and fuzzy program. Successively, L. Biacino and G. Gerla showed that such a definition is not adequate and therefore proposed the following one. Ü denotes the set of rational numbers in [0,1]. A fuzzy subset s : S ⟶[0,1] of a set S is recursively enumerable if a recursive map h : S×N ⟶Ü exists such that, for every x in S, the function h(x,n) is increasing with respect to n and s(x) = lim h(x,n). We say that s is decidable if both s and its complement –s are recursively enumerable. An extension of such a theory to the general case of the L-subsets is proposed in Gerla 2006. The proposed definitions are well related with fuzzy logic. Indeed, the following theorem holds true (provided that the deduction apparatus of the fuzzy logic satisfies some obvious effectiveness property).

Theorem. Any axiomatizable fuzzy theory is recursively enumerable. In particular, the fuzzy set of logically true formulas is recursively enumerable in spite of the fact that the crisp set of valid formulas is not recursively enumerable, in general. Moreover, any axiomatizable and complete theory is decidable.

It is an open question to give supports for a Church thesis for fuzzy logic claiming that the proposed notion of recursive enumerability for fuzzy subsets is the adequate one. To this aim, further investigations on the notions of fuzzy grammar and fuzzy Turing machine should be necessary. Another open question is to start from this notion to find an extension of Gödel's theorems to fuzzy logic.

Fuzzy databases

Once fuzzy relations are defined, it is possible to develop fuzzy relational databases. The first fuzzy relational database, FRDB, appeared in Maria Zemankova's dissertation. Later, some other models arose like the Buckles-Petry model, the Prade-Testemale Model, the Umano-Fukami model or the GEFRED model by J.M. Medina, M.A. Vila et al. In the context of fuzzy databases, some fuzzy querying languages have been defined, highlighting the SQLf by P. Bosc et al. and the FSQL by J. Galindo et al. These languages define some structures in order to include fuzzy aspects in the SQL statements, like fuzzy conditions, fuzzy comparators, fuzzy constants, fuzzy constraints, fuzzy thresholds, linguistic labels and so on.

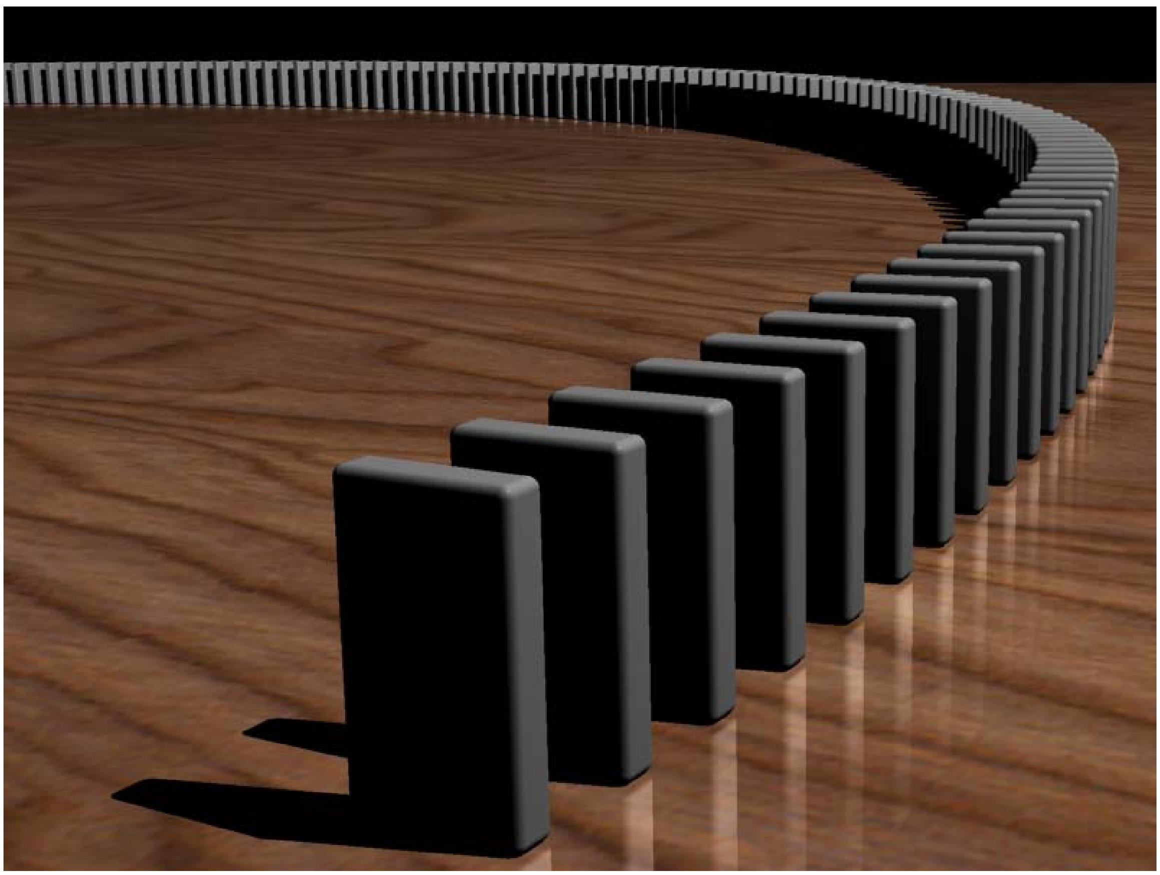
Comparison to probability

Fuzzy logic and probability are different ways of expressing uncertainty. While both fuzzy logic and probability theory can be used to represent subjective belief, fuzzy set theory uses the concept of fuzzy set membership (i.e., how much a variable is in a set), probability theory uses the concept of subjective probability (i.e., how probable do I think that a variable is in a set). While this distinction is mostly philosophical, the fuzzy-logic-derived possibility measure is inherently different from the probability measure, hence they are not directly equivalent. However, many statisticians are persuaded by the work of Bruno de Finetti that only one kind of mathematical uncertainty is needed and thus fuzzy logic is unnecessary. On the other hand, Bart Kosko argues that probability is a subtheory of fuzzy logic, as probability only handles one kind of uncertainty. He also claims to have proven a derivation of Bayes' theorem from the concept of fuzzy subsethood. Lotfi Zadeh argues that fuzzy logic is different in character from probability, and is not a replacement for it. He fuzzified probability to fuzzy probability and also generalized it to what is called possibility theory.

Chapter- 7

Mathematical Induction and Automated Theorem Proving

Mathematical induction



An informal description of mathematical induction can be illustrated by reference to the sequential effect of falling dominoes.

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true of all natural numbers (non-negative integers). It is done by proving that the **first** statement in the infinite sequence of statements is true, and then proving that if **any one** statement in the infinite sequence of statements is true, then so is the **next** one.

The method can be extended to prove statements about more general well-founded structures, such as trees; this generalization, known as structural induction, is used in mathematical logic and computer science. Mathematical induction in this extended sense is closely related to recursion.

Mathematical induction should not be misconstrued as a form of inductive reasoning, which is considered non-rigorous in mathematics. In fact, mathematical induction is a form of rigorous deductive reasoning.

History

In 370 BC, Plato's Parmenides may have contained an early example of an implicit inductive proof. The earliest implicit traces of mathematical induction can be found in Euclid's proof that the number of primes is infinite and in Bhaskara's "cyclic method". An opposite iterated technique, counting down rather than up, is found in the Sorites paradox, where one argued that if 1,000,000 grains of sand formed a heap, and removing one grain from a heap left it a heap, then a single grain of sand (or even no grains) forms a heap.

An implicit proof by mathematical induction for arithmetic sequences was introduced in the al-Fakhri written by al-Karaji around 1000 AD, who used it to prove the binomial theorem and properties of Pascal's triangle.

None of these ancient mathematicians, however, explicitly stated the inductive hypothesis. Another similar case (contrary to what Vacca has written, as Freudenthal carefully showed) was that of Francesco Maurolico in his Arithmeticorum libri duo (1575), who used the technique to prove that the sum of the first n odd integers is n^2. The first explicit formulation of the principle of induction was given by Pascal in his Traité du triangle arithmétique (1665). Another Frenchman, Fermat, made ample use of a related principle, indirect proof by infinite descent. The inductive hypothesis was also employed by the Swiss Jakob Bernoulli, and from then on it became more or less well known. The modern rigorous and systematic treatment of the principle came only in the 19th century, with George Boole , Giuseppe Peano and above all with Richard Dedekind.

Description

The simplest and most common form of mathematical induction proves that a statement involving a natural number n holds for all values of n. The proof consists of two steps:

1. The **basis (base case)**: showing that the statement holds when n is equal to the **lowest** value that n is given in the question. Usually, n = 0 or n = 1.
2. The **inductive step**: showing that **if** the statement holds for some n, **then** the statement also holds when n + 1 is substituted for n.

The assumption in the inductive step that the statement holds for some n is called the **induction hypothesis** (or **inductive hypothesis**). To perform the inductive step, one assumes the induction hypothesis and then uses this assumption to prove the statement for n + 1.

The choice between n = 0 and n = 1 in the base case is specific to the context of the proof: If 0 is considered a natural number, as is common in the fields of combinatorics and mathematical logic, then n = 0. If, on the other hand, 1 is taken as the first natural number, then the base case is given by n = 1.

This method works by first proving the statement is true for a starting value, and then proving that the process used to go from one value to the next is valid. If these are both proven, then any value can be obtained by performing the process repeatedly. It may be helpful to think of the domino effect; if one is presented with a long row of dominoes standing on end, one can be sure that:

1. The first domino will fall
2. Whenever a domino falls, its next neighbor will also fall,

so it is concluded that all of the dominoes will fall, and that this fact is inevitable.

Another analogy can be to consider a set of identical lily pads, all equally spaced in a line across a pond, with the first and last lily pads adjacent to the two sides of the pond. If a frog wishes to traverse the pond, it must:

1. Determine if the first lily pad will hold its weight.
2. Prove that it can jump from one lily pad to another.

Thus, it can conclude that it can jump to all of the lily pads, however many lily pads there are, and cross the pond.

Axiom of induction

The basic assumption or axiom of induction is, in logical symbols,

$$(\forall P)[P(0) \land (\forall k \in \mathbb{N})(P(k) \Rightarrow P(k+1))] \Rightarrow (\forall n \in \mathbb{N})[P(n)]$$

where P is any proposition and k and n are both natural numbers.

In other words, the basis P(0) being true along with the inductive case ("P(k) is true implies P(k + 1) is true" for all natural k) being true together imply that P(n) is true for any natural number n. A proof by induction is then a proof that these two conditions hold, thus implying the required conclusion.

This works because k is used to represent an arbitrary natural number. Then, using the inductive hypothesis, i.e. that P(k) is true, show P(k + 1) is also true. This allows us to "carry" the fact that P(0) is true to the fact that P(1) is also true, and carry P(1) to P(2), etc., thus proving P(n) holds for every natural number n.

Note that the first quantifier in the axiom ranges over predicates rather than over individual numbers. This is called a second-order quantifier, which means that the axiom is stated in second-order logic. Axiomatizing arithmetic induction in first-order logic requires an axiom schema containing a separate axiom for each possible predicate. The article Peano axioms contains further discussion of this issue.

Example

Mathematical induction can be used to prove that the following statement holds for all natural numbers n.

$$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

It gives a formula for the sum of the natural numbers less than or equal to number n. The proof that the statement is true for all natural numbers n proceeds as follows.

Call this statement P(n).

Basis: Show that the statement holds for n = 0.
P(0) amounts to the statement:

$$0 = \frac{0 \cdot (0+1)}{2} \,.$$

In the left-hand side of the equation, the only term is 0, and so the left-hand side is simply equal to 0.
In the right-hand side of the equation, 0·(0 + 1)/2 = 0.
The two sides are equal, so the statement is true for n = 0. Thus it has been shown that P(0) holds.

Inductive step: Show that if P(n) holds, then also P(n + 1) holds. This can be done as follows.

Assume P(n) holds (for some unspecified value of n). It must then be shown that P(n + 1) holds, that is:

$$(0 + 1 + 2 + \cdots + n) + (n+1) = \frac{(n+1)((n+1)+1)}{2}$$

Using the induction hypothesis that P(n) holds, the left-hand side can be rewritten to:

$$\frac{n(n+1)}{2} + (n+1).$$

Algebraically:

$$\begin{aligned}\frac{n(n+1)}{2} + (n+1) &= \frac{n(n+1)+2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2} \\ &= \frac{(n+1)((n+1)+1)}{2}.\end{aligned}$$

thereby showing that indeed P(n + 1) holds.

Since both the basis and the inductive step have been proved, it has now been proved by mathematical induction that P(n) holds for all natural n. Q.E.D.

Variants

In practice, proofs by induction are often structured differently, depending on the exact nature of the property to be proved.

Starting at some other number

If we want to prove a statement not for all natural numbers but only for all numbers greater than or equal to a certain number b then:

1. Showing that the statement holds when n = b.
2. Showing that if the statement holds for n = m ≥ b then the same statement also holds for n = m + 1.

This can be used, for example, to show that $n^2 \geq 3n$ for $n \geq 3$. A more substantial example is a proof that

$$\frac{n^n}{3^n} < n! < \frac{n^n}{2^n} \text{ for } n \geq 6.$$

In this way we can prove that P(n) holds for all n ≥1, or even n ≥−5. This form of mathematical induction is actually a special case of the previous form because if the statement that we intend to prove is P(n) then proving it with these two rules is equivalent with proving P(n + b) for all natural numbers n with the first two steps.

Building on n = 2

In mathematics, many standard functions, including operations such as "+" and relations such as "=", are binary, meaning that they take two arguments. Often these functions possess properties that implicitly extend them to more than two arguments. For example, once addition a + b is defined and is known to satisfy the associativity property (a + b) + c = a + (b + c), then the trinary addition a + b + c makes sense, either as (a + b) + c or as a + (b + c). Similarly, many axioms and theorems in mathematics are stated only for the binary versions of mathematical operations and relations, and implicitly extend to higher-arity versions.

Suppose that we wish to prove a statement about an n-ary operation implicitly defined from a binary operation, using mathematical induction on n. Then it should come as no surprise that the n = 2 case carries special weight. Here are some examples.

Example: product rule for the derivative

In this example, the binary operation in question is multiplication (of functions). The usual product rule for the derivative taught in calculus states:

$$(fg)' = f'g + g'f.$$

or in logarithmic derivative form

$$(fg)'/(fg) = f'/f + g'/g.$$

This can be generalized to a product of n functions. One has

$$\begin{aligned} &(f_1 f_2 f_3 \cdots f_n)' \\ &= (f_1' f_2 f_3 \cdots f_n) + (f_1 f_2' f_3 \cdots f_n) + (f_1 f_2 f_3' \cdots f_n) + \cdots + (f_1 f_2 \cdots f_{n-1} f_n'). \end{aligned}$$

or in logarithmic derivative form

$$\begin{aligned} &(f_1 f_2 f_3 \cdots f_n)'/(f_1 f_2 f_3 \cdots f_n) \\ &= (f_1'/f_1) + (f_2'/f_2) + (f_3'/f_3) + \cdots + (f_n'/f_n). \end{aligned}$$

In each of the n terms of the usual form, just one of the factors is a derivative; the others are not.

When this general fact is proved by mathematical induction, the n = 0 case is trivial, $(1)' = 0$ (since the empty product is 1, and the empty sum is 0). The n = 1 case is also trivial, $f_1' = f_1'$. And for each n ≥ 3, the case is easy to prove from the preceding n − 1 case. The real difficulty lies in the n = 2 case, which is why that is the one stated in the standard product rule.

Example: Pólya's proof that there is no "horse of a different color"

In this example, the binary relation in question is an equivalence relation applied to horses, such that two horses are equivalent if they are the same color. The argument is essentially identical to the one above, but the crucial $n = 2$ case fails, causing the entire argument to be invalid.

In the middle of the 20th century, a commonplace colloquial locution to express the idea that something is unexpectedly different from the usual was "That's a horse of a different color!". George Pólya posed the following exercise: Find the error in the following argument, which purports to prove by mathematical induction that all horses are of the same color:

- Basis: If there is only one horse, there is only one color.
- Induction step: Assume as induction hypothesis that within any set of n horses, there is only one color. Now look at any set of $n + 1$ horses. Number them: 1, 2, 3, ..., n, $n + 1$. Consider the sets $\{1, 2, 3, ..., n\}$ and $\{2, 3, 4, ..., n + 1\}$. Each is a set of only n horses, therefore within each there is only one color. But the two sets overlap, so there must be only one color among all $n + 1$ horses.

In general, the $n = 1$ case is trivial (as any horse is the same color as itself), and the inductive step is correct in all cases $n \geq 3$. However, the logic of the inductive step is incorrect when $n = 2$, because the statement that "the two sets overlap" is false (there are only two horses). Indeed, the $n = 2$ case is clearly the crux of the matter; if one could prove the $n = 2$ case directly, then all higher cases would follow from the inductive hypothesis.

Induction on more than one counter

It is sometimes desirable to prove a statement involving two natural numbers, n and m, by iterating the induction process. That is, one performs a basis step and an inductive step for n, and in each of those performs a basis step and an inductive step for m.

Infinite descent

In mathematics, a proof by **infinite descent** is a particular kind of proof by contradiction which relies on the fact that the natural numbers are well ordered. One typical application is to show that a given equation has no solutions. Assuming a solution exists, one shows that another exists, that is in some sense 'smaller'. Then one must show, usually with greater ease, that the infinite descent implied by having a whole sequence of solutions that are ever smaller, by our chosen measure, is an impossibility. This is a contradiction, so no such initial solution can exist.

This illustrative description can be restated in terms of a minimal counterexample, giving a more common type of formulation of an induction proof. We suppose a 'smallest' solution - then derive a smaller one. That again is a contradiction.

The method can be seen at work in one of the proofs of the irrationality of the square root of two. It was developed by and much used for Diophantine equations by Fermat. Two typical examples are solving the diophantine equation $x^4 + y^4 = z^2$ and proving a prime p ≡ 1 (mod 4) can be expressed as a sum of two squares. In some cases, to a modern eye, what he was using was (in effect) the doubling mapping on an elliptic curve. More precisely, his method of infinite descent was an exploitation in particular of the possibility of halving rational points on an elliptic curve E by inversion of the doubling formulae. The context is of a hypothetical rational point on E with large co-ordinates. Doubling a point on E roughly doubles the length of the numbers required to write it (as number of digits): so that a 'halved' point is quite clearly smaller. In this way Fermat was able to show the non-existence of solutions in many cases of Diophantine equations of classical interest (for example, the problem of four perfect squares in arithmetic progression).

Number theory

In the number theory of the twentieth century, the infinite descent method was taken up again, and pushed to a point where it connected with the main thrust of algebraic number theory and the study of L-functions. The structural result of Mordell, that the rational points on an elliptic curve E form a finitely-generated abelian group, used an infinite descent argument based on E/2E in Fermat's style.

To extend this to the case of an abelian variety A, André Weil had to make more explicit the way of quantifying the size of a solution, by means of a height function - a concept that became foundational. To show that A(Q)/2A(Q) is finite, which is certainly a necessary condition for the finite generation of the group A(Q) of rational points of A, one must do calculations in what later was recognised as Galois cohomology. In this way, abstractly-defined cohomology groups in the theory become identified with descents in the tradition of Fermat. The Mordell-Weil theorem was at the start of what later became a very extensive theory.

Application examples

Irrationality of √2

Suppose that √2 were rational. Then it could be written as

$$\sqrt{2} = \frac{p}{q},$$

for two natural numbers, p and q. Then,

$$2 = \frac{p^2}{q^2}$$
$$2q^2 = p^2,$$

so 2 | p. Let p = 2r, and

$$2q^2 = (2r)^2 = 4r^2$$
$$q^2 = 2r^2,$$

so 2 | q. Therefore, for both p and q, smaller natural numbers (which would work equally well to form the rational) can be found by dividing them in half. The same must hold for those smaller numbers, ad infinitum. However, this is impossible in the set of natural numbers. Since √2 is a real number, which can be either rational or irrational, the only option left is for √2 to be irrational.

A Diophantine equation

Infinite descent can be used to show that there are no integer solutions to

$$a^2 + b^2 = 3 \cdot (s^2 + t^2),$$

other than a = b = s = t = 0.

Suppose there is a nontrivial integer solution of the equation. Then there is a nontrivial nonnegative integer solution obtained by replacing each of a,b,s,t by its absolute value. So it suffices to show that there are no nontrivial nonnegative integer solutions.

Suppose that a_1,b_1,s_1,t_1 is a nonnegative solution. We have

$$3 \mid a_1^2 + b_1^2$$

This is only true if both a_1 and b_1 are divisible by 3. Let

$$3a_2 = a_1 \text{and } 3b_2 = b_1.$$

Thus we have

$$(3a_2)^2 + (3b_2)^2 = 3 \cdot (s_1^2 + t_1^2)$$

and

$$3(a_2^2 + b_2^2) = s_1^2 + t_1^2,$$

which yields a new nontrivial nonnegative integer solution s_1,t_1,a_2,b_2. Under a suitable notion of size of the solutions, e.g. the sum of the four integers, this new solution is smaller than the original one. This process can be repeated infinitely, producing an infinite decreasing sequence of positive solution sizes. This is a contradiction, because no such sequence exists. This shows that there are no nonzero solutions for this Diophantine equation.

Complete induction

Another generalization, called **complete induction** (or **strong induction** or **course of values induction**), says that in the second step we may assume not only that the statement holds for n = m but also that it is true for **all** n less than or equal to m.

In complete induction it is not necessary to list the base case as a separate assumption. When considering the first case, it is vacuously true that the statement holds for all previous cases; the inductive step of complete induction in this situation corresponds to the base case in ordinary induction. Thus the proof then of the inductive step in complete induction needs to be able to work with an empty antecedent; the first proof above is not of this kind (but can be converted).

Complete induction is most useful when several instances of the inductive hypothesis are required for each inductive step. For example, complete induction can be used to show that

$$\mathrm{F}(n) = \frac{(\varphi_+)^n - (\varphi_-)^n}{(\varphi_+) - (\varphi_-)}$$

where $\mathrm{F}(n)$ is the n[th] Fibonacci number and $\varphi_+ = (1+\sqrt{5})/2$ (the golden ratio) and $\varphi_- = (1-\sqrt{5})/2$ are the roots of $x^2 - x - 1 = 0$. By using the definition $\mathrm{F}(m+1) = \mathrm{F}(m) + \mathrm{F}(m-1)$, the identity above can be verified by direct calculation for $\mathrm{F}(m+1)$ if we assume that it already holds for both $\mathrm{F}(m)$ and $\mathrm{F}(m-1)$. To complete the proof, the identity must be verified in the two base cases n = 0 and n = 1.

Another proof by complete induction uses the hypothesis that the statement holds for all smaller n more thoroughly. Consider the statement that "every natural number greater than 1 is a product of prime numbers", and assume that for a given m > 1 it holds for all smaller n > 1. If m is prime then it is certainly a product of primes, and if not, then by definition it is a product: m = n_1 n_2, where neither of the factors is equal to 1; hence neither is equal to m, and so both are smaller than m. The induction hypothesis now applies to n_1 and n_2, so each one is a product of primes. Then m is a product of products of primes; i.e. a product of primes. Note both that the base case (m equal to 2) was never explicitly considered, and that the hypothesis that all smaller numbers than m are products of primes was used, since the factors of m are a priori unknown.

This generalization, complete induction, can be derived from the ordinary mathematical induction described above. Suppose P(n) is the statement that we intend to prove by complete induction. Let Q(n) mean P(m) holds for all m such that 0 ≤ m ≤ n. Apply mathematical induction to Q(n). Since Q(0) is just P(0), we have the base case. Now suppose Q(n) is given and we wish to show Q(n+1). Notice that Q(n) is the same as P(0) and P(1) and ... and P(n). The hypothesis of complete induction tells us that this implies P(n+1). If we add P(n+1) to Q(n), we get P(0) and P(1) and ... and P(n) and P(n+1), which is just Q(n+1). So using mathematical induction, we get that Q(n) holds for all natural numbers n. But Q(n) implies P(n), so we have the conclusion of strong induction, namely that P(n) holds for all natural numbers n.

Transfinite induction

The last two steps can be reformulated as one step:

1. Showing that if the statement holds for all n < m then the same statement also holds for n = m.

This is in fact the most general form of mathematical induction and it can be shown that it is not only valid for statements about natural numbers, but for statements about elements of any well-founded set, that is, a set with an irreflexive relation < that contains no infinite descending chains.

This form of induction, when applied to ordinals (which form a well-ordered and hence well-founded class), is called transfinite induction. It is an important proof technique in set theory, topology and other fields.

Proofs by transfinite induction typically distinguish three cases:

1. when m is a minimal element, i.e. there is no element smaller than m
2. when m has a direct predecessor, i.e. the set of elements which are smaller than m has a largest element
3. when m has no direct predecessor, i.e. m is a so-called limit-ordinal

Strictly speaking, it is not necessary in transfinite induction to prove the basis, because it is a vacuous special case of the proposition that if P is true of all n < m, then P is true of m. It is vacuously true precisely because there are no values of n < m that could serve as counterexamples.

Proof of mathematical induction

The principle of mathematical induction is usually stated as an axiom of the natural numbers. However, it can be proved in some logical systems. For instance, it can be proved if one assumes:

- The set of natural numbers is well-ordered.
- Every natural number is either zero, or n+1 for some natural number n.
- For any natural number n, n+1 is greater than n.

To derive simple induction from these axioms, we must show that if P(n) is some proposition predicated of n, and if:

- P(0) holds and
- whenever P(k) is true then P(k+1) is also true

then P(n) holds for all n.

Proof. Take S to be the set of all natural numbers for which P(n) is false. Let us see what happens if we assert that S is nonempty. Well-ordering tells us that S has a least element, say t. Moreover, since P(0) is true, t is not 0. Since every natural number is either zero or some n+1, there is some natural number n such that n+1=t. Now n is less than t, and t is the least element of S. It follows that n is not in S, and so P(n) is true. This means that P(n+1) is true, and so P(t) is true. This is a contradiction, since t was in S. Therefore, S is empty.

Automated theorem proving

Automated theorem proving (**ATP**) or **automated deduction**, currently the most well-developed subfield of automated reasoning (AR), is the proving of mathematical theorems by a computer program.

Decidability of the problem

Depending on the underlying logic, the problem of deciding the validity of a formula varies from trivial to impossible. For the frequent case of propositional logic, the problem is decidable but Co-NP-complete, and hence only exponential-time algorithms are believed to exist for general proof tasks. For a first order predicate calculus, with no ("proper") axioms, Gödel's completeness theorem states that the theorems (provable statements) are exactly the logically valid well-formed formulas, so identifying valid formulas is recursively enumerable: given unbounded resources, any valid formula can eventually be proven.

However, invalid formulas (those that are not entailed by a given theory), cannot always be recognized. In addition, a consistent formal theory that contains the first-order theory of the natural numbers (thus having certain "proper axioms"), by Gödel's incompleteness theorem, contains true statements which cannot be proven. In these cases, an automated theorem prover may fail to terminate while searching for a proof. Despite these theoretical limits, in practice, theorem provers can solve many hard problems, even in these undecidable logics.

Related problems

A simpler, but related, problem is proof verification, where an existing proof for a theorem is certified valid. For this, it is generally required that each individual proof step can be verified by a primitive recursive function or program, and hence the problem is always decidable.

Interactive theorem provers require a human user to give hints to the system. Depending on the degree of automation, the prover can essentially be reduced to a proof checker, with the user providing the proof in a formal way, or significant proof tasks can be performed automatically. Interactive provers are used for a variety of tasks, but even fully automatic systems have proven a number of interesting and hard theorems, including some that have eluded human mathematicians for a long time. However, these successes are sporadic, and work on hard problems usually requires a proficient user.

Another distinction is sometimes drawn between theorem proving and other techniques, where a process is considered to be theorem proving if it consists of a traditional proof, starting with axioms and producing new inference steps using rules of inference. Other techniques would include model checking, which is equivalent to brute-force enumeration of many possible states (although the actual implementation of model checkers requires much cleverness, and does not simply reduce to brute force).

There are hybrid theorem proving systems which use model checking as an inference rule. There are also programs which were written to prove a particular theorem, with a (usually informal) proof that if the program finishes with a certain result, then the theorem is true. A good example of this was the machine-aided proof of the four color theorem, which was very controversial as the first claimed mathematical proof which was essentially impossible to verify by humans due to the enormous size of the program's calculation (such proofs are called non-surveyable proofs). Another example would be the proof that the game Connect Four is a win for the first player.

Industrial uses

Commercial use of automated theorem proving is mostly concentrated in integrated circuit design and verification. Since the Pentium FDIV bug, the complicated floating point units of modern microprocessors have been designed with extra scrutiny. In the latest processors from AMD, Intel, and others, automated theorem proving has been used to verify that division and other operations are correct.

First-order theorem proving

First-order theorem proving is one of the most mature subfields of automated theorem proving. The logic is expressive enough to allow the specification of arbitrary problems, often in a reasonably natural and intuitive way. On the other hand, it is still semi-decidable, and a number of sound and complete calculi have been developed, enabling fully automated systems. More expressive logics, such as higher order and modal logics, allow the convenient expression of a wider range of problems than first order logic, but theorem proving for these logics is less well developed.

Benchmarks and competitions

The quality of implemented system has benefited from the existence of a large library of standard benchmark examples — the Thousands of Problems for Theorem Provers (TPTP) Problem Library — as well as from the CADE ATP System Competition (CASC), a yearly competition of first-order systems for many important classes of first-order problems.

Some important systems (all have won at least one CASC competition division) are listed below.

- E is a high-performance prover for full first-order logic, but built on a purely equational calculus, developed primarily in the automated reasoning group of Technical University of Munich.
- Otter, developed at the Argonne National Laboratory, is the first widely used high-performance theorem prover. It is based on first-order resolution and paramodulation. Otter has since been replaced by Prover9, which is paired with Mace4.
- SETHEO is a high-performance system based on the goal-directed model elimination calculus. It is developed in the automated reasoning group of Technical University of Munich. E and SETHEO have been combined (with other systems) in the composite theorem prover E-SETHEO.
- Vampire is developed and implemented at Manchester University by Andrei Voronkov, formerly together with Alexandre Riazanov. It has won the "world cup for theorem provers" (the CADE ATP System Competition) in the most prestigious CNF (MIX) division for eight years (1999, 2001–2007).
- Waldmeister is a specialized system for unit-equational first-order logic. It has won the CASC UEQ division for the last ten years (1997–2006).

Deontic theorem proving

Deontic logic concerns normative propositions, such as those used in law, engineering specifications, and computer programs. In other words, propositions that are translations of commands or "ought" or "must (not)" statements in ordinary language. The deontic character of such logic requires formalism that extends the first-order predicate calculus. Representative of this is the tool KED.

Chapter- 8

Defeasible Reasoning and Default Logic

Defeasible reasoning

Defeasible reasoning is a kind of reasoning that is based on reasons that are defeasible, as opposed to the indefeasible reasons of deductive logic. Defeasible reasoning is a particular kind of non-demonstrative reasoning, where the reasoning does not produce a full, complete, or final demonstration of a claim, i.e., where fallibility and corrigibility of a conclusion are acknowledged. Other kinds of non-demonstrative reasoning are probabilistic reasoning, inductive reasoning, statistical reasoning, abductive reasoning, and paraconsistent reasoning. Defeasible reasoning is also a kind of ampliative reasoning because its conclusions reach beyond the pure meanings of the premises.

The differences between these kinds of reasoning correspond to differences about the conditional that each kind of reasoning uses, and on what premise (or on what authority) the conditional is adopted:

- Deductive (from meaning postulate, axiom, or contingent assertion): if p then q (i.e., q or not-p)
- Defeasible (from authority): if p then (defeasibly) q
- Probabilistic (from combinatorics and indifference): if p then (probably) q
- Statistical (from data and presumption): the frequency of qs among ps is high (or inference from a model fit to data); hence, (in the right context) if p then (probably) q
- Inductive (theory formation; from data, coherence, simplicity, and confirmation): (inducibly) "if p then q"; hence, if p then (deducibly-but-revisably) q
- Abductive (from data and theory): p and q are correlated, and q is sufficient for p; hence, if p then (abducibly) q as cause

Some have thought that defeasible reasoning could be connected to qualitative probabilistic reasoning, but such efforts have not borne great insights.

Defeasible reasoning finds its fullest expression in jurisprudence, ethics and moral philosophy, epistemology, pragmatics and conversational conventions in linguistics, constructivist decision theories, and in knowledge representation and planning in artificial intelligence. It is also closely identified with prima facie (presumptive) reasoning (i.e., reasoning on the "face" of evidence), and ceteris paribus (default) reasoning (i.e., reasoning, all things "being equal").

History

Though Aristotle differentiated the forms of reasoning that are valid for logic and philosophy from the more general ones that are used in everyday life, 20th Century philosophers mainly concentrated on deductive reasoning. At the end of the 19th Century, logic texts would typically survey both demonstrative and non-demonstrative reasoning, often giving more space to the latter. However, after the blossoming of mathematical logic at the hands of Bertrand Russell, Alfred North Whitehead and Willard van Orman Quine, latter-20th Century logic texts paid little attention to the non-deductive modes of inference.

There are several notable exceptions. John Maynard Keynes wrote his dissertation on non-demonstrative reasoning, and influenced the thinking of Ludwig Wittgenstein on this subject. Wittgenstein, in turn, had many admirers, including the positivist legal scholar H.L.A. Hart and the speech act linguist John L. Austin, Stephen Toulmin in rhetoric (Chaim Perelman too), the moral theorists W.D. Ross and C.L. Stevenson, and the vagueness epistemologist/ontologist Friedrich Waismann.

The etymology of defeasible usually refers to Middle English law of contracts, where a condition of defeasance is a clause that can invalidate or annul a contract or deed. Though defeat, dominate, defer, defy, deprecate and derogate are often used in the same contexts as defeasible, the verbs annul and invalidate (and nullify, overturn, rescind, vacate, repeal, debar, void, cancel, countermand, preempt, etc.) are more properly correlated with the concept of defeasibility than those words beginning with the letter d. Many dictionaries do contain the verb, to defease with past participle, defeased.

Philosophers in moral theory and rhetoric had taken defeasibility largely for granted when American epistemologists rediscovered Wittgenstein's thinking on the subject: John Ladd, Roderick Chisholm, Roderick Firth, Ernest Sosa, Robert Nozick, and John L. Pollock all began writing with new conviction about how appearance as red was only a defeasible reason for believing something to be red. More importantly Wittgenstein's orientation toward language games (and away from semantics) emboldened these epistemologists to manage rather than to expurgate prima facie logical inconsistency.

At the same time (in the mid-1960s), two more students of Hart and Austin at Oxford, Brian Barry and David Gauthier, were applying defeasible reasoning to political argument and practical reasoning (of action), respectively. Joel Feinberg and Joseph Raz were beginning to produce equally mature works in ethics and jurisprudence informed by defeasibility.

By far the most significant works on defeasibility by the mid-1970s were in epistemology, where John Pollock's 1974 Knowledge and Justification popularized his terminology of undercutting and rebutting (which mirrored the analysis of Toulmin).

Pollock's work was significant precisely because it brought defeasibility so close to philosophical logicians. The failure of logicians to dismiss defeasibility in epistemology (as Cambridge's logicians had done to Hart decades earlier) landed defeasible reasoning in the philosophical mainstream.

Defeasibility had always been closely related to argument, rhetoric, and law, except in epistemology, where the chains of reasons, and the origin of reasons, were not often discussed. Nicholas Rescher's Dialectics is an example of how difficult it was for philosophers to contemplate more complex systems of defeasible reasoning. This was in part because proponents of informal logic became the keepers of argument and rhetoric while insisting that formalism was anathema to argument.

About this time, researchers in artificial intelligence became interested in non-monotonic reasoning and its semantics. With philosophers such as Pollock and Donald Nute (e.g., defeasible logic), dozens of computer scientists and logicians produced complex systems of defeasible reasoning between 1980 and 2000. No single system of defeasible reasoning would emerge in the same way that Quine's system of logic became a de facto standard. Nevertheless, the 100-year head start on non-demonstrative logical calculi, due to George Boole, Charles Sanders Peirce, and Gottlob Frege was being closed: both demonstrative and non-demonstrative reasoning now have formal calculi.

There are related (and slightly competing) systems of reasoning that are newer than systems of defeasible reasoning, e.g., belief revision and dynamic logic. The dialogue logics of Charles Hamblin and Jim Mackenzie (logician), and their colleagues, can also be tied closely to defeasible reasoning. Belief revision is a non-constructive specification of the desiderata with which, or constraints according to which, epistemic change takes place. Dynamic logic is related mainly because, like paraconsistent logic, the reordering of premises can change the set of justified conclusions. Dialogue logics introduce an adversary, but are like belief revision theories in their adherence to deductively consistent states of belief.

Political and judicial use

Many political philosophers have been fond of the word indefeasible when referring to rights, e.g., that were inalienable, divine, or indubitable. For example, in the 1776 Virginia Declaration of Rights, "community hath an indubitable, inalienable, and indefeasible right to reform, alter or abolish government..." (also attributed to James Madison); and John Adams, "The people have a right, an indisputable, unalienable, indefeasible, divine right to that most dreaded and envied kind of knowledge - I mean of the character and conduct of their rulers." Also, Lord Aberdeen: "indefeasible right inherent in the British Crown" and Gouverneur Morris: "the Basis of our own Constitution is the indefeasible Right of the People." Scholarship about Abraham Lincoln often cites these passages in the justification of secession. Philosophers who use the word defeasible have historically had different world views from those who use the word indefeasible (and this distinction has often been mirrored by Oxford and Cambridge zeitgeist); hence it is rare to find authors who use both words.

In judicial opinions, the use of defeasible is commonplace. There is however disagreement among legal logicians whether defeasible reasoning is central, e.g., in the consideration of open texture, precedent, exceptions, and rationales, or whether it applies only to explicit defeasance clauses. H.L.A. Hart in The Concept of Law gives two famous examples of defeasibility: "No vehicles in the park" (except during parades); and "Offer, acceptance, and memorandum produce a contract" (except when the contract is illegal, the parties are minors, inebriated, or incapacitated, etc.).

Specificity

One of the main disputes among those who produce systems of defeasible reasoning is the status of a rule of specificity. In its simplest form, it is the same rule as subclass inheritance preempting class inheritance:

```
 (R1) if p then (defeasibly) q                  e.g., if penguin then
not-flies
 (R2) if r then (defeasibly) not-q              e.g., if bird then
flies
 (O1) if p then (deductively) r                 e.g., if penguin then
bird
 (M1) arguably, p                                       e.g., arguably,
penguin
 (M2) R1 is a more specific reason than R2              e.g., R1 is
better than R2
 (M3) therefore, arguably, q                            e.g.,
therefore, arguably, not-flies
```

Approximately half of the systems of defeasible reasoning discussed today adopt a rule of specificity, while half expect that such preference rules be written explicitly by whomever provides the defeasible reasons. For example, Rescher's dialectical system uses specificity, as do early systems of multiple inheritance (e.g., David Touretzky) and the early argument systems of Donald Nute and of Guillermo Simari and Ronald Loui. Defeasible reasoning accounts of precedent (stare decisis and case-based reasoning) also make use of specificity (e.g., Joseph Raz and the work of Kevin D. Ashley and Edwina Rissland). Meanwhile, the argument systems of Henry Prakken and Giovanni Sartor, of Bart Verheij and Jaap Hage, and the system of Phan Minh Dung do not adopt such a rule.

Nature of defeasibility

There is a distinct difference between those who theorize about defeasible reasoning as if it were a system of confirmational revision (with affinities to belief revision), and those who theorize about defeasibility as if it were the result of further (non-empirical) investigation. There are at least three kinds of further non-empirical investigation: progress in a lexical/syntactic process, progress in a computational process, and progress in an adversary or legal proceeding.

Defeasibility as corrigibility: Here, a person learns something new that annuls a prior inference. In this case, defeasible reasoning provides a constructive mechanism for belief revision, like a truth maintenance system as envisioned by Jon Doyle.

Defeasibility as shorthand for preconditions: Here, the author of a set of rules or legislative code is writing rules with exceptions. Sometimes a set of defeasible rules can be rewritten, with more cogency, with explicit (local) pre-conditions instead of (non-local) competing rules. Many non-monotonic systems with fixed-point or preferential semantics fit this view. However, sometimes the rules govern a process of argument (the last view on this list), so that they cannot be re-compiled into a set of deductive rules lest they lose their force in situations with incomplete knowledge or incomplete derivation of preconditions.

Defeasibility as an anytime algorithm: Here, it is assumed that calculating arguments takes time, and at any given time, based on a subset of the potentially constructible arguments, a conclusion is defeasibly justified. Isaac Levi has protested against this kind of defeasibility, but it is well-suited to the heuristic projects of, for example, Herbert Simon. On this view, the best move so far in a chess-playing program's analysis at a particular depth is a defeasibly justified conclusion. This interpretation works with either the prior or the next semantical view.

Defeasibility as a means of controlling an investigative or social process: Here, justification is the result of the right kind of procedure (e.g., a fair and efficient hearing), and defeasible reasoning provides impetus for pro and con responses to each other. Defeasibility has to do with the alternation of verdict as locutions are made and cases presented, not the changing of a mind with respect to new (empirical) discovery. Under this view, defeasible reasoning and defeasible argumentation refer to the same phenomenon.

Default logic

Default logic is a non-monotonic logic proposed by Raymond Reiter to formalize reasoning with default assumptions.

Default logic can express facts like “by default, something is true”; by contrast, standard logic can only express that something is true or that something is false. This is a problem because reasoning often involves facts that are true in the majority of cases but not always. A classical example is: “birds typically fly”. This rule can be expressed in standard logic either by “all birds fly”, which is inconsistent with the fact that penguins do not fly, or by “all birds that are not penguins and not ostriches and ... fly”, which requires all exceptions to the rule to be specified. Default logic aims at formalizing inference rules like this one without explicitly mentioning all their exceptions.

Syntax of default logic

A default theory is a pair $\langle D, W \rangle$. W is a set of logical formulae, called the background theory, that formalize the facts that are known for sure. D is a set of default rules, each one being of the form:

$$\frac{\text{Prerequisite} : \text{Justification}_1, \dots, \text{Justification}_n}{\text{Conclusion}}$$

According to this default, if we believe that Prerequisite is true, and each of Justification_i is consistent with our current beliefs, we are led to believe that Conclusion is true.

The logical formulae in W and all formulae in a default were originally assumed to be first-order logic formulae, but they can potentially be formulae in an arbitrary formal logic. The case in which they are formulae in propositional logic is one of the most studied.

Examples

The default rule "birds typically fly" is formalized by the following default:

$$D = \left\{ \frac{Bird(X) : Flies(X)}{Flies(X)} \right\}$$

This rule means that, if X is a bird, and it can be assumed that it flies, then we can conclude that it flies. A background theory containing some facts about birds is the following one:

$$W = \{Bird(Condor), Bird(Penguin), \neg Flies(Penguin), Flies(Eagle)\}.$$

According to this default rule, a condor flies because the precondition Bird(Condor) is true and the justification Flies(Condor) is not inconsistent with what is currently known. On the contrary, Bird(Penguin) does not allow concluding Flies(Penguin): even if the precondition of the default Bird(Penguin) is true, the justification Flies(Penguin) is inconsistent with what is known. From this background theory and this default, Bird(Eagle) cannot be concluded because the default rule only allows deriving Flies(X) from Bird(X), but not vice versa. Deriving the antecedents of an inference rule from the consequences is a form of explanation of the consequences, and is the aim of abductive reasoning.

A common default assumption is that what is not known to be true is believed to be false. This is known as the Closed World Assumption, and is formalized in default logic using a default like the following one for every fact F.

$$\frac{:\neg F}{\neg F}$$

For example, the computer language Prolog uses a sort of default assumption when dealing with negation: if a negative atom cannot be proved to be true, then it is assumed to be false. Note, however, that Prolog uses the so-called negation as failure: when the interpreter has to evaluate the atom $\neg F$, it tries to prove that F is true, and conclude that $\neg F$ is true if it fails. In default logic, instead, a default having $\neg F$ as a justification can only be applied if $\neg F$ is consistent with the current knowledge.

Restrictions

A default is categorical or prerequisite-free if it has no prerequisite (or, equivalently, its prerequisite is tautological). A default is normal if it has a single justification that is equivalent to its conclusion. A default is supernormal if it is both categorical and normal. A default is seminormal if all its justifications entail its conclusion. A default theory is called categorical, normal, supernormal, or seminormal if all defaults it contains are categorical, normal, supernormal, or seminormal, respectively.

Semantics of default logic

A default rule can be applied to a theory if its precondition is entailed by the theory and its justifications are all **consistent with** the theory. The application of a default rule leads to the addition of its consequence to the theory. Other default rules may then be applied to the resulting theory. **When the theory is such that no other default can be applied, the theory is called an extension of the default theory.** The default rules may be applied in different order, and this may lead to different extensions. The Nixon diamond example is a default theory with two extensions:

$$\left\langle \left\{ \frac{Republican(X) : \neg Pacifist(X)}{\neg Pacifist(X)}, \frac{Quaker(X) : Pacifist(X)}{Pacifist(X)} \right\}, \{Republican(Nixon), Quaker(Nixon)\} \right\rangle$$

Since Nixon is both a Republican and a Quaker, both defaults can be applied. However, applying the first default leads to the conclusion that Nixon is not a pacifist, which makes the second default not applicable. In the same way, applying the second default we obtain that Nixon is a pacifist, thus making the first default not applicable. This particular default theory has therefore two extensions, one in which Pacifist(Nixon) is true, and one in which Pacifist(Nixon) is false.

The original semantics of default logic was based on the fixed point of a function. The following is an equivalent algorithmic definition. If a default contains formulae with free variables, it is considered to represent the set of all defaults obtained by giving a value to all these variables. A default $\frac{\alpha : \beta_1, \dots, \beta_n}{\gamma}$ is applicable to a propositional theory T if

$T \models \alpha$ and all theories $T \cup \{\beta_i\}$ are consistent. The application of this default to T leads to the theory $T \cup \{\gamma\}$. An extension can be generated by applying the following algorithm:

```
T=W             /* current theory */
A=0             /* set of defaults applied so far */

                /* apply a sequence of defaults */
while there is a default d that is not in A and is applicable to T
  add the consequence of d to T
  add d to A

                /* final consistency check */
if
  for every default d in A
    T is consistent with all justifications of d
then
  output T
```

This algorithm is non-deterministic, as several defaults can alternatively be applied to a given theory T. In the Nixon diamond example, the application of the first default leads to a theory to which the second default cannot be applied and vice versa. As a result, two extensions are generated: one in which Nixon is a pacifist and one in which Nixon is not a pacifist.

The final check of consistency of the justifications of all defaults that have been applied implies that some theories do not have any extensions. In particular, this happens whenever this check fails for every possible sequence of applicable defaults. The following default theory has no extension:

$$\left\langle \left\{ \frac{: A(b)}{\neg A(b)} \right\}, \emptyset \right\rangle$$

Since A(b) is consistent with the background theory, the default can be applied, thus leading to the conclusion that A(b) is false. This result however undermines the assumption that has been made for applying the first default. Consequently, this theory has no extensions.

In a normal default theory, all defaults are normal: each default has the form $\frac{\phi : \psi}{\psi}$. A normal default theory is guaranteed to have at least one extension.

Entailment

A default theory can have zero, one, or more extensions. Entailment of a formula from a default theory can be defined in two ways:

Skeptical

a formula is entailed by a default theory if it is entailed by all its extensions;

Credulous

a formula is entailed by a default theory if it is entailed by at least one of its extensions.

Thus, the Nixon diamond example theory has two extensions, one in which Nixon is a pacifist and one in which he is not a pacifist. Consequently, neither Pacifist(Nixon) nor $\neg Pacifist(Nixon)$ are skeptically entailed, while both of them are credulously entailed. As this example shows, the credulous consequences of a default theory may be inconsistent with each other.

Alternative default inference rules

The following alternative inference rules for default logic are all based on the same syntax as the original system.

Justified

differs from the original one in that a default is not applied if thereby the set T becomes inconsistent with a justification of an applied default;

Concise

a default is applied only if its consequence is not already entailed by T (the exact definition is more complicated than this one; this is only the main idea behind it);

Constrained

a default is applied only if the set composed of the background theory, the justifications of all applied defaults, and the consequences of all applied defaults (including this one) is consistent;

Rational

similar to constrained default logic, but the consequence of the default to add is not considered in the consistency check;

Cautious

defaults that can be applied but are conflicting with each other (like the ones of the Nixon diamond example) are not applied.

The justified and constrained versions of the inference rule assign at least an extension to every default theory.

Variants of default logic

The following variants of default logic differ from the original one on both syntax and semantics.

Assertional variants

An assertion is a pair $\langle p : \{r_1, \ldots, r_n\} \rangle$ composed of a formula and a set of formulae. Such a pair indicates that p is true while the formulae $r_1, \ldots, r_n$ have been assumed consistent to prove that p is true. An assertional default theory is composed of an assertional theory (a set of assertional formulae) called the background theory and a set of defaults defined as in the original syntax. Whenever a default is applied to an assertional theory, the pair composed of its consequence and its set of justifications is added to the theory. The following semantics use assertional theories:

- Cumulative default logic
- Commitment to assumptions default logic
- Quasi-default logic

Weak extensions
: rather than checking whether the preconditions are valid in the theory composed of the background theory and the consequences of the applied defaults, the preconditions are checked for validity in the extension that will be generated; in other words, the algorithm for generating extensions starts by guessing a theory and using it in place of the background theory; what results from the process of extension generation is actually an extension only if it is equivalent to the theory guessed at the beginning. This variant of default logic is related in principle to autoepistemic logic, where a theory $\Box x \rightarrow x$ has the model in which x is true just because, assuming $\Box x$ true, the formula $\Box x \rightarrow x$ supports the initial assumption.

Disjunctive default logic
: the consequence of a default is a set of formulae instead of a single formula. Whenever the default is applied, at least one of its consequences is nondeterministically chosen and made true.

Priorities on defaults
: the relative priority of defaults can be explicitly specified; among the defaults that are applicable to a theory, only one of the most preferred ones can be applied. Some semantics of default logic do not require priorities to be explicitly specified; rather, more specific defaults (those that are applicable in fewer cases) are preferred over less specific ones.

Statistical variant
: a statistical default is a default with an attached upper bound on its frequency of error; in other words, the default is assumed to be an incorrect inference rule in at most that fraction of times it is applied.

Translations

Default theories can be translated into theories in other logics and vice versa. The following conditions on translations have been considered:

Consequence-Preserving
: the original and the translated theories have the same (propositional) consequences;

Faithful

this condition only makes sense when translating between two variants of default logic or between default logic and a logic in which a concept similar to extension exists, e.g., models in modal logic; a translation is faithful if there exists a mapping (typically, a bijection) between the extensions (or models) of the original and translated theories;

Modular
: a translation from default logic to another logic is modular if the defaults and the background theory can be translated separately; moreover, the addition of formulae to the background theory only leads to adding the new formulae to the result of the translation;

Same-Alphabet
: the original and translated theories are built on the same alphabet;

Polynomial
: the running time of the translation or the size of the generated theory are required to be polynomial in the size of the original theory.

Translations are typically required to be faithful or at least consequence-preserving, while the conditions of modularity and same alphabet are sometimes ignored.

The translatability between propositional default logic and the following logics have been studied:

- classical propositional logic;
- autoepistemic logic;
- propositional default logic restricted to seminormal theories;
- alternative semantics of default logic;
- circumscription.

Translations exist or not depending on which conditions are imposed. Translations from propositional default logic to classical propositional logic cannot always generate a polynomially sized propositional theory, unless the polynomial hierarchy collapses. Translations to autoepistemic logic exists or not depending on whether modularity or the use of the same alphabet is required.

Complexity

The computational complexity of the following problems about default logic is known:

Existence of extensions
: deciding whether a propositional default theory has at least one extension is NP^{NP}-complete;

Skeptical entailment
: deciding whether a propositional default theory skeptically entails a propositional formula is $co - NP^{NP}$-complete;

Credulous entailment

deciding whether a propositional default theory credulously entails a propositional formula is NP^{NP}-complete;

Extension checking

deciding whether a propositional formula is equivalent to an extension of a propositional default theory is P^{NP}-complete;

Model checking

deciding whether a propositional interpretation is a model of an extension of a propositional default theory is NP^{NP}-complete.

Chapter- 9

Argument and Inquiry

Argument

In logic, an **argument** is a set of one or more meaningful declarative sentences (or "propositions") known as the premises along with another meaningful declarative sentence (or "proposition") known as the conclusion. A deductive argument asserts that the truth of the conclusion is a logical consequence of the premises; an inductive argument asserts that the truth of the conclusion is supported by the premises. Deductive arguments are valid or invalid, and sound or not sound. An argument is valid if and only if the truth of the conclusion is a logical consequence of the premises and (consequently) its corresponding conditional is a necessary truth. A sound argument is a valid argument with true premises.

Each premise and the conclusion are only either true or false, i.e. are truth bearers. The sentences composing an argument are referred to as being either true or false, not as being valid or invalid; deductive arguments are referred to as being valid or invalid, not as being true or false. Some authors refer to the premises and conclusion using the terms declarative sentence, statement, proposition, sentence, or even indicative utterance. The reason for the variety is concern about the ontological significance of the terms, proposition in particular. Whichever term is used, each premise and the conclusion must be capable of being true or false and nothing else: they are truthbearers.

Formal and informal arguments

Informal arguments are studied in informal logic, are presented in ordinary language and are intended for everyday discourse. Conversely, formal arguments are studied in formal logic (historically called symbolic logic, more commonly referred to as mathematical logic today) and are expressed in a formal language. Informal logic may be said to emphasize the study of argumentation, whereas formal logic emphasizes implication and inference. Informal arguments are sometimes implicit. That is, the logical structure –the relationship of claims, premises, warrants, relations of implication, and conclusion –is not always spelled out and immediately visible and must sometimes be made explicit by analysis.

Deductive arguments

A deductive argument is one which, if valid, has a conclusion that is entailed by its premises. In other words, the truth of the conclusion is a logical consequence of the premises—if the premises are true, then the conclusion must be true. It would be self-contradictory to assert the premises and deny the conclusion, because the negation of the conclusion is contradictory to the truth of the premises.

Validity

Arguments may be either valid or invalid. If an argument is valid, and its premises are true, the conclusion must be true: a valid argument cannot have true premises and a false conclusion.

The validity of an argument depends, however, not on the actual truth or falsity of its premises and conclusions, but solely on whether or not the argument has a valid logical form. The validity of an argument is not a guarantee of the truth of its conclusion. A valid argument may have false premises and a false conclusion.

Logic seeks to discover the valid forms, the forms that make arguments valid arguments. An argument form is valid if and only if all arguments of that form are valid. Since the validity of an argument depends on its form, an argument can be shown to be invalid by showing that its form is invalid, and this can be done by giving another argument of the same form that has true premises but a false conclusion. In informal logic this is called a counter argument.

The form of argument can be shown by the use of symbols. For each argument form, there is a corresponding statement form, called a corresponding conditional, and an argument form is valid if and only its corresponding conditional is a logical truth. A statement form which is logically true is also said to be a valid statement form. A statement form is a logical truth if it is true under all interpretations. A statement form can be shown to be a logical truth by either (a) showing that it is a tautology or (b) by means of a proof procedure.

The corresponding conditional, of a valid argument is a necessary truth (true in all possible worlds) and so we might say that the conclusion necessarily follows from the premises, or follows of logical necessity. The conclusion of a valid argument is not necessarily true, it depends on whether the premises are true. The conclusion of a valid argument need not be a necessary truth: if it were so, it would be so independently of the premises.

For example:

Some Greeks are logicians; therefore, some logicians are Greeks. Valid argument; it would be self-contradictory to admit that some Greeks are logicians but deny that some (any) logicians are Greeks.

All Greeks are human and all humans are mortal; therefore, all Greeks are mortal. : Valid argument; if the premises are true the conclusion must be true.
Some Greeks are logicians and some logicians are tiresome; therefore, some Greeks are tiresome. Invalid argument: the tiresome logicians might all be Romans (for example).
Either we are all doomed or we are all saved; we are not all saved; therefore, we are all doomed. Valid argument; the premises entail the conclusion. (Remember that this does not mean the conclusion has to be true; it is only true if the premises are true, which they may not be!)

Arguments can be invalid for a variety of reasons. There are well-established patterns of reasoning that render arguments that follow them invalid; these patterns are known as logical fallacies.

Soundness

A sound argument is a valid argument with true premises. A sound argument, being both valid and having true premises, must have a true conclusion. Some authors (especially in earlier literature) use the term sound as synonymous with valid.

Inductive arguments

Non-deductive logic is reasoning using arguments in which the premises support the conclusion but do not entail it. Forms of non-deductive logic include the statistical syllogism, which argues from generalizations true for the most part, and induction, a form of reasoning that makes generalizations based on individual instances. An inductive argument is said to be cogent if and only if the truth of the argument's premises would render the truth of the conclusion probable (i.e., the argument is strong), and the argument's premises are, in fact, true. Cogency can be considered inductive logic's analogue to deductive logic's "soundness." Despite its name, mathematical induction is not a form of inductive reasoning. The problem of induction is the philosophical question of whether inductive reasoning is valid.

Defeasible arguments

An argument is defeasible when additional information (such as new counterreasons) can have the effect that it no longer justifies its conclusion. The term "defeasibility" goes back to the legal theorist H.L.A. Hart, although he focused on concepts instead of arguments. Stephen Toulmin's influential argument model includes the possibility of counterreasons that is characteristic of defeasible arguments, but he did not discuss the evaluation of defeasible arguments. Defeasible arguments give rise to defeasible reasoning.

Argument by analogy

Argument by analogy may be thought of as argument from the particular to particular. An argument by analogy may use a particular truth in a premise to argue towards a similar particular truth in the conclusion. For example, if A. Plato was mortal, and B. Plato was just like Socrates, then asserting that C. Socrates was mortal is an example of argument by analogy because the reasoning employed in it proceeds from a particular truth in a premise (Plato was mortal) to a similar particular truth in the conclusion, namely that Socrates was mortal.

Explanations and arguments

While arguments attempt to show that something is, will be, or should be the case, explanations try to show why or how something is or will be. If Fred and Joe address the issue of whether or not Fred's cat has fleas, Joe may state: "Fred, your cat has fleas. Observe the cat is scratching right now." Joe has made an argument that the cat has fleas. However, if Fred and Joe agree on the fact that the cat has fleas, they may further question why this is so and put forth an explanation: "The reason the cat has fleas is that the weather has been damp." The difference is that the attempt is not to settle whether or not some claim is true, it is to show why it is true.

Arguments and explanations largely resemble each other in rhetorical use. This is the cause of much difficulty in thinking critically about claims. There are several reasons for this difficulty.

- People often are not themselves clear on whether they are arguing for or explaining something.
- The same types of words and phrases are used in presenting explanations and arguments.
- The terms 'explain' or 'explanation,' et cetera are frequently used in arguments.
- Explanations are often used within arguments and presented so as to serve as arguments.

Fallacies and non arguments

A fallacy is an invalid argument that appears valid, or a valid argument with disguised assumptions. First the premises and the conclusion must be statements, capable of being true and false. Secondly it must be asserted that the conclusion follows from the premises. In English the words therefore, so, because and hence typically separate the premises from the conclusion of an argument, but this is not necessarily so. Thus: Socrates is a man, all men are mortal therefore Socrates is mortal is clearly an argument (a valid one at that), because it is clear it is asserted that that Socrates is mortal follows from the preceding statements. However I was thirsty and therefore I drank is NOT an argument, despite its appearance. It is not being claimed that I drank is logically entailed by I was thirsty. The therefore in this sentence indicates for that reason not it follows that.

- Elliptical arguments

Often an argument is invalid because there is a missing premise the supply of which would make it valid. Speakers and writers will often leave out a strictly necessary premise in their reasonings if it is widely accepted and the writer does not wish to state the blindingly obvious. Example: All metals expand when heated, therefore iron will expand when heated. (Missing premise: iron is a metal). On the other hand a seemingly valid argument may be found to lack a premise – a 'hidden assumption' – which if highlighted can show a fault in reasoning. Example: A witness reasoned: Nobody came out the front door except the milkman therefore the murderer must have left by the back door. (Hidden assumption- the milkman was not the murderer).

Inquiry

An **inquiry** is any process that has the aim of augmenting knowledge, resolving doubt, or solving a problem. A theory of inquiry is an account of the various types of inquiry and a treatment of the ways that each type of inquiry achieves its aim.

Classical sources

Deduction

When three terms are so related to one another that the last is wholly contained in the middle and the middle is wholly contained in or excluded from the first, the extremes must admit of perfect syllogism. By 'middle term' I mean that which both is contained in another and contains another in itself, and which is the middle by its position also; and by 'extremes' (a) that which is contained in another, and (b) that in which another is contained. For if A is predicated of all B, and B of all C, A must necessarily be predicated of all C. ... I call this kind of figure the First. (Aristotle, Prior Analytics, 1.4)

Induction

Inductive reasoning consists in establishing a relation between one extreme term and the middle term by means of the other extreme; for example, if B is the middle term of A and C, in proving by means of C that A applies to B; for this is how we effect inductions. (Aristotle, Prior Analytics, 2.23)

Abduction

The locus classicus for the study of abductive reasoning is found in Aristotle's Prior Analytics, Book 2, Chapt. 25. It begins this way:

We have Reduction (απαγωγη, abduction):

1. When it is obvious that the first term applies to the middle, but that the middle applies to the last term is not obvious, yet is

nevertheless more probable or not less probable than the conclusion;
2. Or if there are not many intermediate terms between the last and the middle;

For in all such cases the effect is to bring us nearer to knowledge.

By way of explanation, Aristotle supplies two very instructive examples, one for each of the two varieties of abductive inference steps that he has just described in the abstract:

1. For example, let A stand for "that which can be taught", B for "knowledge", and C for "morality". Then that knowledge can be taught is evident; but whether virtue is knowledge is not clear. Then if BC is not less probable or is more probable than AC, we have reduction; for we are nearer to knowledge for having introduced an additional term, whereas before we had no knowledge that AC is true.
2. Or again we have reduction if there are not many intermediate terms between B and C; for in this case too we are brought nearer to knowledge. For example, suppose that D is "to square", E "rectilinear figure", and F "circle". Assuming that between E and F there is only one intermediate term — that the circle becomes equal to a rectilinear figure by means of lunules — we should approximate to knowledge. (Aristotle, "Prior Analytics", 2.25, with minor alterations)

Aristotle's latter variety of abductive reasoning, though it will take some explaining in the sequel, is well worth our contemplation, since it hints already at streams of inquiry that course well beyond the syllogistic source from which they spring, and into regions that Peirce will explore more broadly and deeply.

Inquiry in the pragmatic paradigm

In the pragmatic philosophies of Charles Sanders Peirce, William James, John Dewey, and others, inquiry is closely associated with the normative science of logic. In its inception, the pragmatic model or theory of inquiry was extracted by Peirce from its raw materials in classical logic, with a little bit of help from Kant, and refined in parallel with the early development of symbolic logic by Boole, De Morgan, and Peirce himself to address problems about the nature and conduct of scientific reasoning. Borrowing a brace of concepts from Aristotle, Peirce examined three fundamental modes of reasoning that play a role in inquiry, commonly known as abductive, deductive, and inductive inference.

In rough terms, abduction is what we use to generate a likely hypothesis or an initial diagnosis in response to a phenomenon of interest or a problem of concern, while deduction is used to clarify, to derive, and to explicate the relevant consequences of the selected hypothesis, and induction is used to test the sum of the predictions against the sum of the data. It needs to be observed that the classical and pragmatic treatments of the types of reasoning, dividing the generic territory of inference as they do into three special parts, arrive at a different characterization of the environs of reason than do those accounts that count only two.

These three processes typically operate in a cyclic fashion, systematically operating to reduce the uncertainties and the difficulties that initiated the inquiry in question, and in this way, to the extent that inquiry is successful, leading to an increase in knowledge or in skills.

In the pragmatic way of thinking everything has a purpose, and the purpose of each thing is the first thing we should try to note about it. The purpose of inquiry is to reduce doubt and lead to a state of belief, which a person in that state will usually call knowledge or certainty. As they contribute to the end of inquiry, we should appreciate that the three kinds of inference describe a cycle that can be understood only as a whole, and none of the three makes complete sense in isolation from the others. For instance, the purpose of abduction is to generate guesses of a kind that deduction can explicate and that induction can evaluate. This places a mild but meaningful constraint on the production of hypotheses, since it is not just any wild guess at explanation that submits itself to reason and bows out when defeated in a match with reality. In a similar fashion, each of the other types of inference realizes its purpose only in accord with its proper role in the whole cycle of inquiry. No matter how much it may be necessary to study these processes in abstraction from each other, the integrity of inquiry places strong limitations on the effective modularity of its principal components.

Art and science of inquiry

For our present purposes, the first feature to note in distinguishing the three principal modes of reasoning from each other is whether each of them is exact or approximate in character. In this light, deduction is the only one of the three types of reasoning that can be made exact, in essence, always deriving true conclusions from true premises, while abduction and induction are unavoidably approximate in their modes of operation, involving elements of fallible judgment in practice and inescapable error in their application.

The reason for this is that deduction, in the ideal limit, can be rendered a purely internal process of the reasoning agent, while the other two modes of reasoning essentially demand a constant interaction with the outside world, a source of phenomena and problems that will no doubt continue to exceed the capacities of any finite resource, human or machine, to master. Situated in this larger reality, approximations can be judged appropriate only in relation to their context of use and can be judged fitting only with regard to a purpose in view.

A parallel distinction that is often made in this connection is to call deduction a demonstrative form of inference, while abduction and induction are classed as non-demonstrative forms of reasoning. Strictly speaking, the latter two modes of reasoning are not properly called inferences at all. They are more like controlled associations of words or ideas that just happen to be successful often enough to be preserved as useful heuristic strategies in the repertoire of the agent. But non-demonstrative ways of thinking are inherently subject to error, and must be constantly checked out and corrected as needed in practice.

In classical terminology, forms of judgment that require attention to the context and the purpose of the judgment are said to involve an element of "art", in a sense that is judged to distinguish them from "science", and in their renderings as expressive judgments to implicate arbiters in styles of rhetoric, as contrasted with logic.

In a figurative sense, this means that only deductive logic can be reduced to an exact theoretical science, while the practice of any empirical science will always remain to some degree an art.

Zeroth order inquiry

Many aspects of inquiry can be recognized and usefully studied in very basic logical settings, even simpler than the level of syllogism, for example, in the realm of reasoning that is variously known as Boolean algebra, propositional calculus, sentential calculus, or zeroth-order logic. By way of approaching the learning curve on the gentlest availing slope, we may well begin at the level of zeroth-order inquiry, in effect, taking the syllogistic approach to inquiry only so far as the propositional or sentential aspects of the associated reasoning processes are concerned. One of the bonuses of doing this in the context of Peirce's logical work is that it provides us with doubly instructive exercises in the use of his logical graphs, taken at the level of his so-called "alpha graphs".

In the case of propositional calculus or sentential logic, deduction comes down to applications of the transitive law for conditional implications and the approximate forms of inference hang on the properties that derive from these. In describing the various types of inference I will employ a few old "terms of art" from classical logic that are still of use in treating these kinds of simple problems in reasoning.

Deduction takes a Case, the minor premise $X \Rightarrow Y$
and combines it with a Rule, the major premise $Y \Rightarrow Z$
to arrive at a Fact, the demonstrative conclusion $X \Rightarrow Z$.
Induction takes a Case of the form $X \Rightarrow Y$
and matches it with a Fact of the form $X \Rightarrow Z$
to infer a Rule of the form $Y \Rightarrow Z$.
Abduction takes a Fact of the form $X \Rightarrow Z$
and matches it with a Rule of the form $Y \Rightarrow Z$
to infer a Case of the form $X \Rightarrow Y$.

For ease of reference, Figure 1 and the Legend beneath it summarize the classical terminology for the three types of inference and the relationships among them.

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|                   |     /   C A S E             |
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|                   o                             |
|                   X                             |
|                                                 |
| Deduction takes a Case of the form X → Y,       |
| matches it with a Rule of the form Y → Z,       |
| then adverts to a Fact of the form X → Z.       |
|                                                 |
| Induction takes a Case of the form X → Y,       |
| matches it with a Fact of the form X → Z,       |
| then adverts to a Rule of the form Y → Z.       |
|                                                 |
| Abduction takes a Fact of the form X → Z,       |
| matches it with a Rule of the form Y → Z,       |
| then adverts to a Case of the form X → Y.       |
|                                                 |
| Even more succinctly:                           |
|                                                 |
|           Abduction  Deduction  Induction       |
|                                                 |
| Premise:     Fact       Case       Case         |
| Premise:     Rule       Rule       Fact         |
| Outcome:     Case       Fact       Rule         |
|                                                 |
o-------------------------------------------------o
```
Figure 1. Elementary Structure and Terminology

In its original usage a statement of Fact has to do with a deed done or a record made, that is, a type of event that is openly observable and not riddled with speculation as to its very occurrence. In contrast, a statement of Case may refer to a hidden or a hypothetical cause, that is, a type of event that is not immediately observable to all concerned. Obviously, the distinction is a rough one and the question of which mode applies can depend on the points of view that different observers adopt over time. Finally, a statement of a Rule is called that because it states a regularity or a regulation that governs a whole class of situations, and not because of its syntactic form. So far in this discussion, all three types of constraint are expressed in the form of conditional propositions, but this is not a fixed requirement. In practice, these modes of statement are distinguished by the roles that they play within an argument, not by their style of expression. When the time comes to branch out from the syllogistic framework, we will find that propositional constraints can be discovered and represented in arbitrary syntactic forms.

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Example of inquiry

Examples of inquiry, that illustrate the full cycle of its abductive, deductive, and inductive phases, and yet are both concrete and simple enough to be suitable for a first (or zeroth) exposition, are somewhat rare in Peirce's writings, and so let us draw one from the work of fellow pragmatician John Dewey, analyzing it according to the model of zeroth-order inquiry that we developed above.

A man is walking on a warm day. The sky was clear the last time he observed it; but presently he notes, while occupied primarily with other things, that the air is cooler. It occurs to him that it is probably going to rain; looking up, he sees a dark cloud between him and the sun, and he then quickens his steps. What, if anything, in such a situation can be called thought? Neither the act of walking nor the noting of the cold is a thought. Walking is one direction of activity; looking and noting are other modes of activity. The likelihood that it will rain is, however, something suggested. The pedestrian feels the cold; he thinks of clouds and a coming shower. (John Dewey, How We Think, pp. 6-7).

Once over quickly

Let's first give Dewey's elegant example of inquiry in everyday life the quick once over, hitting just the high points of its analysis into Peirce's three kinds of reasoning.

Abductive phase

In Dewey's "Rainy Day" or "Sign of Rain" story, we find our peripatetic hero presented with a surprising Fact:

- Fact: $C \rightarrow A$, In the Current situation the Air is cool.

Responding to an intellectual reflex of puzzlement about the situation, his resource of common knowledge about the world is impelled to seize on an approximate Rule:

- Rule: $B \rightarrow A$, Just Before it rains, the Air is cool.

This Rule can be recognized as having a potential relevance to the situation because it matches the surprising Fact, $C \rightarrow A$, in its consequential feature A.

All of this suggests that the present Case may be one in which it is just about to rain:

- Case: $C \rightarrow B$, The Current situation is just Before it rains.

The whole mental performance, however automatic and semi-conscious it may be, that leads up from a problematic Fact and a previously settled knowledge base of Rules to the plausible suggestion of a Case description, is what we are calling an abductive inference.

Deductive phase

The next phase of inquiry uses deductive inference to expand the implied consequences of the abductive hypothesis, with the aim of testing its truth. For this purpose, the inquirer needs to think of other things that would follow from the consequence of his precipitate explanation. Thus, he now reflects on the Case just assumed:

- Case: $C \rightarrow B$, The Current situation is just Before it rains.

He looks up to scan the sky, perhaps in a random search for further information, but since the sky is a logical place to look for details of an imminent rainstorm, symbolized in our story by the letter B, we may safely suppose that our reasoner has already detached the consequence of the abduced Case, $C \rightarrow B$, and has begun to expand on its further implications. So let us imagine that our up-looker has a more deliberate purpose in mind, and that his search for additional data is driven by the new-found, determinate Rule:

- Rule: $B \rightarrow D$, Just Before it rains, Dark clouds appear.

Contemplating the assumed Case in combination with this new Rule leads him by an immediate deduction to predict an additional Fact:

- Fact: $C \rightarrow D$, In the Current situation Dark clouds appear.

The reconstructed picture of reasoning assembled in this second phase of inquiry is true to the pattern of deductive inference.

Inductive phase

Whatever the case, our subject observes a Dark cloud, just as he would expect on the basis of the new hypothesis. The explanation of imminent rain removes the discrepancy between observations and expectations and thereby reduces the shock of surprise that made this process of inquiry necessary.

Looking more closely

Seeding hypotheses

Figure 4 gives a graphical illustration of Dewey's example of inquiry, isolating for the purposes of the present analysis the first two steps in the more extended proceedings that go to make up the whole inquiry.

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|        \  *                                   *  /        |
|         \   *                               *   /         |
|          \    *                           *    /          |
|           \     *                       *     /           |
|            \   R u l e             R u l e   /            |
|             \       *               *       /             |
|              \        *           *        /              |
|               \         *       *         /               |
|                \          * B *          /                |
|             F a c t        o        F a c t              |
|                  \          *          /                  |
|                   \         *         /                   |
|                    \        *        /                    |
|                     \       *       /                     |
|                      \   C a s e   /                      |
|                       \     *     /                       |
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|                          \  *  /                          |
|                           \ * /                           |
|                            \*/                            |
|                             o                             |
|                             C                             |
|                                                           |
| A  =  the Air is cool                                     |
| B  =  just Before it rains                                |
| C  =  the Current situation                               |
| D  =  a Dark cloud appears                                |
|                                                           |
| A is a major term                                         |
| B is a middle term                                        |
| C is a minor term                                         |
| D is a major term, associated with A                      |
|                                                           |
o-----------------------------------------------------------o
```
Figure 4. Dewey's 'Rainy Day' Inquiry

In this analysis of the first steps of Inquiry, we have a complex or a mixed form of inference that can be seen as taking place in two steps:

- The first step is an Abduction that abstracts a Case from the consideration of a Fact and a Rule.

Fact: C → A, In the Current situation the Air is cool.
Rule: B → A, Just Before it rains, the Air is cool.
Case: C → B, The Current situation is just Before it rains.

- The final step is a Deduction that admits this Case to another Rule and so arrives at a novel Fact.

Case: C → B, The Current situation is just Before it rains.
Rule: B → D, Just Before it rains, a Dark cloud will appear.
Fact: C → D, In the Current situation, a Dark cloud will appear.

This is nowhere near a complete analysis of the Rainy Day inquiry, even insofar as it might be carried out within the constraints of the syllogistic framework, and it covers only the first two steps of the relevant inquiry process, but maybe it will do for a start.

One other thing needs to be noticed here, the formal duality between this expansion phase of inquiry and the argument from analogy. This can be seen most clearly in the propositional lattice diagrams shown in Figures 3 and 4, where analogy exhibits a rough "A" shape and the first two steps of inquiry exhibit a rough "V" shape, respectively. Since we find ourselves repeatedly referring to this expansion phase of inquiry as a unit, let's give it a name that suggests its duality with analogy—"catalogy" will do for the moment. This usage is apt enough if one thinks of a catalogue entry for an item as a text that lists its salient features. Notice that analogy has to do with the examples of a given quality, while catalogy has to do with the qualities of a given example. Peirce noted similar forms of duality in many of his early writings, leading to the consummate treatment in his 1867 paper "On a New List of Categories" (CP 1.545-559, W 2, 49-59).

Weeding hypotheses

In order to comprehend the bearing of inductive reasoning on the closing phases of inquiry there are a couple of observations that we need to make:

- First, we need to recognize that smaller inquiries are typically woven into larger inquiries, whether we view the whole pattern of inquiry as carried on by a single agent or by a complex community.
- Further, we need to consider the different ways in which the particular instances of inquiry can be related to ongoing inquiries at larger scales. Three modes of inductive interaction between the micro-inquiries and the macro-inquiries that are salient here can be described under the headings of the "Learning", the "Transfer", and the "Testing" of rules.

Analogy of experience

Throughout inquiry the reasoner makes use of rules that have to be transported across intervals of experience, from the masses of experience where they are learned to the moments of experience where they are applied. Inductive reasoning is involved in the learning and the transfer of these rules, both in accumulating a knowledge base and in carrying it through the times between acquisition and application.

- Learning. The principal way that induction contributes to an ongoing inquiry is through the learning of rules, that is, by creating each of the rules that goes into the knowledge base, or ever gets used along the way.
- Transfer. The continuing way that induction contributes to an ongoing inquiry is through the exploit of analogy, a two-step combination of induction and deduction that serves to transfer rules from one context to another.
- Testing. Finally, every inquiry that makes use of a knowledge base constitutes a "field test" of its accumulated contents. If the knowledge base fails to serve any live inquiry in a satisfactory manner, then there is a prima facie reason to reconsider and possibly to amend some of its rules.

Let's now consider how these principles of learning, transfer, and testing apply to John Dewey's "Sign of Rain" example.

Learning

Rules in a knowledge base, as far as their effective content goes, can be obtained by any mode of inference.

For example, a rule like:

- Rule: $B \rightarrow A$, Just Before it rains, the Air is cool,

is usually induced from a consideration of many past events, in a manner that can be rationally reconstructed as follows:

- Case: $C \rightarrow B$, In Certain events, it is just Before it rains,
- Fact: $C \rightarrow A$, In Certain events, the Air is cool,

----

- Rule: $B \rightarrow A$, Just Before it rains, the Air is cool.

However, the very same proposition could also be abduced as an explanation of a singular occurrence or deduced as a conclusion of a presumptive theory.

Transfer

What is it that gives a distinctively inductive character to the acquisition of a knowledge base? It is evidently the "analogy of experience" that underlies its useful application. Whenever we find ourselves prefacing an argument with the phrase "If past experience is any guide..." then we can be sure that this principle has come into play. We are invoking an analogy between past experience, considered as a totality, and present experience, considered as a point of application. What we mean in practice is this: "If past experience is a fair sample of possible experience, then the knowledge gained in it applies to present experience". This is the mechanism that allows a knowledge base to be carried across gulfs of experience that are indifferent to the effective contents of its rules.

Here are the details of how this notion of transfer works out in the case of the "Sign of Rain" example:

Let K(pres) be a portion of the reasoner's knowledge base that is logically equivalent to the conjunction of two rules, as follows:

- K(pres) = (B → A) and (B → D).

K(pres) is the present knowledge base, expressed in the form of a logical constraint on the present universe of discourse.

It is convenient to have the option of expressing all logical statements in terms of their logical models, that is, in terms of the primitive circumstances or the elements of experience over which they hold true.

- Let E(past) be the chosen set of experiences, or the circumstances that we have in mind when we refer to "past experience".
- Let E(poss) be the collective set of experiences, or the projective total of possible circumstances.
- Let E(pres) be the present experience, or the circumstances that are present to the reasoner at the current moment.

If we think of the knowledge base K(pres) as referring to the "regime of experience" over which it is valid, then all of these sets of models can be compared by the simple relations of set inclusion or logical implication.

Figure 5 schematizes this way of viewing the "analogy of experience".

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|                         K(pres)                           |
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|                          / | \                            |
|                         /  |  \                           |
|                        /   |   \                          |
```

```
|                        /  Rule   \                        |
|                       /     |     \                       |
|                      /      |      \                      |
|                     /       |       \                     |
|                    /     E(poss)     \                    |
|              Fact /         o         \ Fact              |
|                  /        *   *        \                  |
|                 /       *       *       \                 |
|                /      *           *      \                |
|               /     *               *     \               |
|              /    *                   *    \              |
|             /   *  Case           Case  *   \             |
|            /  *                           *  \            |
|           / *                               * \           |
|          /*                                   *\          |
|         o<<<---------------<<<---------------<<<o         |
|      E(past)        Analogy Morphism         E(pres)      |
|    More Known                              Less Known     |
|                                                           |
o-----------------------------------------------------------o
```
Figure 5. Analogy of Experience

In these terms, the "analogy of experience" proceeds by inducing a Rule about the validity of a current knowledge base and then deducing a Fact, its applicability to a current experience, as in the following sequence:

Inductive Phase:

- Given Case: E(past) → E(poss), Chosen events fairly sample Collective events.
- Given Fact: E(past) → K(pres), Chosen events support the Knowledge regime.

- Induce Rule: E(poss) → K(pres), Collective events support the Knowledge regime.

Deductive Phase:

- Given Case: E(pres) → E(poss), Current events fairly sample Collective events.
- Given Rule: E(poss) → K(pres), Collective events support the Knowledge regime.

- Deduce Fact: E(pres) → K(pres), Current events support the Knowledge regime.

Testing

If the observer looks up and does not see dark clouds, or if he runs for shelter but it does not rain, then there is fresh occasion to question the utility or the validity of his knowledge base. But we must leave our foulweather friend for now and defer the logical analysis of this testing phase to another occasion.