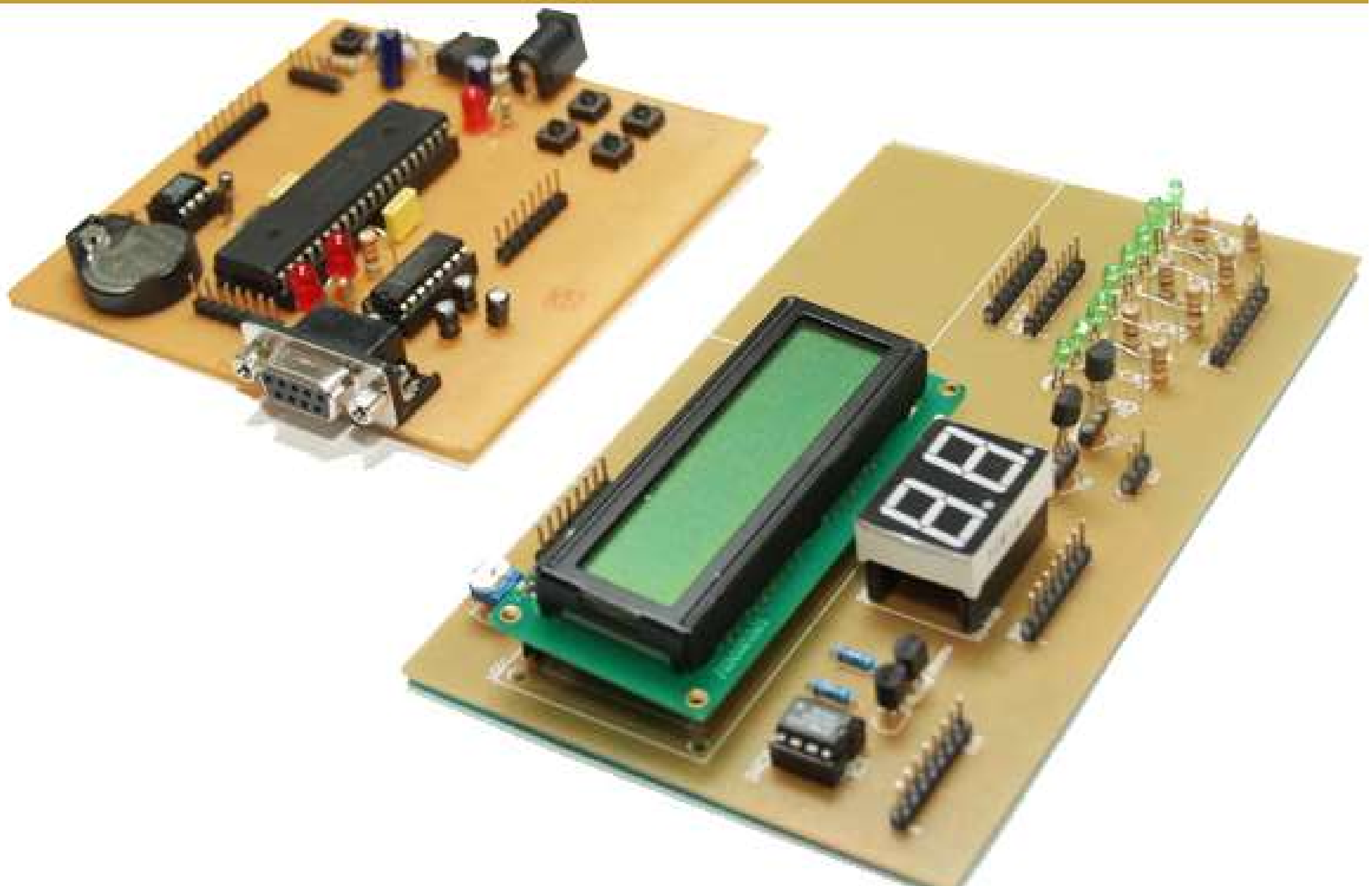
Network Synthesis and Simple Filters

(Linear Analog Electronic Filters)



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First Edition, 2012

ISBN 978-81-323-2881-0

Published by:
Orange Apple
4735/22 Prakashdeep Bldg,
Ansari Road, Darya Ganj,
Delhi - 110002
Email: info@wtbooks.com

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Network Synthesis Filters

Network synthesis is a method of designing signal processing filters. It has produced several important classes of filter including the Butterworth filter, the Chebyshev filter and the Elliptic filter. It was originally intended to be applied to the design of passive linear analogue filters but its results can also be applied to implementations in active filters and digital filters. The essence of the method is to obtain the component values of the filter from a given mathematical polynomial ratio expression representing the desired transfer function.

Description of method

The method can be viewed as the inverse problem of network analysis. Network analysis starts with a network and by applying the various electric circuit theorems predicts the response of the network. Network synthesis on the other hand, starts with a desired response and its methods produce a network that outputs, or approximates to, that response.

Network synthesis was originally intended to produce filters of the kind formerly described as "wave filters" but now usually just called filters. That is, filters whose purpose is to pass waves of certain wavelengths while rejecting waves of other wavelengths. Network synthesis starts out with a specification for the transfer function of the filter, H(s), as a function of complex frequency, s. This is used to generate an expression for the input impedance of the filter (the driving point impedance) which then, by a process of continued fraction or partial fraction expansions results in the required values of the filter components. In a digital implementation of a filter, H(s) can be implemented directly.

The advantages of the method are best understood by comparing it to the filter design methodology that was used before it, the image method. The image method considers the characteristics of an individual filter section in an infinite chain (ladder topology) of identical sections. The filters produced by this method suffer from innaccuracies due to the theoretical termination impedance, the image impedance, not generally being equal to the actual termination impedance. This is not the case with network synthesis filters, the terminations are included in the design from the start. The image method also requires a certain amount of experience on the part of the designer. The designer must first decide how many sections and of what type should be used, and then after calculation, will obtain the transfer function of the filter. This may not be what is required and there can be a number of iterations. The network synthesis method, on the other hand, starts out with the required function and outputs the sections needed to build the corresponding filter.

In general, the sections of a network synthesis filter are identical topology (usually the simplest ladder type) but different component values are used in each section. By contrast, the structure of an image filter has identical values at each section - this is a consequence of the infinite chain approach - but may vary the topology from section to section to achieve various desirable characteristics. Both methods make use of low-pass prototype filters followed by frequency transformations and impedance scaling to arrive at the final desired filter.

Important filter classes

The class of a filter refers to the class of polynomials from which the filter is mathematically derived. The order of the filter is the number of filter elements present in the filter's ladder implementation. Generally speaking, the higher the order of the filter, the steeper the cut-off transition between passband and stopband. Filters are often named after the mathematician or mathematics on which they are based rather than the discoverer or inventor of the filter.

Butterworth filter

The Butterworth class of filter was first described in a 1930 paper by the British engineer Stephen Butterworth after whom it is named. The filter response is described by Butterworth polynomials, also due to Butterworth.

Chebyshev filter

A Chebyshev filter has a faster cut-off transition than a Butterworth, but at the expense of there being ripples in the frequency response of the passband. There is a compromise to be had between the maximum allowed attenuation in the passband and the steepness of the cut-off response. This is also sometimes called a type I Chebyshev, the type 2 being a filter with no ripple in the passband but ripples in the stopband. The filter is named after Pafnuty Chebyshev whose Chebyshev polynomials are used in the derivation of the transfer function.

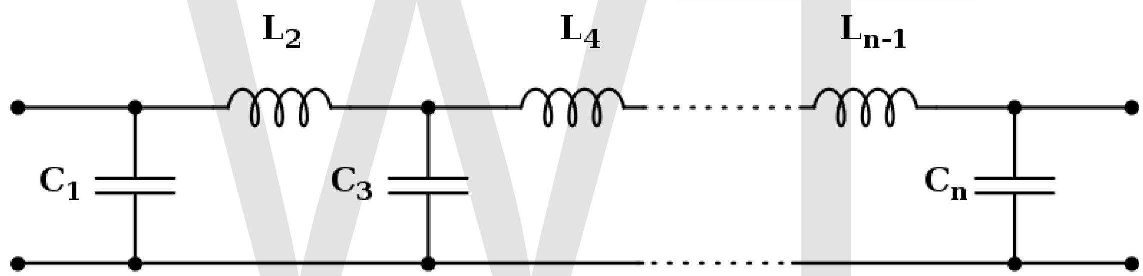
Cauer filter

Cauer filters have equal maximum ripple in the passband and the stopband. The Cauer filter has a faster transition from the passband to the stopband than any other class of network synthesis filter. The term Cauer filter can be used interchangeably with elliptical filter, but the general case of elliptical filters can have unequal ripples in the passband and stopband. An elliptical filter in the limit of zero ripple in the passband is identical to a Chebyshev Type 1 filter. An elliptical filter in the limit of zero ripple in the stopband is identical to a Chebyshev Type 2 filter. An elliptical filter in the limit of zero ripple in both passbands is identical to a Butterworth filter. The filter is named after Wilhelm Cauer and the transfer function is based on elliptic rational functions.

Bessel filter

- The Bessel filter has a maximally flat time-delay (group delay) over its passband. This gives the filter a linear phase response and results in it passing waveforms with minimal distortion. The Bessel filter has minimal distortion in the time domain due to the phase response with frequency as opposed to the Butterworth filter which has minimal distortion in the frequency domain due to the attenuation response with frequency. The Bessel filter is named after Friedrich Bessel and the transfer function is based on Bessel polynomials.

Driving point impedance



Low-pass filter implemented as a ladder (Cauer) topology

The driving point impedance is a mathematical representation of the input impedance of a filter in the frequency domain using one of a number of notations such as Laplace transform (s-domain) or Fourier transform (jω-domain). Treating it as a one-port network, the expression is expanded using continued fraction or partial fraction expansions. The resulting expansion is transformed into a network (usually a ladder network) of electrical elements. Taking an output from the end of this network, so realised, will transform it into a two-port network filter with the desired transfer function.

Not every possible mathematical function for driving point impedance can be realised using real electrical components. Wilhelm Cauer (following on from R. M. Foster) did much of the early work on what mathematical functions could be realised and in which filter topologies. The ubiquitous ladder topology of filter design is named after Cauer.

There are a number of canonical forms of driving point impedance that can be used to express all (except the simplest) realisable impedances. The most well known ones are;

- Cauer's first form of driving point impedance consists of a ladder of shunt capacitors and series inductors and is most useful for low-pass filters.
- Cauer's second form of driving point impedance consists of a ladder of series capacitors and shunt inductors and is most useful for high-pass filters.
- Foster's first form of driving point impedance consists of parallel connected LC resonators and is most useful for band-pass filters.
- Foster's second form of driving point impedance consists of series connected LC anti-resonators and is most useful for band-stop filters.

Prototype filters

Prototype filters are used to make the process of filter design less labour intensive. The prototype is usually designed to be a low-pass filter of unity nominal impedance and unity cut-off frequency, although other schemes are possible. The full design calculations from the relevant mathematical functions and polynomials are carried out only once. The actual filter required is obtained by a process of scaling and transforming the prototype.

Values of prototype elements are published in tables, one of the first being due to Sidney Darlington. Both modern computing power and the practice of directly implementing filter transfer functions in the digital domain have largely rendered this practice obsolete.

A different prototype is required for each order of filter in each class. For those classes in which there is attenuation ripple, a different prototype is required for each value of ripple. The same prototype may be used to produce filters which have a different bandform from the prototype. For instance low-pass, high-pass, band-pass and band-stop filters can all be produced from the same prototype.

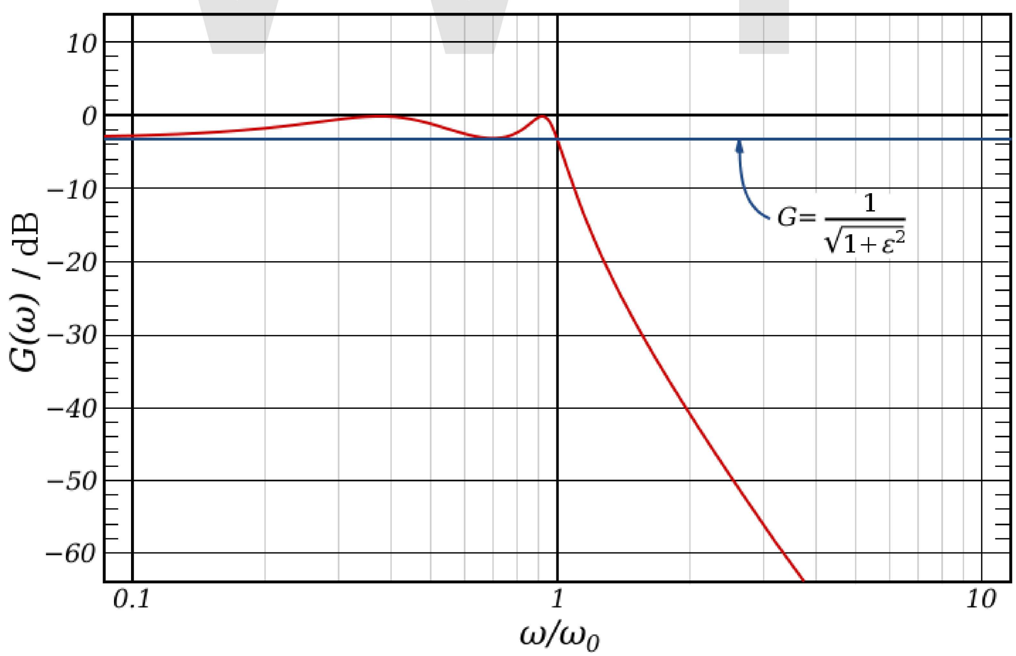
Chapter- 2

Chebyshev Filter

Chebyshev filters are analog or digital filters having a steeper roll-off and more passband ripple (type I) or stopband ripple (type II) than Butterworth filters. Chebyshev filters have the property that they minimize the error between the idealized and the actual filter characteristic over the range of the filter, but with ripples in the passband. This type of filter is named in honor of Pafnuty Chebyshev because their mathematical characteristics are derived from Chebyshev polynomials.

Because of the passband ripple inherent in Chebyshev filters, filters which have a smoother response in the passband but a more irregular response in the stopband are preferred for some applications.

Type I Chebyshev filters



The frequency response of a fourth-order type I Chebyshev low-pass filter with $\varepsilon = 1$

These are the most common Chebyshev filters. The gain (or amplitude) response as a function of angular frequency ω of the *n*th order low pass filter is

$$G_n(\omega) = |H_n(j\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 T_n^2\left(\frac{\omega}{\omega_0}\right)}}$$

where ε is the ripple factor, ω_0 is the cutoff frequency and $T_n()$ is a Chebyshev polynomial of the *n*th order.

The passband exhibits equiripple behavior, with the ripple determined by the ripple factor ε. In the passband, the Chebyshev polynomial alternates between 0 and 1 so the filter gain will alternate between maxima at $G = 1$ and minima at $G = 1/\sqrt{1+\varepsilon^2}$. At the cutoff frequency ω_0 the gain again has the value $1/\sqrt{1+\varepsilon^2}$ but continues to drop into the stop band as the frequency increases. This behavior is shown in the diagram on the right. (*note*: the common definition of the cutoff frequency to −3 dB does *not* hold for Chebyshev filters!)

The order of a Chebyshev filter is equal to the number of reactive components (for example, inductors) needed to realize the filter using analog electronics.

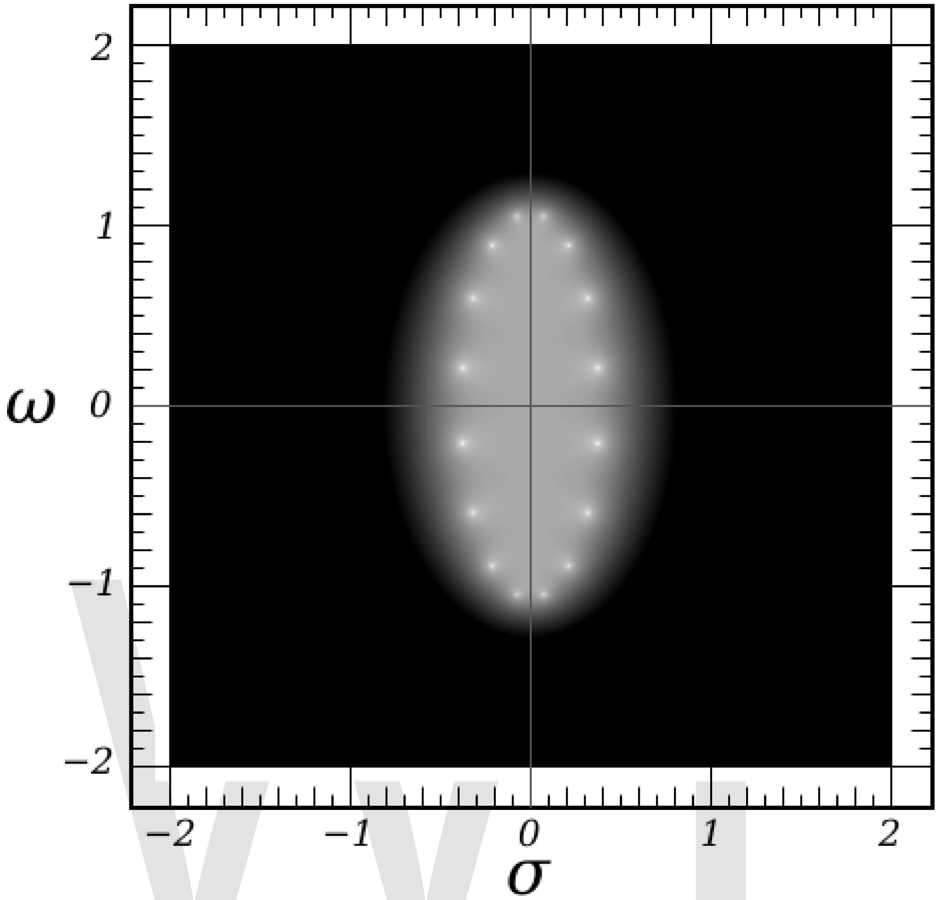
The ripple is often given in dB:

Ripple in dB = $20\log_{10}\frac{1}{\sqrt{1+\varepsilon^2}}$

so that a ripple amplitude of 3 dB results from $\varepsilon = 1.$

An even steeper roll-off can be obtained if we allow for ripple in the stop band, by allowing zeroes on the $j\omega$-axis in the complex plane. This will however result in less suppression in the stop band. The result is called an elliptic filter, also known as Cauer filters.

Poles and zeroes



Log of the absolute value of the gain of an 8th order Chebyshev type I filter in complex frequency space ($s = \sigma + j\omega$) with $\varepsilon = 0.1$ and $\omega_0 = 1$. The white spots are poles and are arranged on an ellipse with a semi-axis of 0.3836... in σ and 1.071... in ω. The transfer function poles are those poles in the left half plane. Black corresponds to a gain of 0.05 or less, white corresponds to a gain of 20 or more.

For simplicity, assume that the cutoff frequency is equal to unity. The poles (ω_{pm}) of the gain of the Chebyshev filter will be the zeroes of the denominator of the gain. Using the complex frequency s:

$$1 + \varepsilon^2 T_n^2(-js) = 0.$$

Defining $-js = \cos(\theta)$ and using the trigonometric definition of the Chebyshev polynomials yields:

$$1 + \varepsilon^2 T_n^2(\cos(\theta)) = 1 + \varepsilon^2 \cos^2(n\theta) = 0.$$

Solving for θ

$$\theta = \frac{1}{n}\arccos\left(\frac{\pm j}{\varepsilon}\right) + \frac{m\pi}{n}$$

where the multiple values of the arc cosine function are made explicit using the integer index *m*. The poles of the Chebyshev gain function are then:

$$\begin{aligned} s_{pm} &= j\cos(\theta) \\ &= j\cos\left(\frac{1}{n}\arccos\left(\frac{\pm j}{\varepsilon}\right) + \frac{m\pi}{n}\right) \end{aligned}$$

Using the properties of the trigonometric and hyperbolic functions, this may be written in explicitly complex form:

$$\begin{aligned} s_{pm}^{\pm} = &\pm \sinh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\sin(\theta_m) \\ &+ j\cosh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\cos(\theta_m) \end{aligned}$$

where $m = 1, 2, ..., n$ and

$$\theta_m = \frac{\pi}{2}\,\frac{2m-1}{n}$$

This may be viewed as an equation parametric in θ_n and it demonstrates that the poles lie on an ellipse in *s*-space centered at $s = 0$ with a real semi-axis of length $\sinh(\mathrm{arsinh}(1/\varepsilon)/n)$ and an imaginary semi-axis of length of $\cosh(\mathrm{arsinh}(1/\varepsilon)/n)$.

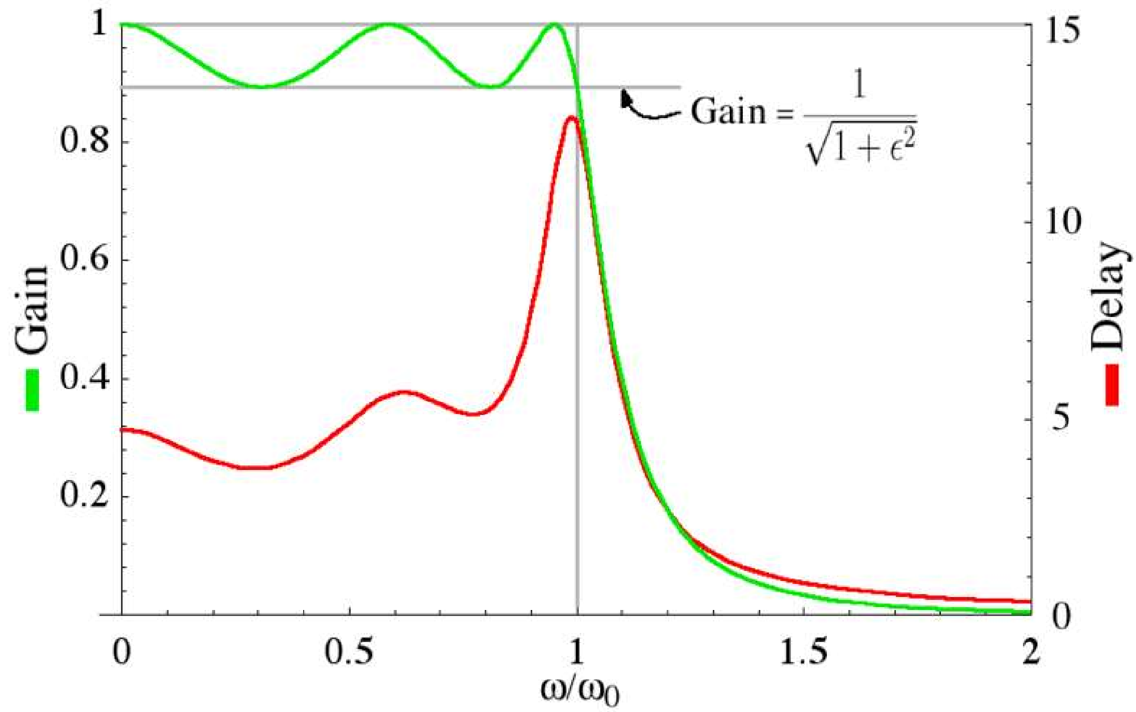
The transfer function

The above expression yields the poles of the gain *G*. For each complex pole, there is another which is the complex conjugate, and for each conjugate pair there are two more that are the negatives of the pair. The transfer function must be stable, so that its poles will be those of the gain that have negative real parts and therefore lie in the left half plane of complex frequency space. The transfer function is then given by

$$H(s) = \frac{1}{2^{n-1}\varepsilon}\ \prod_{m=1}^{n} \frac{1}{(s - s_{pm}^{-})}$$

where s_{pm}^{-} are only those poles with a negative sign in front of the real term in the above equation for the poles.

The group delay



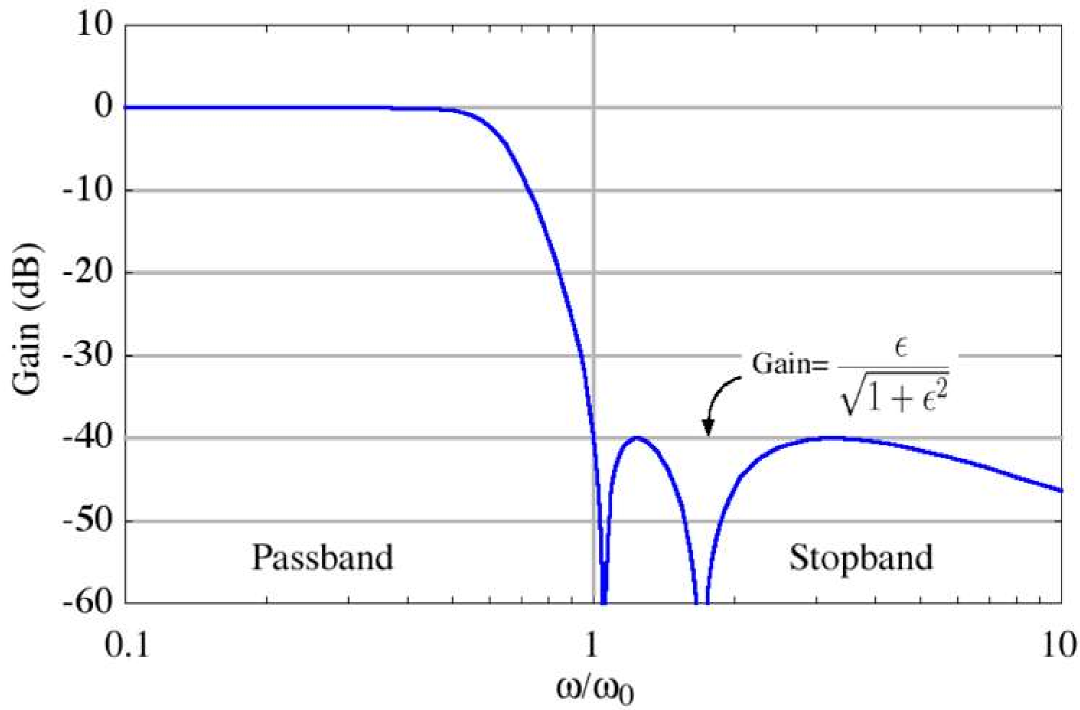
Gain and group delay of a fifth-order type I Chebyshev filter with ε = 0.5.

The group delay is defined as the derivative of the phase with respect to angular frequency and is a measure of the distortion in the signal introduced by phase differences for different frequencies.

$$\tau_g = -\frac{d}{d\omega}\arg(H(j\omega))$$

The gain and the group delay for a fifth order type I Chebyshev filter with ε=0.5 are plotted in the graph on the left. It can be seen that there are ripples in the gain and the group delay in the passband but not in the stop band.

Type II Chebyshev filters



The frequency response of a fifth-order type II Chebyshev low-pass filter with $\varepsilon = 0.01$

Also known as inverse Chebyshev, this type is less common because it does not roll off as fast as type I, and requires more components. It has no ripple in the passband, but does have equiripple in the stopband. The gain is:

$$G_n(\omega, \omega_0) = \frac{1}{\sqrt{1 + \frac{1}{\varepsilon^2 T_n^2(\omega_0/\omega)}}}.$$

In the stop band, the Chebyshev polynomial will oscillate between 0 and 1 so that the gain will oscillate between zero and

$$\frac{1}{\sqrt{1 + \frac{1}{\varepsilon^2}}}$$

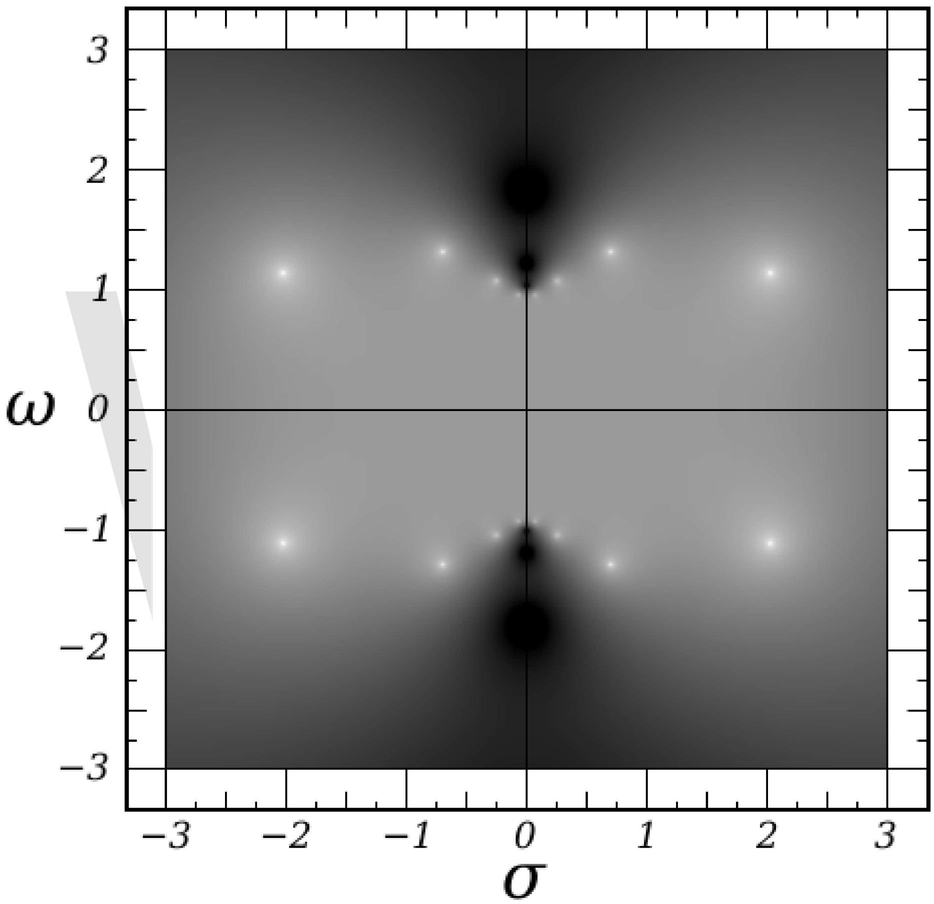
and the smallest frequency at which this maximum is attained will be the cutoff frequency ω_0. The parameter ε is thus related to the stopband attenuation γ in decibels by:

$$\varepsilon = \frac{1}{\sqrt{10^{0.1\gamma} - 1}}.$$

For a stopband attenuation of 5dB, ε = 0.6801; for an attenuation of 10dB, ε = 0.3333. The frequency $f_C = \omega_C/2\pi$ is the cutoff frequency. The 3dB frequency f_H is related to f_C by:

$$f_H = \frac{f_C}{\cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)}.$$

Poles and zeroes



Log of the absolute value of the gain of an 8th order Chebyshev type II filter in complex frequency space (s=σ+jω) with ε = 0.1 and $\omega_0 = 1$. The white spots are poles and the black spots are zeroes. All 16 poles are shown. Each zero has multiplicity of two, and 12 zeroes are shown and four are located outside the picture, two on the positive ω axis, and two on the negative. The poles of the transfer function will be poles on the left half plane and the zeroes of the transfer function will be the zeroes, but with multiplicity 1. Black corresponds to a gain of 0.05 or less, white corresponds to a gain of 20 or more.

Again, assuming that the cutoff frequency is equal to unity, the poles (ω_{pm}) of the gain of the Chebyshev filter will be the zeroes of the denominator of the gain:

$$1+\varepsilon^2 T_n^2(-1/js_{pm})=0$$

The poles of gain of the type II Chebyshev filter will be the inverse of the poles of the type I filter:

$$\frac{1}{s_{pm}^{\pm}}=\pm\sinh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\sin(\theta_m)$$
$$+j\cosh\left(\frac{1}{n}\mathrm{arsinh}\left(\frac{1}{\varepsilon}\right)\right)\cos(\theta_m)$$

where $m = 1, 2, ..., n$. The zeroes (ω_{zm}) of the type II Chebyshev filter will be the zeroes of the numerator of the gain:

$$\varepsilon^2 T_n^2(-1/js_{zm})=0.$$

The zeroes of the type II Chebyshev filter will thus be the inverse of the zeroes of the Chebyshev polynomial.

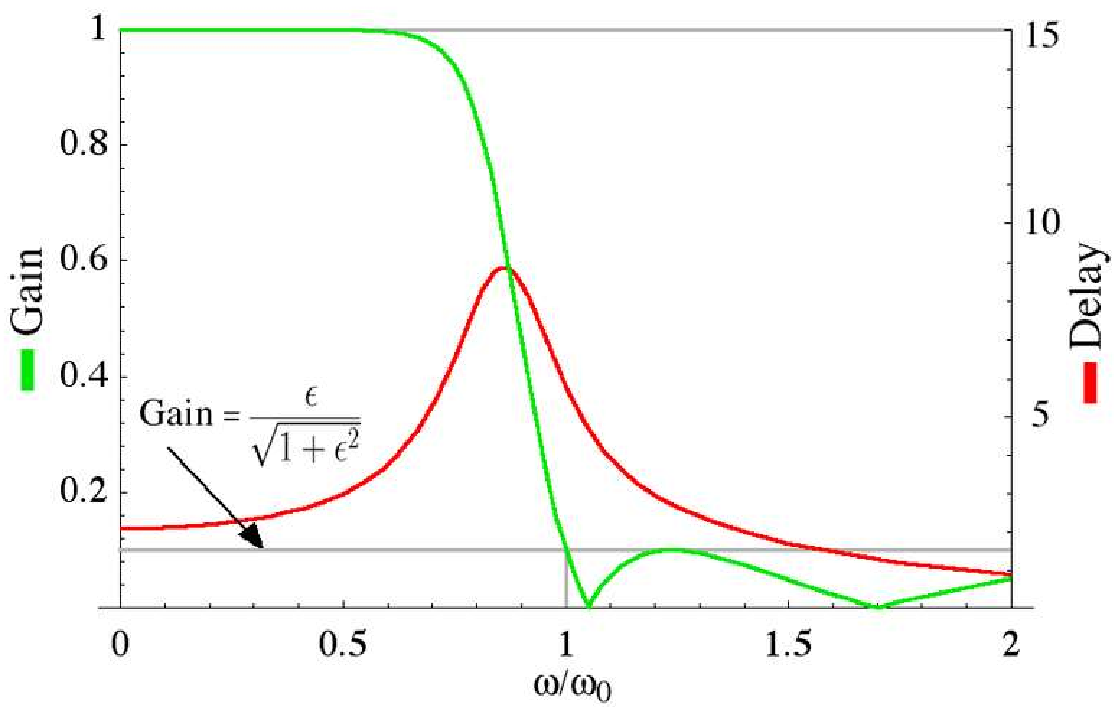
$$1/s_{zm}=-j\cos\left(\frac{\pi}{2}\,\frac{2m-1}{n}\right)$$

for $m = 1, 2, ..., n$.

The transfer function

The transfer function will be given by the poles in the left half plane of the gain function, and will have the same zeroes but these zeroes will be single rather than double zeroes.

The group delay



Gain and group delay of a fifth-order type II Chebyshev filter with ε = 0.1.

The gain and the group delay for a fifth order type II Chebyshev filter with ε=0.1 are plotted in the graph on the left. It can be seen that there are ripples in the gain in the stop band but not in the pass band.

Implementation

Cauer topology

A passive LC Chebyshev low-pass filter may be realized using a Cauer topology. Inductor or capacitor values of a nth-order Chebyshev filter may be calculated from the following equations:

$$G_1 = \frac{2A_1\cosh(f_H)}{Y}$$

$$G_k = \frac{4A_{k-1}A_k\cosh^2(f_H)}{B_{k-1}G_{k-1}}, \qquad k = 1,2,3,\ldots,n$$

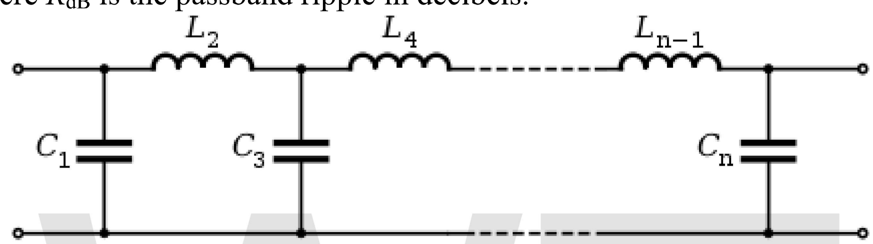
G_1, G_k are the capacitor or inductor element values.

f_H, the 3 dB frequency is calculated with: $f_H = f_C\cosh\left(\frac{1}{n}\cosh^{-1}\frac{1}{\varepsilon}\right)$

The coefficients A, Y, β, A_k, and B_k may be calculated from the following equations:

$$Y = \sinh\left(\frac{\beta}{2n}\right)$$
$$\beta = \ln\left(\coth(R_{db}/17.37)\right)$$
$$A_k = \sin\frac{(2k-1)\pi}{2n}, \qquad k = 1, 2, 3, \ldots, n$$
$$B_k = Y^2 + \sin^2\left(\frac{k\pi}{n}\right), \qquad k = 1, 2, 3, \ldots, n$$

where R_{dB} is the passband ripple in decibels.



The calculated G_k values may then be converted into shunt capacitors and top inductors as shown on the right, or they may be converted into top capacitors and shunt inductors.

- For example, $C_{1\ \text{shunt}} = G_1$, $L_{2\ \text{top}} = G_2$, ...
- or $L_{1\ \text{shunt}} = G_1$, $C_{1\ \text{top}} = G_2$, ...

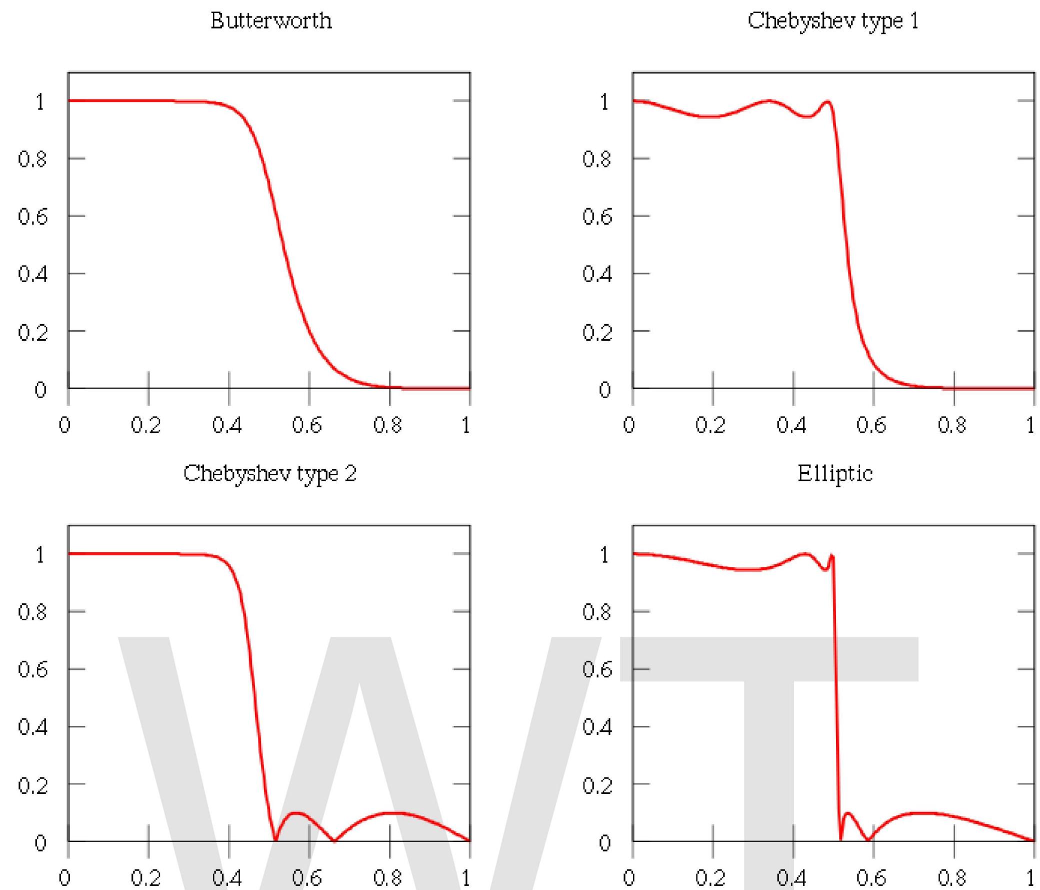
The resulting circuit is a normalized low-pass filter. Using frequency transformations and impedance scaling, the normalized low-pass filter may be transformed into high-pass, band-pass, and band-stop filters of any desired cutoff frequency or bandwidth.

Digital

As with most analog filters, the Chebyshev may be converted to a digital (discrete-time) recursive form via the bilinear transform. However, as digital filters have a finite bandwidth, the response shape of the transformed Chebyshev will be warped. Alternatively, the Matched Z-transform method may be used, which does not warp the response.

Comparison with other linear filters

Here is an image showing the Chebyshev filters next to other common kind of filters obtained with the same number of coefficients (all filters are fifth order):



As is clear from the image, Chebyshev filters are sharper than the Butterworth filter; they are not as sharp as the elliptic one, but they show fewer ripples over the bandwidth.

Chapter- 3

Butterworth Filter

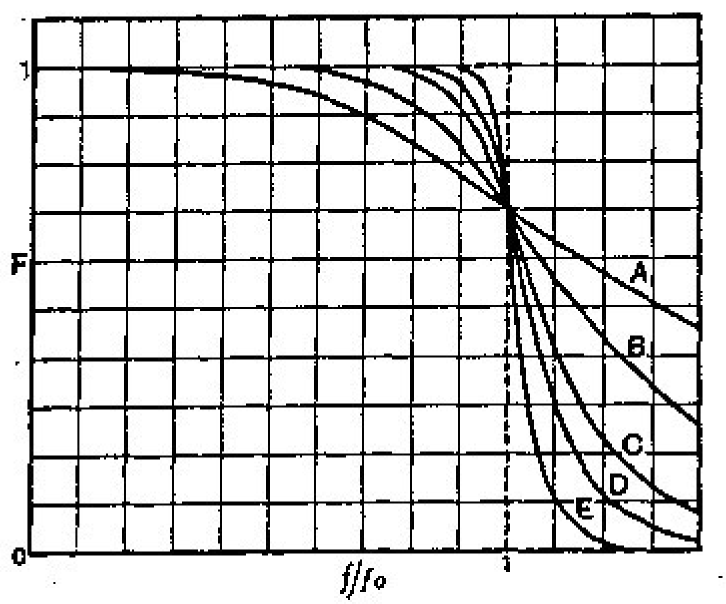


Fig. 3.

The frequency response plot from Butterworth's 1930 paper.

The **Butterworth filter** is a type of signal processing filter designed to have as flat a frequency response as possible in the passband so that it is also termed a **maximally flat magnitude filter**. It was first described by the British engineer Stephen Butterworth in his paper entitled "On the Theory of Filter Amplifiers".

Original paper

Butterworth had a reputation for solving "impossible" mathematical problems. At the time filter design was largely by trial and error because of their mathematical complexity.

His paper was far ahead of its time: the filter was not in common use for over 30 years after its publication. Butterworth stated that;

"An ideal electrical filter should not only completely reject the unwanted frequencies but should also have uniform sensitivity for the wanted frequencies."

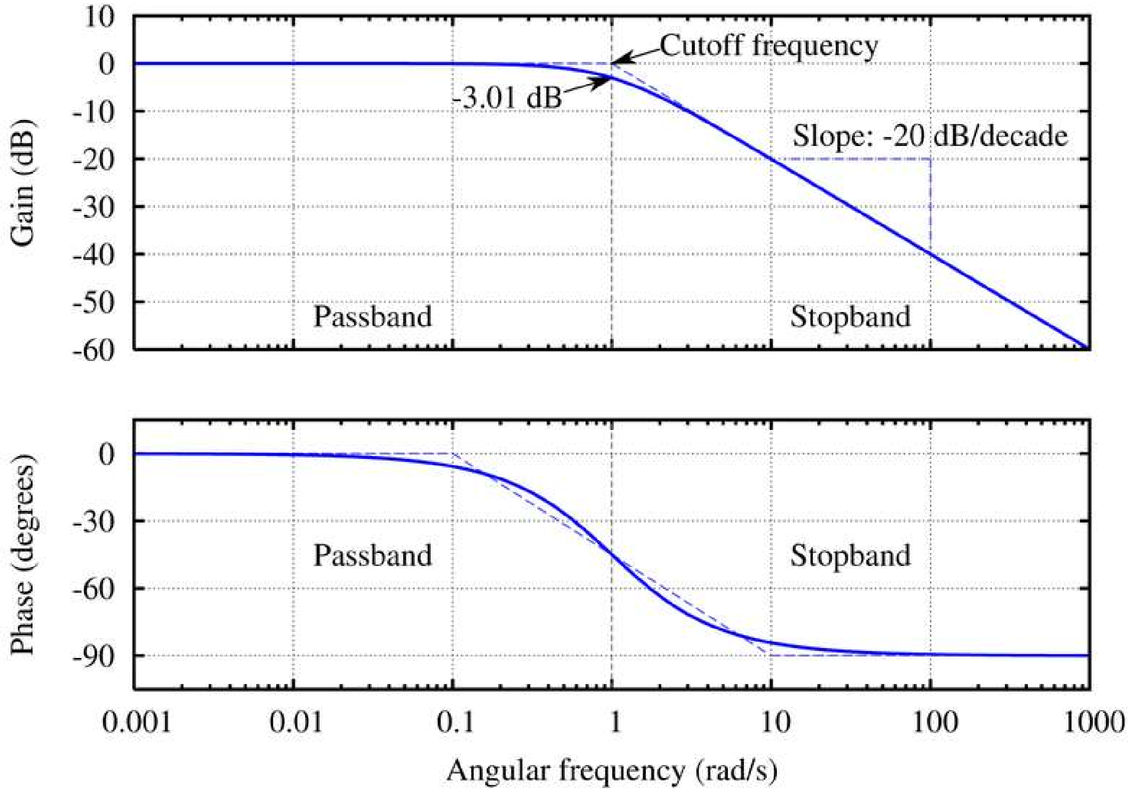
At the time filters generated substantial ripple in the passband and the choice of component values was highly interactive. Butterworth showed that low pass filters could be designed whose frequency response (gain) was;

$$G = \sqrt{\frac{1}{1 + \omega^{2n}}}$$

where ω is the angular frequency in radians per second and n is the number of reactive elements (poles) in the filter. Butterworth only dealt with filters with an even number of poles in his paper: he may have been unaware that such filters could be designed with an odd number of poles. His plot of the frequency response of 2, 4, 6, 8, and 10 pole filters is shown as A, B, C, D, and E in his original graph.

Butterworth solved the equations for two- and four-pole filters, showing how the latter could be cascaded when separated by vacuum tube amplifiers and so enabling the construction of higher-order filters despite inductor losses. In 1930 low-loss core materials such as molypermalloy had not been discovered and air-cored audio inductors were rather lossy. Butterworth discovered that it was possible to adjust the component values of the filter to compensate for the winding resistance of the inductors. He also showed that his basic low-pass filter could be modified to give low-pass, high-pass, band-pass and band-stop functionality.

Overview

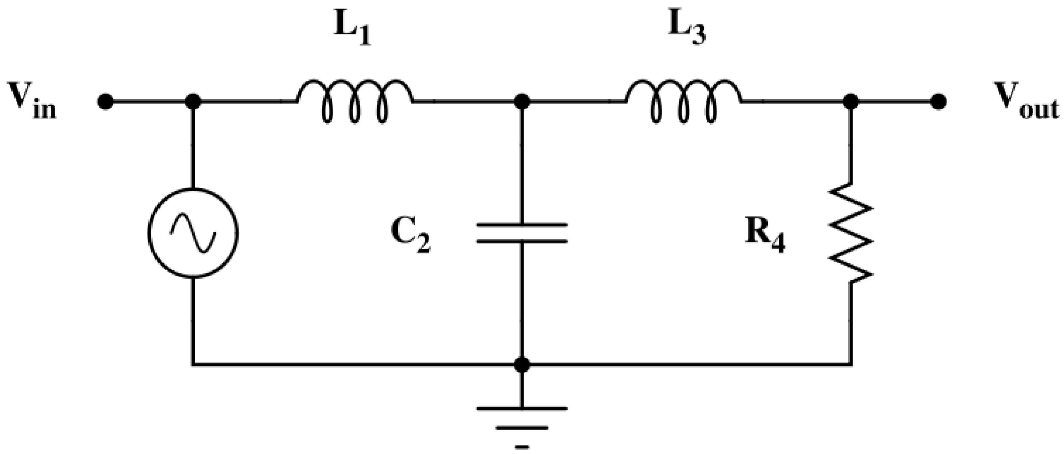


The Bode plot of a first-order Butterworth low-pass filter

The frequency response of the Butterworth filter is maximally flat (has no ripples) in the passband and rolls off towards zero in the stopband.` When viewed on a logarithmic Bode plot the response slopes off linearly towards negative infinity. A first-order filter's response rolls off at −6 dB per octave (−20 dB per decade) (all first-order lowpass filters have the same normalized frequency response). A second-order filter decreases at −12 dB per octave, a third-order at −18 dB and so on. Butterworth filters have a monotonically changing magnitude function with ω, unlike other filter types that have non-monotonic ripple in the passband and/or the stopband.

Compared with a Chebyshev Type I/Type II filter or an elliptic filter, the Butterworth filter has a slower roll-off, and thus will require a higher order to implement a particular stopband specification, but Butterworth filters have a more linear phase response in the pass-band than Chebyshev Type I/Type II and elliptic filters can achieve.

A simple example



A third-order low-pass filter (Cauer topology). The filter becomes a Butterworth filter with cutoff frequency ω_c=1 when (for example) C_2=4/3 farad, R_4=1 ohm, L_1=3/2 henry and L_3=1/2 henry.

A simple example of a Butterworth filter is the third-order low-pass design shown in the figure on the right, with $C_2 = 4 / 3$ farad, $R_4 = 1$ ohm, $L_1 = 3 / 2$ and $L_3 = 1 / 2$ henry. Taking the impedance of the capacitors C to be $1/Cs$ and the impedance of the inductors L to be Ls, where $s = \sigma + j\omega$ is the complex frequency, the circuit equations yield the transfer function for this device;

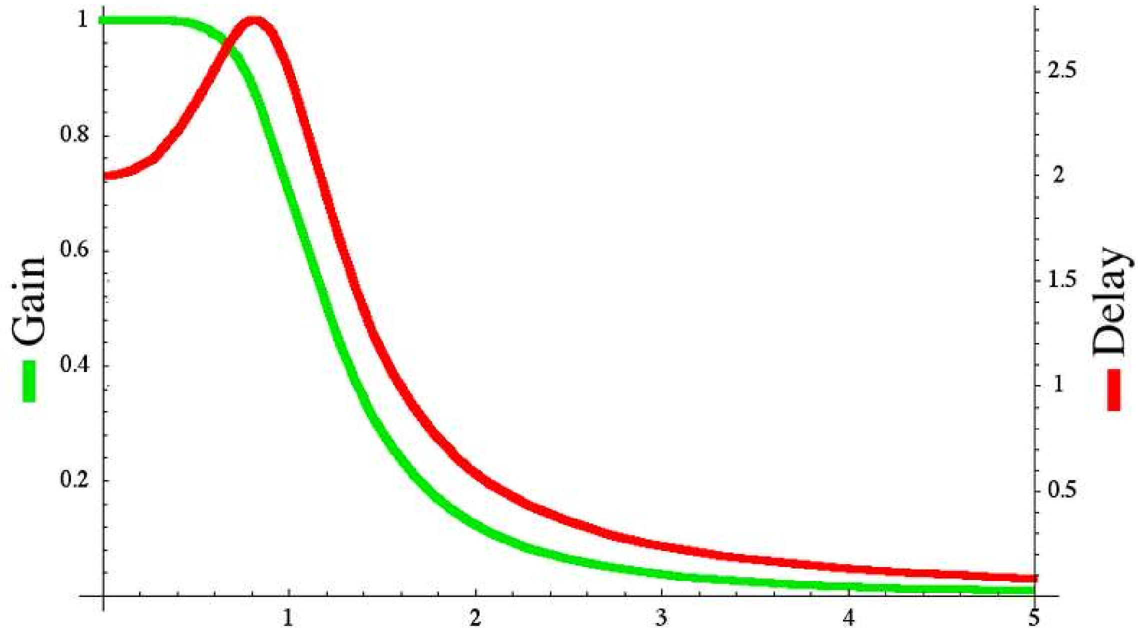
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + 2s + 2s^2 + s^3}$$

The magnitude of the frequency response (gain) $G(\omega)$ is given by;

$$G^2(\omega) = |H(j\omega)|^2 = \frac{1}{1 + \omega^6}$$

and the phase is given by;

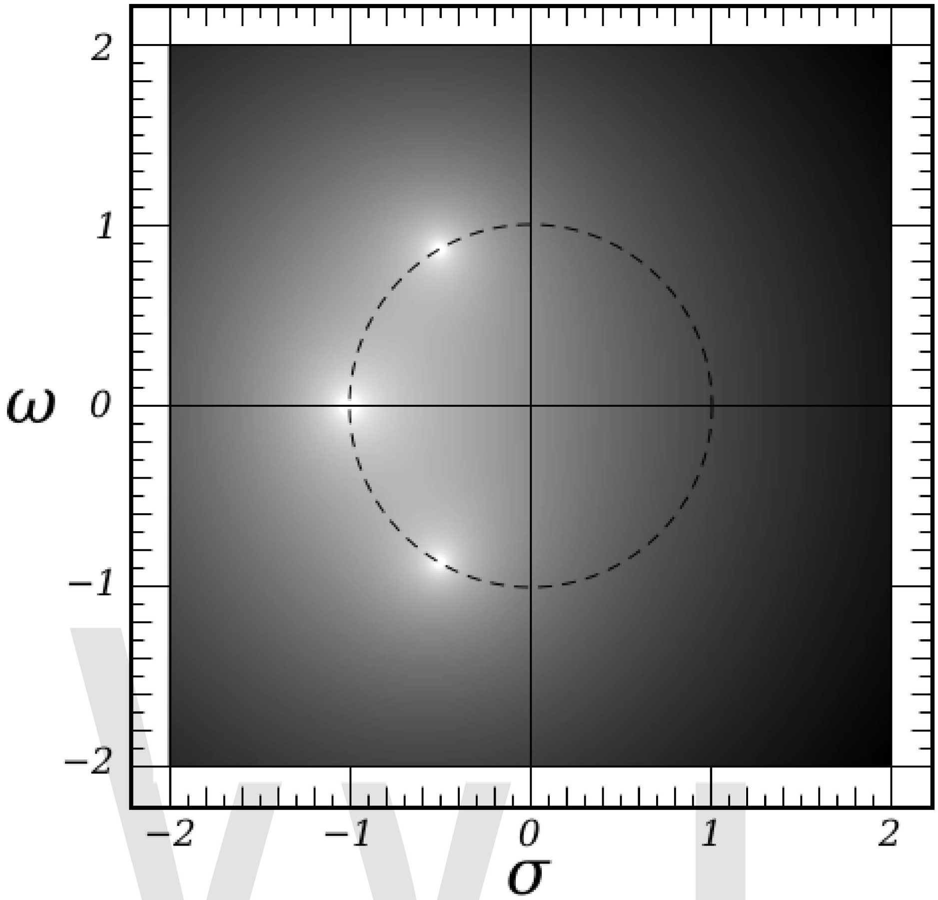
$$\Phi(\omega) = \arg(H(j\omega))$$



Gain and group delay of the third-order Butterworth filter with ω_c=1

The group delay is defined as the derivative of the phase with respect to angular frequency and is a measure of the distortion in the signal introduced by phase differences for different frequencies. The gain and the delay for this filter are plotted in the graph on the left. It can be seen that there are no ripples in the gain curve in either the passband or the stop band.

The log of the absolute value of the transfer function *H(s)* is plotted in complex frequency space in the second graph on the right. The function is defined by the three poles in the left half of the complex frequency plane.



Log density plot of the transfer function H(s) in complex frequency space for the third-order Butterworth filter with ω_c=1. The three poles lie on a circle of unit radius in the left half-plane.

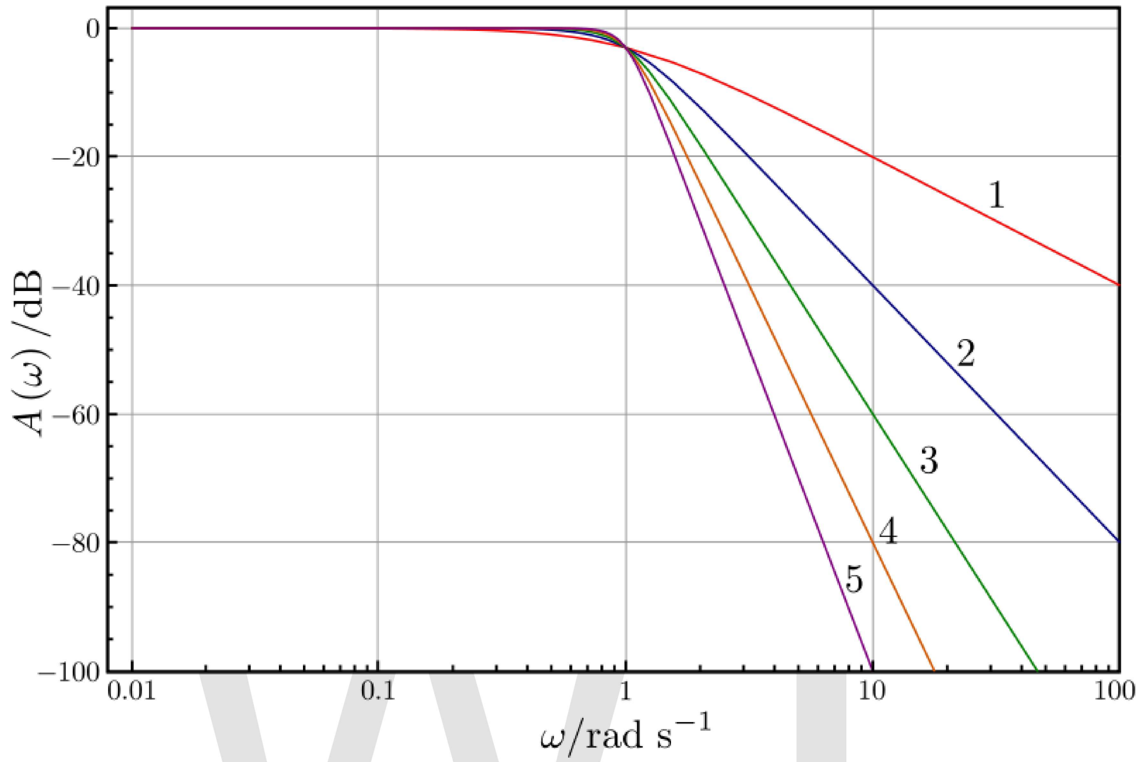
These are arranged on a circle of radius unity, symmetrical about the real *s* axis. The gain function will have three more poles on the right half plane to complete the circle.

By replacing each inductor with a capacitor and each capacitor with an inductor, a high-pass Butterworth filter is obtained.

A band-pass Butterworth filter is obtained by placing a capacitor in series with each inductor and an inductor in parallel with each capacitor to form resonant circuits. The value of each new component must be selected to resonate with the old component at the frequency of interest.

A band-stop Butterworth filter is obtained by placing a capacitor in parallel with each inductor and an inductor in series with each capacitor to form resonant circuits. The value of each new component must be selected to resonate with the old component at the frequency to be rejected.

Transfer function



Plot of the gain of Butterworth low-pass filters of orders 1 through 5, with cutoff frequency $\omega_0 = 1$. Note that the slope is 20*n* dB/decade where *n* is the filter order.

Like all filters, the typical prototype is the low-pass filter, which can be modified into a high-pass filter, or placed in series with others to form band-pass and band-stop filters, and higher order versions of these.

The gain $G(\omega)$ of an *n*-order Butterworth low pass filter is given in terms of the transfer function *H(s)* as;

$$G^2(\omega) = |H(j\omega)|^2 = \frac{{G_0}^2}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

where

- n = order of filter
- ω_c = cutoff frequency (approximately the -3dB frequency)
- G_0 is the DC gain (gain at zero frequency)

It can be seen that as n approaches infinity, the gain becomes a rectangle function and frequencies below ω_c will be passed with gain G_0, while frequencies above ω_c will be suppressed. For smaller values of n, the cutoff will be less sharp.

We wish to determine the transfer function $H(s)$ where $s = \sigma + j\omega$ (from Laplace transform). Since $H(s)H(-s)$ evaluated at $s = j\omega$ is simply equal to $|H(j\omega)|^2$, it follows that;

$$H(s)H(-s) = \frac{{G_0}^2}{1 + \left(\frac{-s^2}{\omega_c^2}\right)^n}$$

The poles of this expression occur on a circle of radius ω_c at equally spaced points. The transfer function itself will be specified by just the poles in the negative real half-plane of s. The *k-th* pole is specified by;

$$-\frac{s_k^2}{\omega_c^2} = (-1)^{\frac{1}{n}} = e^{\frac{j(2k-1)\pi}{n}} \qquad \mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$$

and hence;

$$s_k = \omega_c e^{\frac{j(2k+n-1)\pi}{2n}} \qquad \mathrm{k} = 1, 2, 3, \ldots, \mathrm{n}$$

The transfer function may be written in terms of these poles as;

$$H(s) = \frac{G_0}{\prod_{k=1}^{n}(s - s_k)/\omega_c}$$

The denominator is a Butterworth polynomial in s.

Normalized Butterworth polynomials

The Butterworth polynomials may be written in complex form as above, but are usually written with real coefficients by multiplying pole pairs which are complex conjugates, such as s_1 and s_n. The polynomials are normalized by setting $\omega_c = 1$. The normalized Butterworth polynomials then have the general form;

$$B_n(s) = \prod_{k=1}^{\frac{n}{2}} \left[s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\,\pi\right) + 1\right]_{\text{for n even}}$$

$$B_n(s) = (s+1)\prod_{k=1}^{\frac{n-1}{2}} \left[s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\,\pi\right) + 1\right]_{\text{for n odd}}$$

To four decimal places, they are;

n	Factors of Polynomial $B_n(s)$
1	$(s + 1)$
2	$s^2 + 1.4142s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$
5	$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.4142s + 1)(s^2 + 1.9319s + 1)$
7	$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2470s + 1)(s^2 + 1.8019s + 1)$
8	$(s^2 + 0.3902s + 1)(s^2 + 1.1111s + 1)(s^2 + 1.6629s + 1)(s^2 + 1.9616s + 1)$

The normalized Butterworth polynomials can be used to determine the transfer function for any low-pass filter cut-off frequency ω_c, as follows

$$H(s) = \frac{G_0}{B_n(a)}, \text{ where } a = \frac{s}{\omega_c}$$

Maximal flatness

Assuming $\omega_c = 1$ and $G_0 = 1$, the derivative of the gain with respect to frequency can be shown to be;

$$\frac{dG}{d\omega} = -nG^3\omega^{2n-1}$$

which is monotonically decreasing for all ω since the gain G is always positive. The gain function of the Butterworth filter therefore has no ripple. Furthermore, the series expansion of the gain is given by;

$$G(\omega) = 1 - \frac{1}{2}\omega^{2n} + \frac{3}{8}\omega^{4n} + \ldots$$

In other words all derivatives of the gain up to but not including the $2n$-th derivative are zero, resulting in "maximal flatness". If the requirement to be monotonic is limited to the passband only and ripples are allowed in the stopband, then it is possible to design a filter of the same order, such as the inverse Chebyshev filter, that is flatter in the passband than the "maximally flat" Butterworth.

High-frequency roll-off

Again assuming $\omega_c = 1$, the slope of the log of the gain for large ω is;

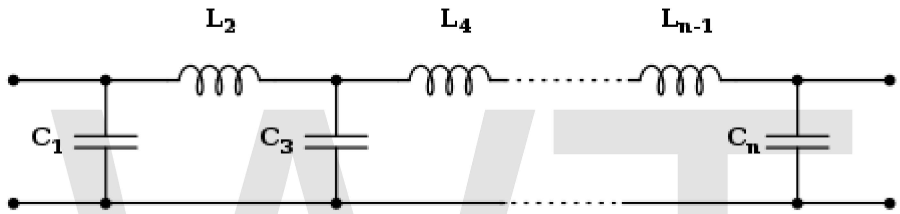
$$\lim_{\omega \to \infty} \frac{d \log(G)}{d \log(\omega)} = -n$$

In decibels, the high-frequency roll-off is therefore 20*n* dB/decade, or 6*n* dB/octave (The factor of 20 is used because the power is proportional to the square of the voltage gain.)

Filter design

There are a number of different filter topologies available to implement a linear analogue filter. The most often used topology for a passive realisation is Cauer topology and the most often used topology for an active realisation is Sallen-Key topology.

Cauer topology



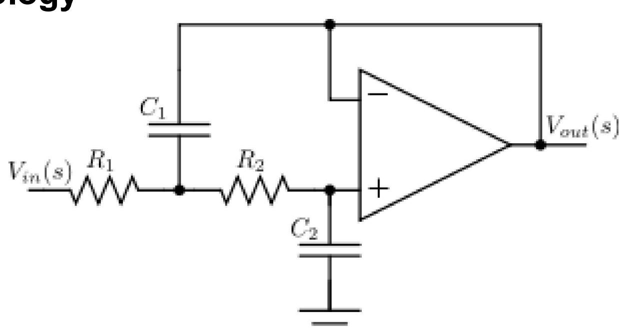
The Cauer topology uses passive components (shunt capacitors and series inductors) to implement a linear analog filter. The Butterworth filter having a given transfer function can be realised using a Cauer 1-form. The kth element is given by;

$$C_k = 2 \sin \left[\frac{(2k-1)}{2n} \pi \right]; \text{ k = odd}$$

$$L_k = 2 \sin \left[\frac{(2k-1)}{2n} \pi \right]; \text{ k = even}$$

The filter may start with a series inductor if desired, in which case the L_k are k odd and the C_k are k even.

Sallen–Key topology



The Sallen–Key topology uses active and passive components (noninverting buffers, usually op amps, resistors, and capacitors) to implement a linear analog filter. Each Sallen-Key stage implements a conjugate pair of poles; the overall filter is implemented by cascading all stages in series. If there is a real pole (in the case where *n* is odd), this must be implemented separately, usually as an RC circuit, and cascaded with the active stages.

For the second order Sallen-Key circuit shown to the right the transfer function is given by;

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{1}{1 + C_2(R_1 + R_2)s + C_1C_2R_1R_2s^2}$$

We wish the denominator to be one of the quadratic terms in a Butterworth polynomial. Assuming that $\omega_c = 1$, this will mean that;

$$C_1C_2R_1R_2 = 1$$

and;

$$C_2(R_1 + R_2) = -2\cos\left(\frac{2k+n-1}{2n}\pi\right)$$

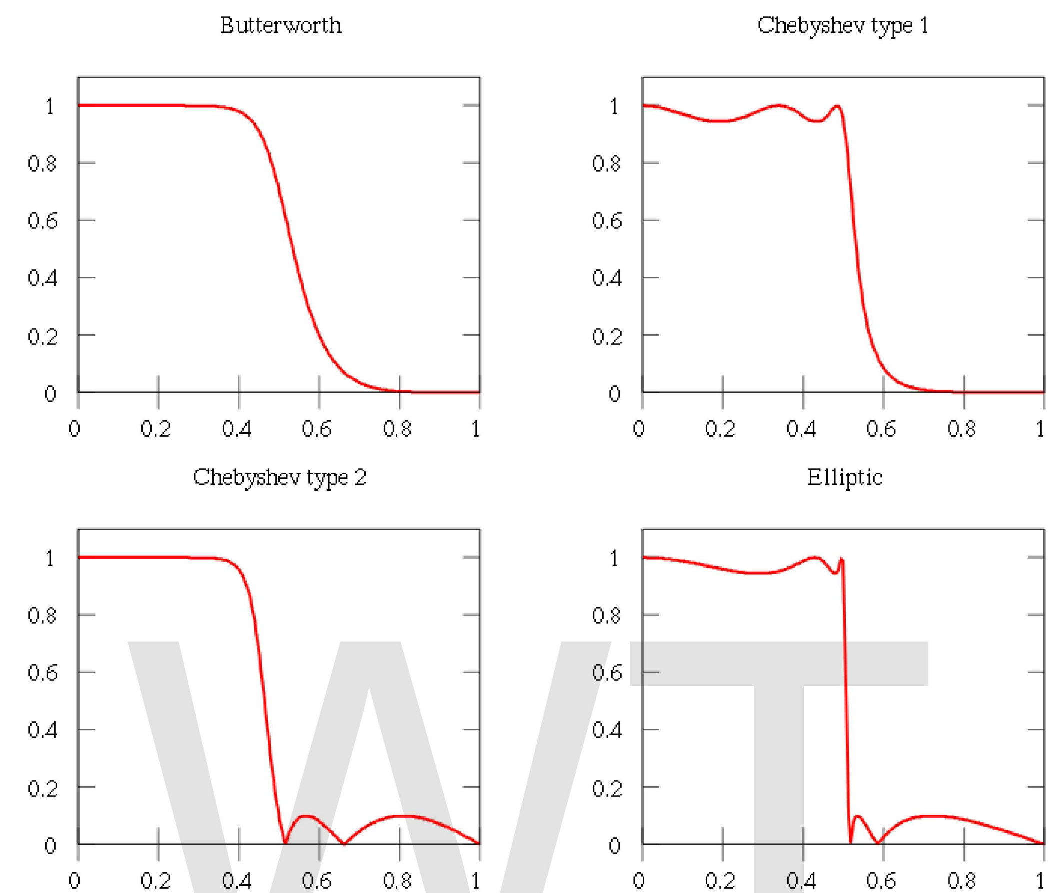
This leaves two undefined component values that may be chosen at will.

Digital implementation

Digital implementations of Butterworth and other filters are often based on the bilinear transform method or the matched Z-transform method, two different methods to discretize an analog filter design. In the case of all-pole filters such as the Butterworth, the matched Z-transform method is equivalent to the impulse invariance method. For higher orders, digital filters are sensitive to quantization errors, so they are often calculated as cascaded biquad sections, plus one first-order or third-order section for odd orders.

Comparison with other linear filters

Here is an image showing the gain of a discrete-time Butterworth filter next to other common filter types. All of these filters are fifth-order.



The Butterworth filter rolls off more slowly around the cutoff frequency than the Chebyshev filter or the Elliptic filter, but without ripple.

Chapter- 4

Elliptic Filter and Optimum "L" Filter

Elliptic filter

An **elliptic filter** (also known as a **Cauer filter**, named after Wilhelm Cauer) is a signal processing filter with equalized ripple (equiripple) behavior in both the passband and the stopband. The amount of ripple in each band is independently adjustable, and no other filter of equal order can have a faster transition in gain between the passband and the stopband, for the given values of ripple (whether the ripple is equalized or not). Alternatively, one may give up the ability to independently adjust the passband and stopband ripple, and instead design a filter which is maximally insensitive to component variations.

As the ripple in the stopband approaches zero, the filter becomes a type I Chebyshev filter. As the ripple in the passband approaches zero, the filter becomes a type II Chebyshev filter and finally, as both ripple values approach zero, the filter becomes a Butterworth filter.

The gain of a lowpass elliptic filter as a function of angular frequency ω is given by:

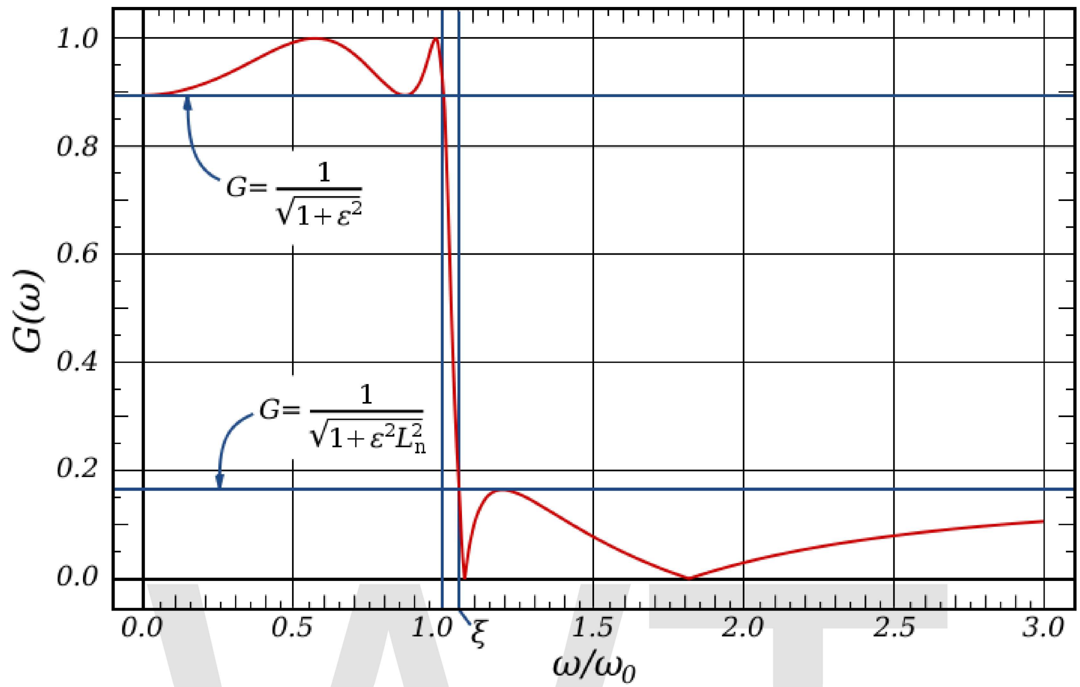
$$G_n(\omega) = \frac{1}{\sqrt{1 + \epsilon^2 R_n^2(\xi, \omega/\omega_0)}}$$

where R_n is the *n*th-order elliptic rational function (sometimes known as a Chebyshev rational function) and

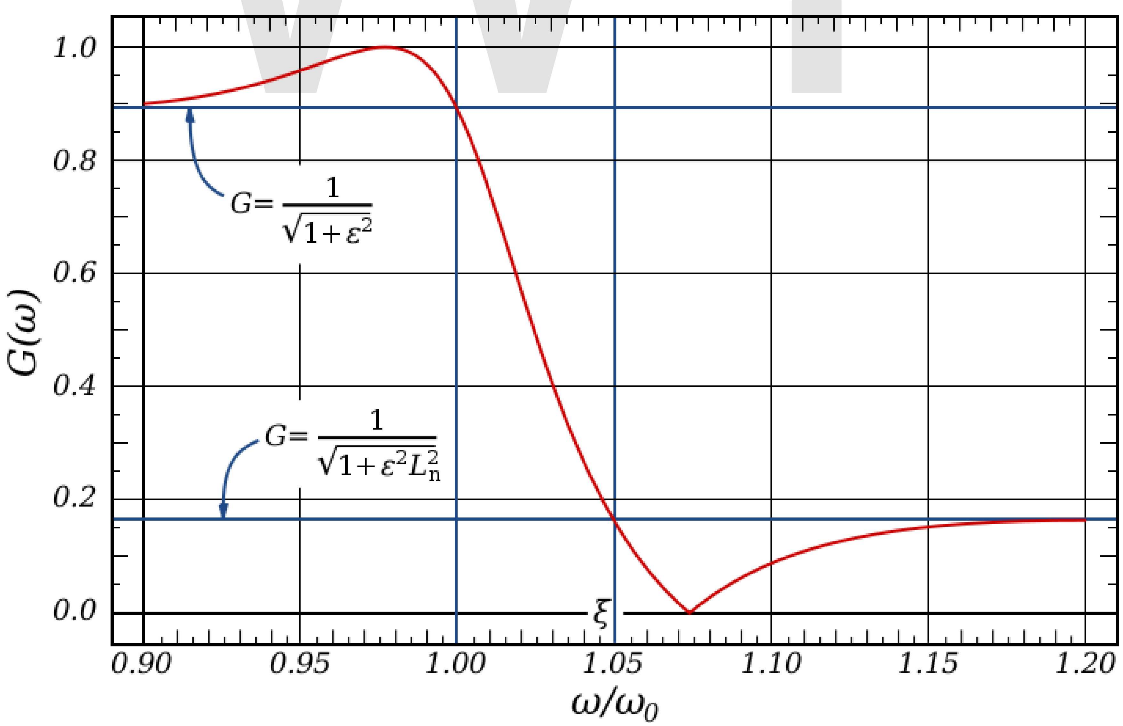
ω_0 is the cutoff frequency
ε is the ripple factor
ξ is the selectivity factor

The value of the ripple factor specifies the passband ripple, while the combination of the ripple factor and the selectivity factor specify the stopband ripple.

Properties



The frequency response of a fourth-order elliptic low-pass filter with ε=0.5 and ξ=1.05. Also shown are the minimum gain in the passband and the maximum gain in the stopband, and the transition region between normalized frequency 1 and ξ



A closeup of the transition region of the above plot.

- In the passband, the elliptic rational function varies between zero and unity. The passband of the gain therefore will vary between 1 and $1/\sqrt{1+\epsilon^2}$.

- In the stopband, the elliptic rational function varies between infinity and the discrimination factor L_n which is defined as:

  $$L_n = R_n(\xi,\xi)$$

  The gain of the stopband therefore will vary between 0 and $1/\sqrt{1+\epsilon^2 L_n^2}$.

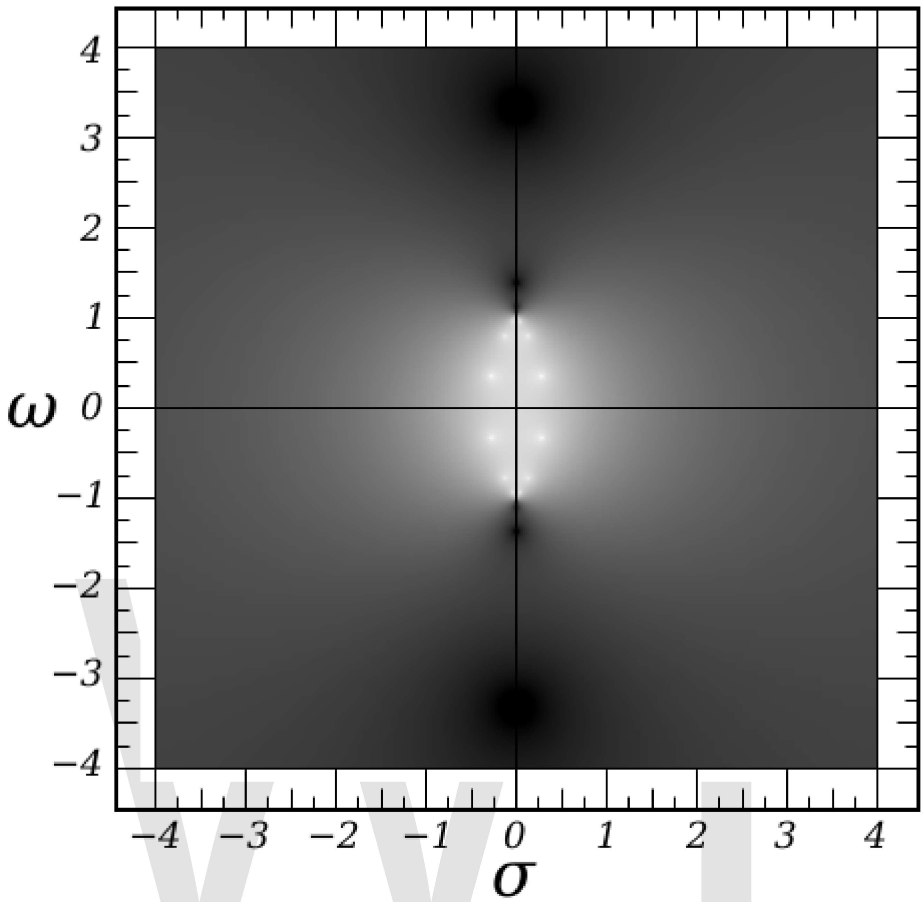
- In the limit of $\xi \rightarrow \infty$ the elliptic rational function becomes a Chebyshev polynomial, and therefore the filter becomes a Chebyshev type I filter, with ripple factor ε

- Since the Butterworth filter is a limiting form of the Chebyshev filter, it follows that in the limit of $\xi \rightarrow \infty$, $\omega_0 \rightarrow 0$ and $\epsilon \rightarrow 0$ such that $\epsilon\, R_n(\xi, 1/\omega_0) = 1$ the filter becomes a Butterworth filter

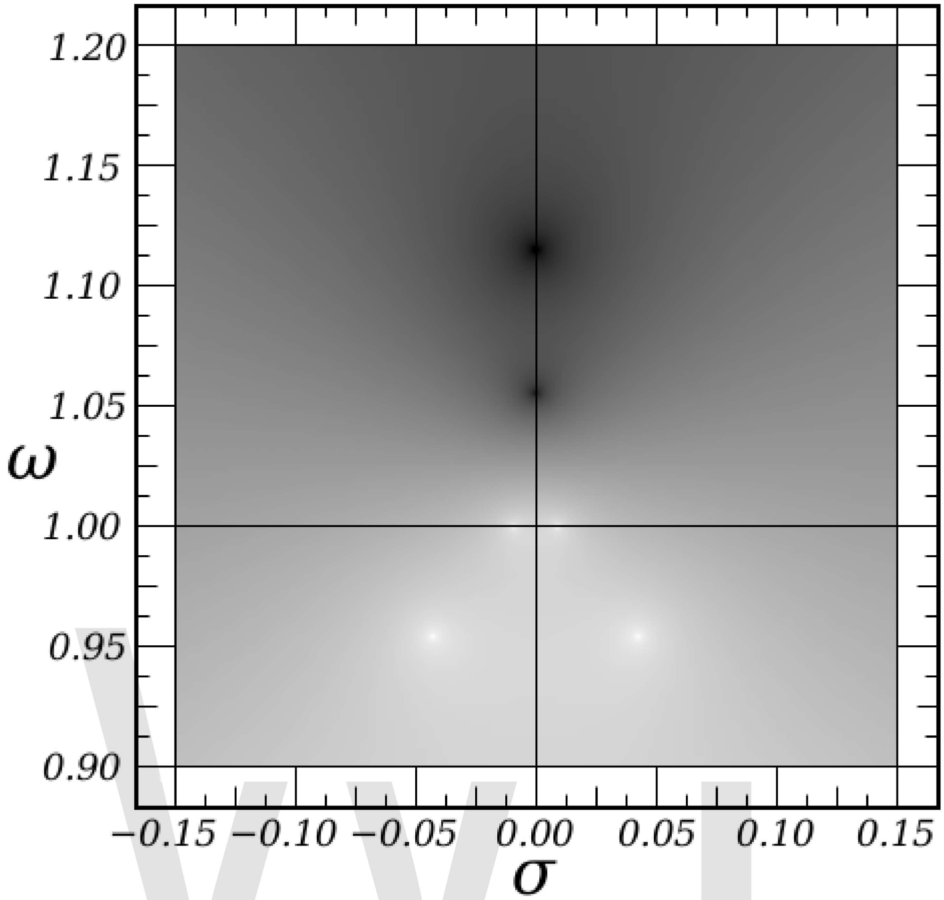
- In the limit of $\xi \rightarrow \infty$, $\epsilon \rightarrow 0$ and $\omega_0 \rightarrow 0$ such that $\xi\omega_0 = 1$ and $\epsilon L_n = \alpha$, the filter becomes a Chebyshev type II filter with gain

  $$G(\omega) = \frac{1}{\sqrt{1+\frac{1}{\alpha^2 T_n^2(1/\omega)}}}$$

Poles and zeroes



Log of the absolute value of the gain of an 8th order elliptic filter in complex frequency space ($s=\sigma+j\omega$) with $\varepsilon=0.5$, $\xi=1.05$ and $\omega_0 = 1$. The white spots are poles and the black spots are zeroes. There are a total of 16 poles and 8 double zeroes. What appears to be a single pole and zero near the transition region is actually four poles and two double zeroes as shown in the expanded view below. In this image, black corresponds to a gain of 0.0001 or less and white corresponds to a gain of 10 or more.



An expanded view in the transition region of the above image, resolving the four poles and two double zeroes.

The zeroes of the gain of an elliptic filter will coincide with the poles of the elliptic rational function, which are derived in elliptic rational functions.

The poles of the gain of an elliptic filter may be derived in a manner very similar to the derivation of the poles of the gain of a type I Chebyshev filter. For simplicity, assume that the cutoff frequency is equal to unity. The poles (ω_{pm}) of the gain of the elliptical filter will be the zeroes of the denominator of the gain. Using the complex frequency $s = \sigma + j\omega$ this means that:

$$1+\epsilon^2 R_n^2(-js,\xi)=0$$

Defining $-js = \mathrm{cd}(w,1/\xi)$ where cd() is the Jacobi elliptic cosine function and using the definition of the elliptic rational functions yields:

$$1+\epsilon^2\mathrm{cd}^2\left(\frac{nwK_n}{K},\frac{1}{L_n}\right)=0$$

where $K = K(1 / \xi)$ and $K_n = K(1 / L_n)$. Solving for w

$$w = \frac{K}{nK_n}\mathrm{cd}^{-1}\left(\frac{\pm j}{\epsilon}, \frac{1}{L_n}\right) + \frac{mK}{n}$$

where the multiple values of the inverse cd() function are made explicit using the integer index m.

The poles of the elliptic gain function are then:

$$s_{pm} = i\,\mathrm{cd}(w, 1/\xi)$$

As is the case for the Chebyshev polynomials, this may be expressed in explicitly complex form (Lutovac & et al. 2001, § 12.8)

$$\begin{aligned}
s_{pm} &= \frac{a + jb}{c} \\
a &= -\zeta_n\sqrt{1-\zeta_n^2}\sqrt{1-x_m^2}\sqrt{1-x_m^2/\xi^2} \\
b &= x_m\sqrt{1-\zeta_n^2(1-1/\xi^2)} \\
c &= 1-\zeta_n^2+x_i^2\zeta_n^2/\xi^2
\end{aligned}$$

where ζ_n is a function of n, ϵ and ξ and x_m are the zeroes of the elliptic rational function. ζ_n is expressible for all n in terms of Jacobi elliptic functions, or algebraically for some orders, especially orders 1,2, and 3. For orders 1 and 2 we have

$$\begin{aligned}
\zeta_1 &= \frac{1}{\sqrt{1+\epsilon^2}} \\
\zeta_2 &= \frac{2}{(1+t)\sqrt{1+\epsilon^2}+\sqrt{(1-t)^2+\epsilon^2(1+t)^2}}
\end{aligned}$$

where

$$t = \sqrt{1-1/\xi^2}$$

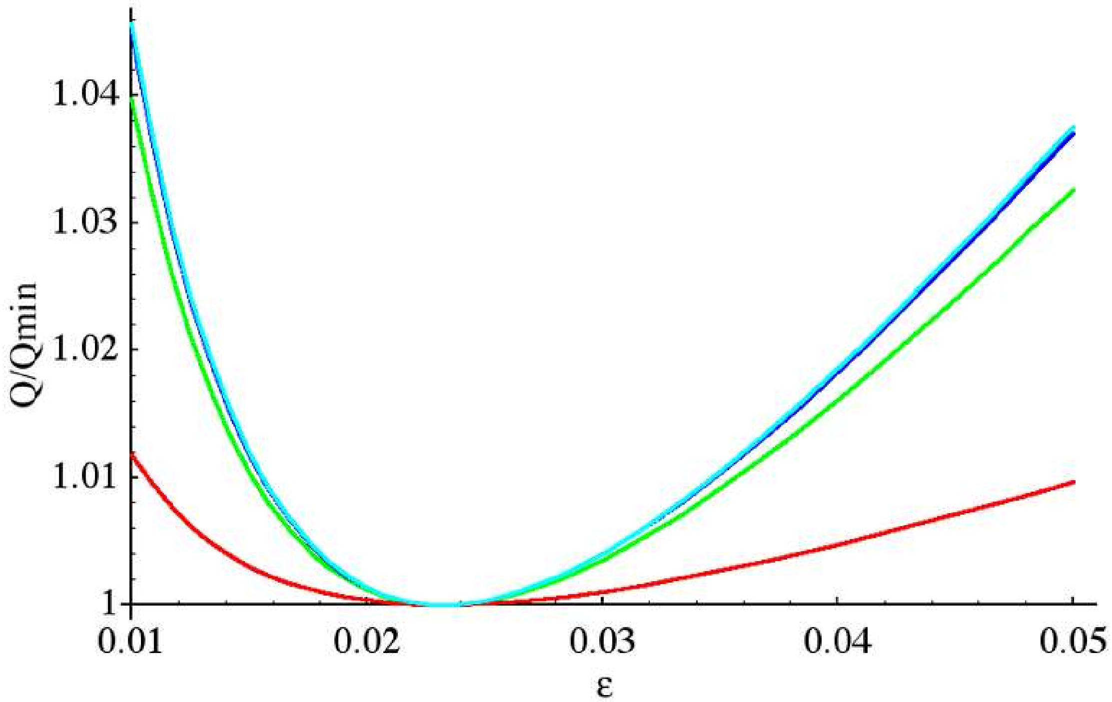
The algebraic expression for ζ_3 is rather involved.

The nesting property of the elliptic rational functions can be used to build up higher order expressions for ζ_n:

$$\zeta_{m\cdot n}(\xi, \epsilon) = \zeta_m\left(\xi, \sqrt{\frac{1}{\zeta_n^2(L_m, \epsilon)} - 1}\right)$$

where $L_m = R_m(\xi,\xi)$.

Minimum Q-factor elliptic filters



The normalized Q-factors of the poles of an 8-th order elliptic filter with ξ=1.1 as a function of ripple factor ε. Each curve represents four poles, since complex conjugate pole pairs and positive-negative pole pairs have the same Q-factor. (The blue and cyan curves nearly coincide). The Q-factor of all poles are simultaneously minimized at $\varepsilon_{Qmin}=1/\sqrt{L_n}=0.02323...$

Elliptic filters are generally specified by requiring a particular value for the passband ripple, stopband ripple and the sharpness of the cutoff. This will generally specify a minimum value of the filter order which must be used. Another design consideration is the sensitivity of the gain function to the values of the electronic components used to build the filter. This sensitivity is inversely proportional to the quality factor (Q-factor) of the poles of the transfer function of the filter. The Q-factor of a pole is defined as:

$$Q=-\frac{|s_{pm}|}{2\mathrm{Re}(s_{pm})}=-\frac{1}{2\cos(\arg(s_{pm}))}$$

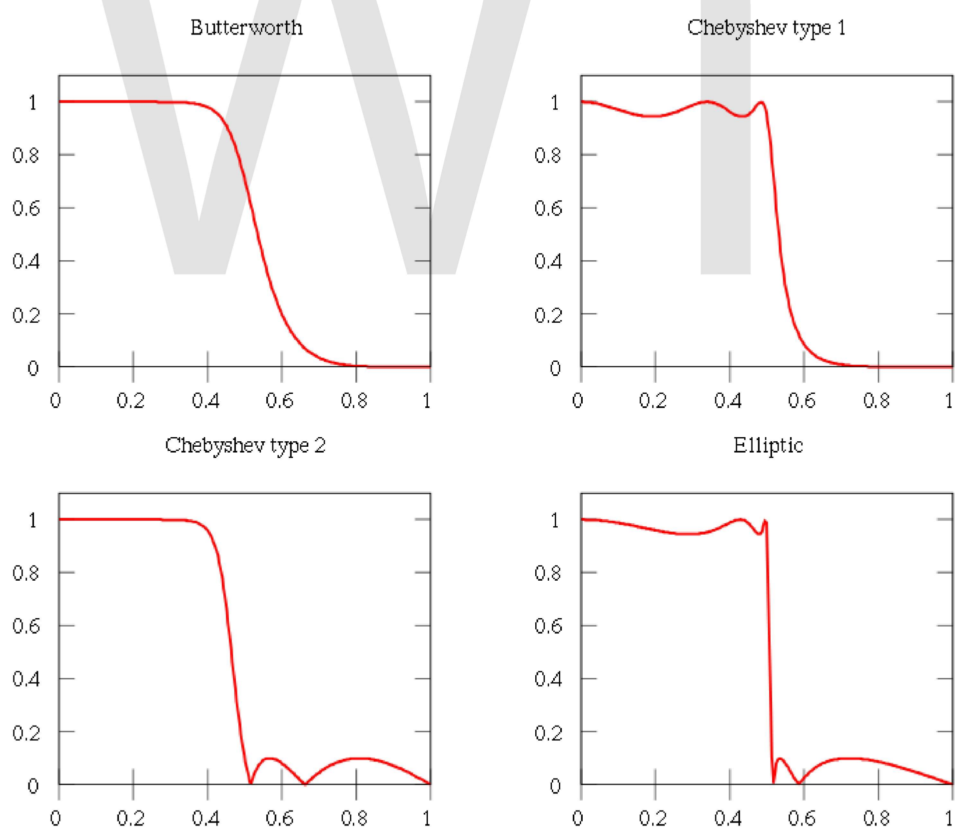
and is a measure of the influence of the pole on the gain function. For an elliptic filter, it happens that, for a given order, there exists a relationship between the ripple factor and selectivity factor which simultaneously minimizes the Q-factor of all poles in the transfer function:

$$\epsilon_{Qmin} = \frac{1}{\sqrt{L_n(\xi)}}$$

This results in a filter which is maximally insensitive to component variations, but the ability to independently specify the passband and stopband ripples will be lost. For such filters, as the order increases, the ripple in both bands will decrease and the rate of cutoff will increase. If one decides to use a minimum-Q elliptic filter in order to achieve a particular minimum ripple in the filter bands along with a particular rate of cutoff, the order needed will generally be greater than the order one would otherwise need without the minimum-Q restriction. An image of the absolute value of the gain will look very much like the image in the previous section, except that the poles are arranged in a circle rather than an ellipse. They will not be evenly spaced and there will be zeroes on the ω axis, unlike the Butterworth filter, whose poles are also arranged in a circle.

Comparison with other linear filters

Here is an image showing the elliptic filter next to other common kind of filters obtained with the same number of coefficients:



As is clear from the image, elliptic filters are sharper than all the others, but they show ripples on the whole bandwidth.

Optimum "L" filter

The **Optimum "L" filter** (also known as a **Legendre filter**) was proposed by Athanasios Papoulis in 1958. It has the maximum roll off rate for a given filter order while maintaining a monotonic frequency response. It provides a compromise between the Butterworth filter which is monotonic but has a slower roll off and the Chebyshev filter which has a faster roll off but has ripple in either the pass band or stop band. The filter design is based on Legendre polynomials which is the reason for its alternate name and the "L" in Optimum "L".

Chapter- 5

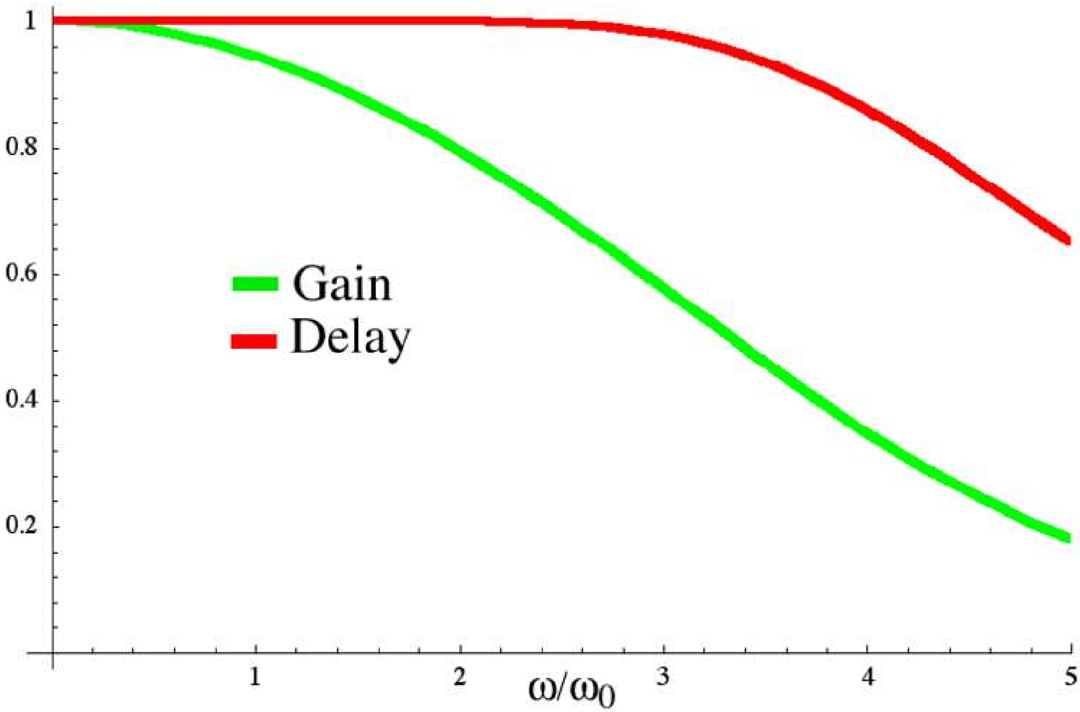
Bessel Filter, Gaussian Filter and Linkwitz–Riley Filter

Bessel filter

In electronics and signal processing, a **Bessel filter** is a type of linear filter with a maximally flat group delay (maximally linear phase response). Bessel filters are often used in audio crossover systems. Analog Bessel filters are characterized by almost constant group delay across the entire passband, thus preserving the wave shape of filtered signals in the passband.

The filter's name is a reference to Friedrich Bessel, a German mathematician (1784–1846), who developed the mathematical theory on which the filter is based. The filters are also called Bessel-Thomson filters in recognition of W. E. Thomson, who worked out how to apply Bessel functions to filter design.

The transfer function



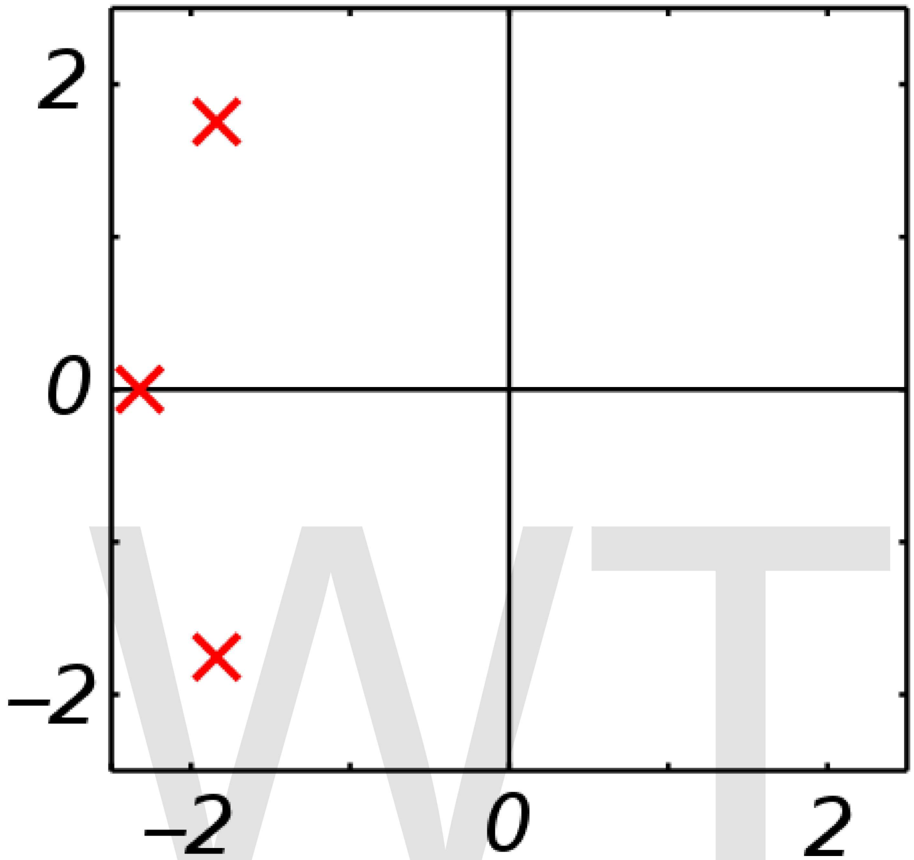
A plot of the gain and group delay for a fourth-order low pass Bessel filter. Note that the transition from the pass band to the stop band is much slower than for other filters, but the group delay is practically constant in the passband. The Bessel filter maximizes the flatness of the group delay curve at zero frequency.

A Bessel low-pass filter is characterized by its transfer function:

$$H(s) = \frac{\theta_n(0)}{\theta_n(s/\omega_0)}$$

where $\theta_n(s)$ is a reverse Bessel polynomial from which the filter gets its name and ω_0 is a frequency chosen to give the desired cut-off frequency. The filter has a low-frequency group delay of $1 / \omega_0$.

Bessel polynomials



The roots of the third-order Bessel polynomial are the poles of filter transfer function in the *s* plane, here plotted as crosses.

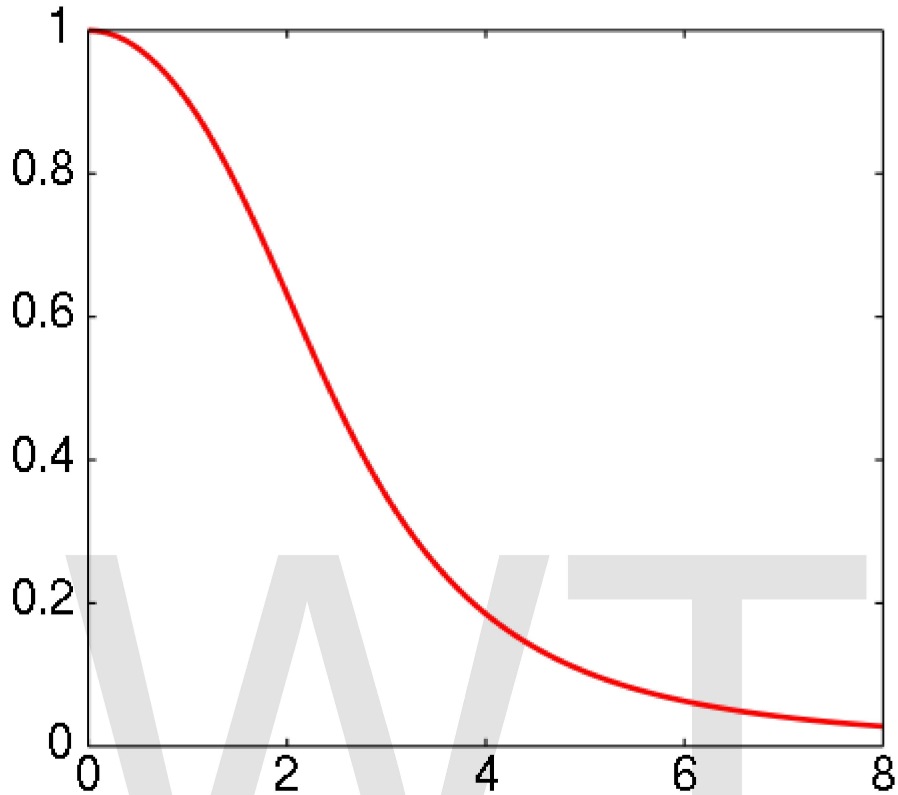
The transfer function of the Bessel filter is a rational function whose denominator is a reverse Bessel polynomial, such as the following:

$$n = 1; \quad s + 1$$
$$n = 2; \quad s^2 + 3s + 3$$
$$n = 3; \quad s^3 + 6s^2 + 15s + 15$$

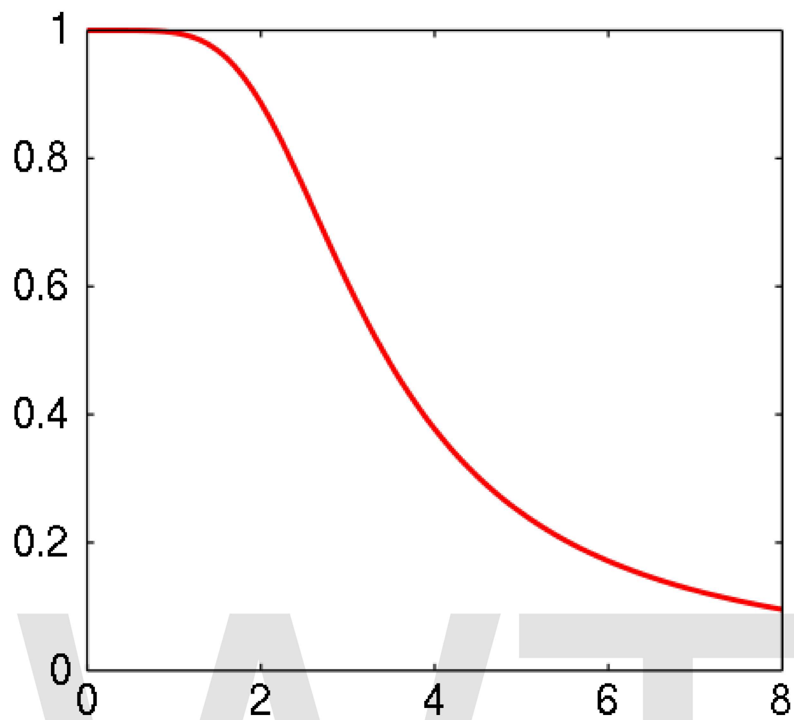
The reverse Bessel polynomials are given by:

$$\theta_n(s) = \sum_{k=0}^{n} a_k s^k$$ where
$$a_k = \frac{(2n-k)!}{2^{n-k} k!(n-k)!} \quad k = 0, 1, ..., n$$

Example



Gain plot of the third-order Bessel filter, versus normalized frequency



Group delay plot of the third-order Bessel filter, illustrating flat unit delay in the passband

The transfer function for a third-order (three-pole) Bessel low-pass filter, normalized to have unit group delay, is

$$H(s) = \frac{15}{s^3 + 6s^2 + 15s + 15}\,.$$

The roots of the denominator polynomial, the filter's poles, include a real pole at $s = -2.3222$, and a complex-conjugate pair of poles at $s = -1.8389 \pm j1.7544$, plotted above. The numerator 15 is chosen to give a gain of 1 at DC (at $s = 0$).

The gain is then

$$G(\omega) = |H(j\omega)| = \frac{15}{\sqrt{\omega^6 + 6\omega^4 + 45\omega^2 + 225}}\,.$$

The phase is

$$\phi(\omega) = -\arg(H(j\omega)) = -\arctan\left(\frac{15\omega - \omega^3}{15 - 6\omega^2}\right)\,.$$

The group delay is

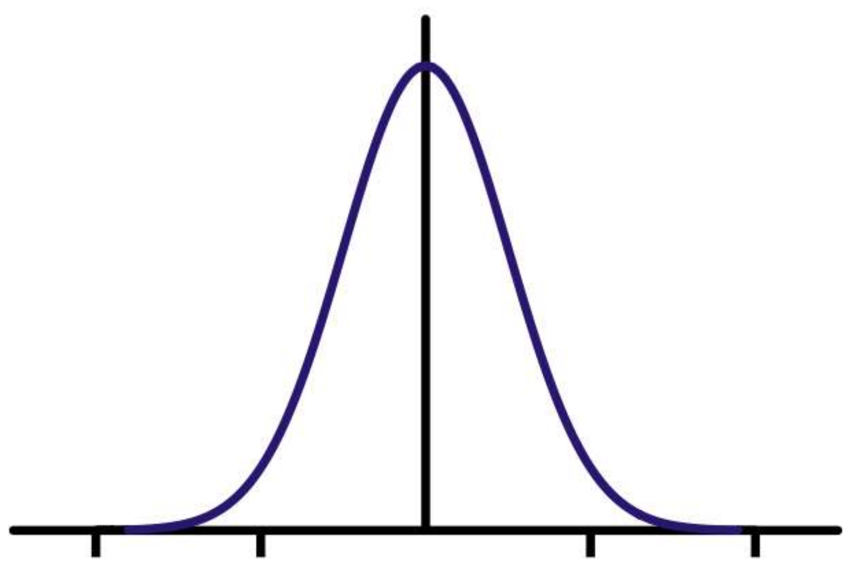
$$D(\omega) = -\frac{d\phi}{d\omega} = \frac{6\omega^4 + 45\omega^2 + 225}{\omega^6 + 6\omega^4 + 45\omega^2 + 225}.$$

The Taylor series expansion of the group delay is

$$D(\omega) = 1 - \frac{\omega^6}{225} + \frac{\omega^8}{1125} + \cdots.$$

Note that the two terms in ω^2 and ω^4 are zero, resulting in a very flat group delay at $\omega = 0$. This is the greatest number of terms that can be set to zero, since there are a total of four coefficients in the third order Bessel polynomial, requiring four equations in order to be defined. One equation specifies that the gain be unity at $\omega = 0$ and a second specifies that the gain be zero at $\omega = \infty$, leaving two equations to specify two terms in the series expansion to be zero. This is a general property of the group delay for a Bessel filter of order n: the first $n - 1$ terms in the series expansion of the group delay will be zero, thus maximizing the flatness of the group delay at $\omega = 0$.

Gaussian filter



Shape of a typical Gaussian filter

In electronics and signal processing, a **Gaussian filter** is a filter whose impulse response is a Gaussian function. Gaussian filters are designed to give no overshoot to a step function input while minimizing the rise and fall time. This behavior is closely connected to the fact that the Gaussian filter has the minimum possible group delay. Mathematically, a Gaussian filter modifies the input signal by convolution with a Gaussian function; this transformation is also known as the Weierstrass transform.

Definition

The one-dimensional Gaussian filter has an impulse response given by

$$g(x) = \sqrt{\frac{a}{\pi}} \cdot e^{-a \cdot x^2}$$

or with the standard deviation as parameter

$$g(x) = \frac{1}{\sqrt{2 \cdot \pi} \cdot \sigma} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

In two dimensions, it is the product of two such Gaussians, one per direction:

$$g(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} \text{...}$$

where x is the distance from the origin in the horizontal axis, y is the distance from the origin in the vertical axis, and σ is the standard deviation of the Gaussian distribution.

Digital implementation

The Gaussian function is non-zero for $x \in [-\infty, \infty]$and would theoretically require an infinite window length. However, since it decays rapidly, it is often reasonable to truncate the filter window and implement the filter directly for narrow windows, in effect by using a simple rectangular window function. In other cases, the truncation may introduce significant errors.

Filtering involves convolution. The filter function is said to be the kernel of an integral transform. The Gaussian kernel is continuous. Most commonly, the discrete equivalent is the sampled Gaussian kernel that is produced by sampling points from the continuous Gaussian. An alternate method is to use the discrete Gaussian kernel which has superior characteristics for some purposes. Unlike the sample Gaussian kernel, the discrete Gaussian kernel is the solution to the discrete diffusion equation.

Since the Fourier transform of the Gaussian function yields a Gaussian function, the signal (preferably after being divided into overlapping windowed blocks) can be transformed with a Fast Fourier transform, multiplied with a Gaussian function and transformed back. This is the standard procedure of applying an arbitrary finite impulse response filter, with the only difference that the Fourier transform of the filter window is explicitly known.

Due to the central limit theorem, the Gaussian can be approximated by several runs of a very simple filter such as the moving average. The simple moving average corresponds to convolution with the constant B-spline, and, for example, four iterations of a moving average yields a cubic B-spline as filter window which approximates the Gaussian quite well.

Borrowing the terms from statistics, the standard deviation of a filter can be interpreted as a measure of its size. The cut-off frequency of the filter can be considered as the ratio between the sample rate F_s and the standard deviation σ

$$f_c = \frac{F_s}{\sigma}$$

A simple moving average corresponds to a uniform probability distribution and thus its filter width of size n has standard deviation $\sqrt{(\sigma^2 - 1)/12}$. Thus m moving averages with sizes $\sigma_1, \dots, \sigma_m$ yield a standard deviation of

$$\sigma = \sqrt{\frac{\sigma_1^2 + \dots + \sigma_m^2 - m}{12}}.$$

(Note that standard deviations do not sum up, but variances do.)

When applied in two dimensions, this formula produces a Gaussian surface that has a maximum at the origin, whose contours are concentric circles with the origin as center. A two dimensional convolution matrix is precomputed from the formula and convolved with two dimensional data. Each element in the resultant matrix new value is set to a weighted average of that elements neighborhood. The focal element receives the heaviest weight (having the highest Gaussian value) and neighboring elements receive smaller weights as their distance to the focal element increases. In Image processing, each element in the matrix represents a pixel attribute such as brightness or a color intensity, and the overall effect is called Gaussian blur.

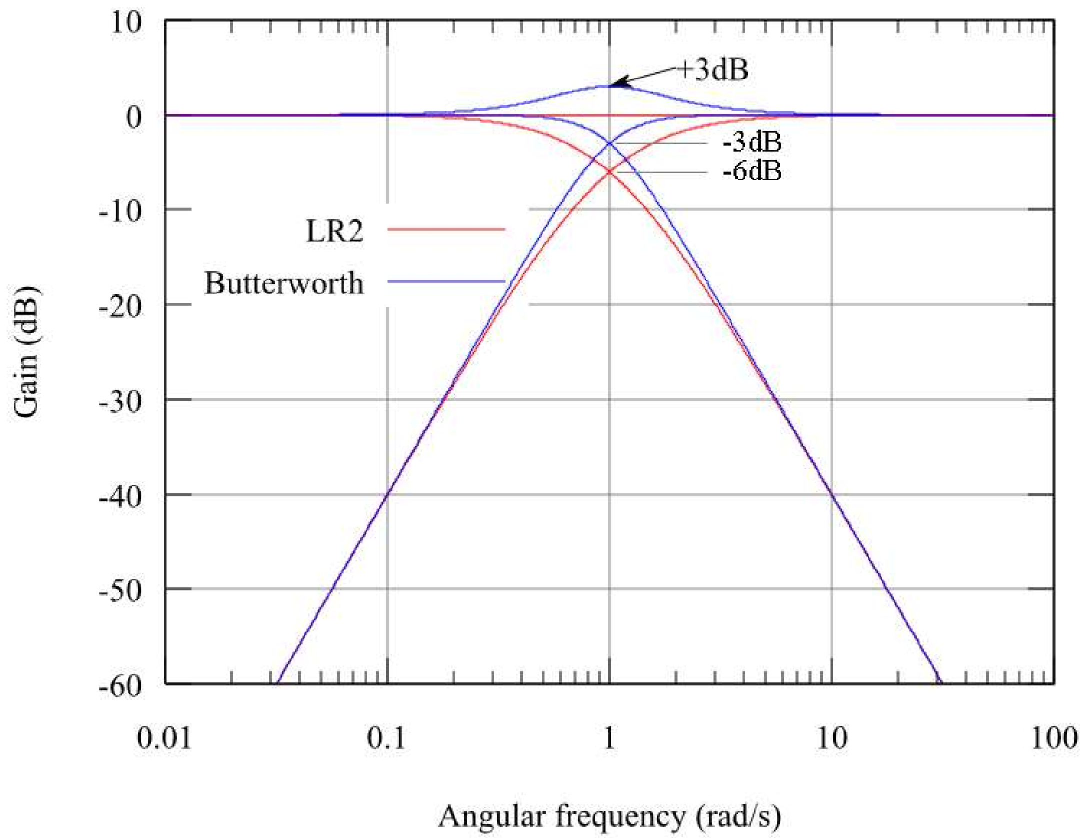
The Gaussian filter is non-causal which means the filter window is symmetric about the origin. This is usually of no consequence for most applications. In real-time systems, a delay is incurred because incoming samples need to fill the filter window before the filter can be applied to the signal.

Communications applications

- It is used in GSM since it applies GMSK modulation

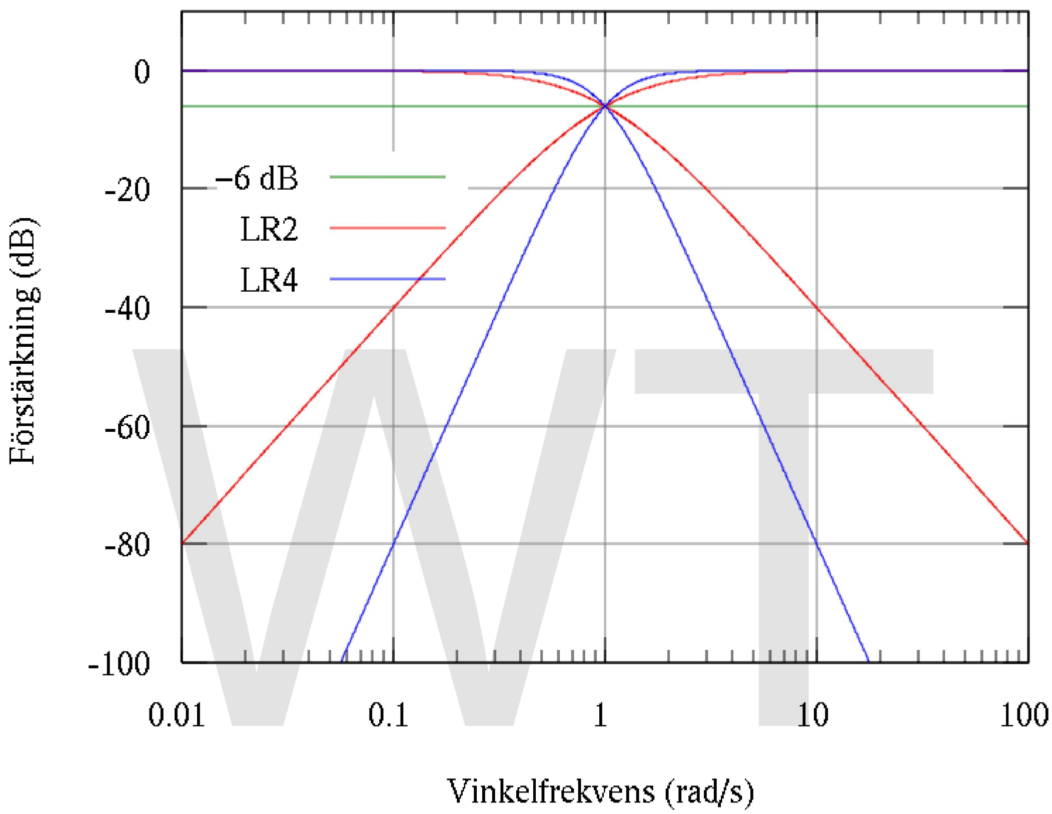
- the Gaussian filter is also used in GFSK.

Linkwitz–Riley filter

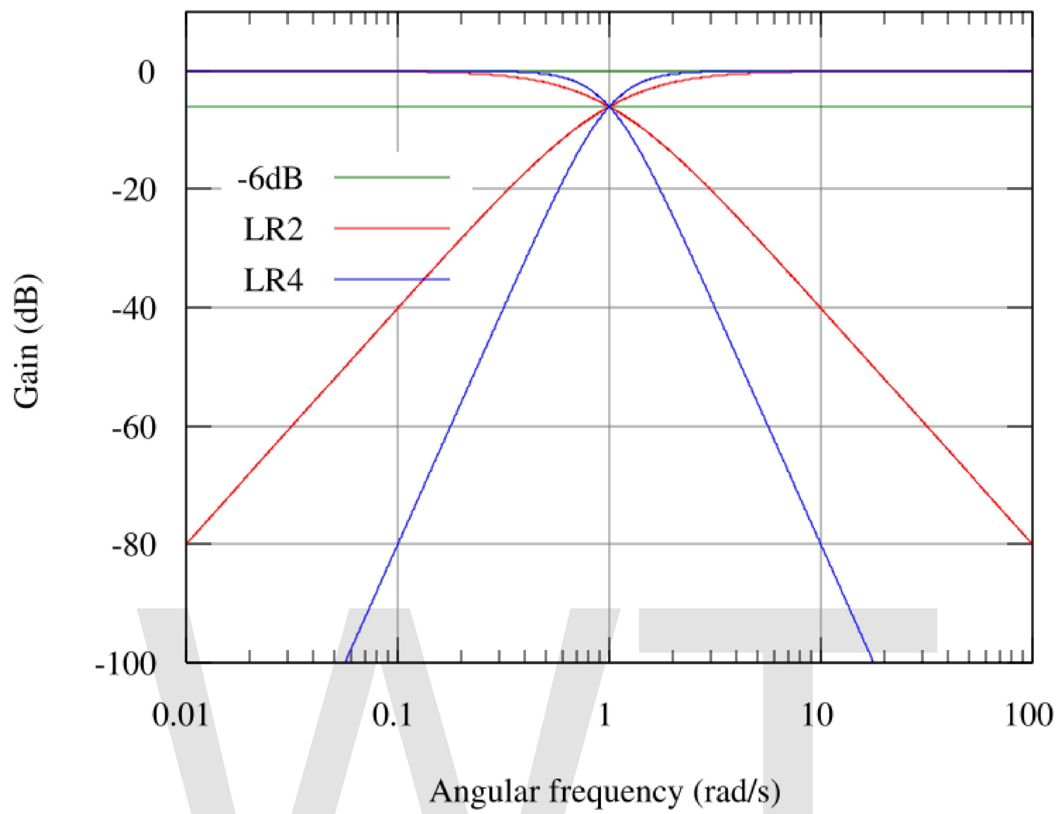


Comparison of the magitude response of the summed Butterworth and Linkwitz–Riley crossover filters. The Butterworth crossovers have a +3dB peak at the crossover frequency, whereas the L-R filters have a flat summed output.

A **Linkwitz–Riley (L-R) filter** is an infinite impulse response filter used in Linkwitz–Riley audio crossovers, named after its inventors Siegfried Linkwitz and Russ Riley, which was originally described in *Passive Crossover Networks for Noncoincident Drivers* in JAES Volume 26 Number 3 pp. 149-150; March 1978. It is also known as a *Butterworth squared* filter. An L-R crossover consists of a parallel combination of a low-pass and a high-pass L-R filter. The filters are usually designed by cascading two Butterworth filters, each of which has −3 dB gain at the cut-off frequency. The resulting Linkwitz–Riley filter has a −6 dB gain at the cutoff frequency. This means that summing the low-pass and high-pass outputs, the gain at the crossover frequency will be 0 dB, so the crossover behaves like an all-pass filter, having a flat amplitude response with a smoothly changing phase response. This is the biggest advantage of L-R crossovers compared to Butterworth crossovers, whose summed output has a +3 dB peak around the crossover frequency. Since cascading two nth order Butterworth filters will give a 2nth order Linkwitz–Riley filter, theoretically any 2nth order Linkwitz–Riley crossover can be designed. However, crossovers of higher order than 4th may have less usability due to their increasing peak in group delay around crossover frequency and complexity.



Linkwitz sv



Linkwitz

Common types

Second order Linkwitz–Riley crossover (LR2, LR-2)

Second order Linkwitz–Riley crossovers (LR2) have a 12 dB/octave (40 dB/decade) slope. They can be realized by cascading two one-pole filters, or using a Sallen Key filter topology with a Q_0 value of 0.5. There is a 180° phase difference between the lowpass and highpass output of the filter, which can be corrected by inverting one signal. In loudspeakers this is usually done by reversing the polarity of one driver if the crossover is passive. For active crossovers inversion is usually done using a unity gain inverting op-amp.

Fourth order Linkwitz–Riley crossover (LR4, LR-4)

Fourth order Linkwitz–Riley crossovers (LR4) are probably today's most commonly used type of audio crossover. They are constructed by cascading two second order Butterworth filters. Their steepness is 24 dB/octave (80 dB/decade). The phase difference amounts to 360°, i.e. the two drives appear in phase, albeit with a full period time delay for the low-pass section.

Eighth order Linkwitz–Riley crossover (LR8, LR-8)

Eighth order Linkwitz–Riley crossovers (LR8) have a very steep, 48 dB/octave (160 dB/decade) slope. They can be constructed by cascading two 4th order Butterworth filters.

Chapter- 6

RL Circuit

A **resistor-inductor circuit (RL circuit)**, or **RL filter** or **RL network**, is one of the simplest analogue infinite impulse response electronic filters. It consists of a resistor and an inductor, either in series or in parallel, driven by a voltage source.

Introduction

The fundamental passive linear circuit elements are the resistor (R), capacitor (C) and inductor (L). These circuit elements can be combined to form an electrical circuit in four distinct ways: the RC circuit, the RL circuit, the LC circuit and the RLC circuit with the abbreviations indicating which components are used. These circuits exhibit important types of behaviour that are fundamental to analogue electronics. In particular, they are able to act as passive filters.

In practice, however, capacitors (and RC circuits) are usually preferred to inductors since they can be more easily manufactured and are generally physically smaller, particularly for higher values of components.

Complex Impedance

The complex impedance Z_L (in ohms) of an inductor with inductance L (in henries) is

$$Z_L = Ls$$

The complex frequency s is a complex number,

$$s = \sigma + j\omega$$

where

- j represents the imaginary unit:

  $j^2 = -1$

- σ is the exponential decay constant (in radians per second), and
- ω is the angular frequency (in radians per second).

Eigenfunctions

The complex-valued **eigenfunctions** of ANY linear time-invariant (LTI) system are of the following forms:

$V(t) = \mathbf{A}e^{st} = \mathbf{A}e^{(\sigma+j\omega)t}$, or letting $\mathbf{A} = Ae^{j\phi}$ and rewriting;
$= Ae^{j\phi}e^{(\sigma+j\omega)t}$, and collecting terms is $= Ae^{\sigma t}e^{j(\omega t+\phi)}$

From Euler's formula, the **real-part** of these eigenfunctions are exponentially-decaying sinusoids:

$$v(t) = \mathrm{Re}\{V(t)\} = Ae^{\sigma t}\cos(\omega t + \phi)$$

Sinusoidal Steady State

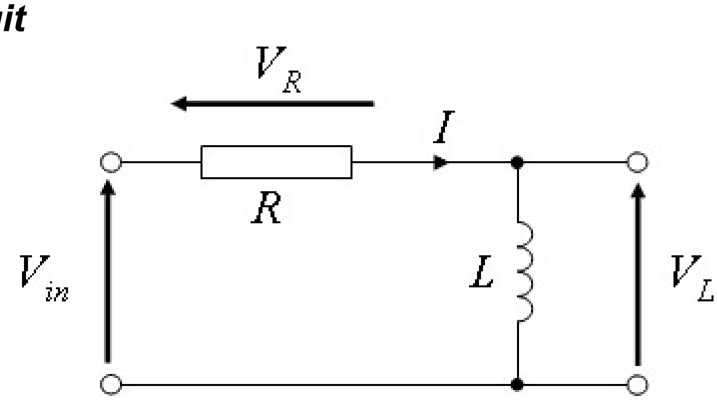
Sinusoidal steady state is a special case in which the input voltage consists of a pure sinusoid (with no exponential decay). As a result,

$$\sigma = 0$$

and the evaluation of *s* becomes

$$s = j\omega$$

Series circuit



Series RL circuit

By viewing the circuit as a voltage divider, we see that the voltage across the inductor is:

$$V_L(s) = \frac{Ls}{R + Ls} V_{in}(s)$$

and the voltage across the resistor is:

$$V_R(s) = \frac{R}{R + Ls} V_{in}(s).$$

Current

The current in the circuit is the same everywhere since the circuit is series:

$$I(s) = \frac{V_{in}(s)}{R + Ls}.$$

Transfer functions

The transfer function for the inductor is

$$H_L(s) = \frac{V_L(s)}{V_{in}(s)} = \frac{Ls}{R + Ls} = G_L e^{j\phi_L}$$

Similarly, the transfer function for the resistor is

$$H_R(s) = \frac{V_R(s)}{V_{in}(s)} = \frac{R}{R + Ls} = G_R e^{j\phi_R}$$

Poles and zeros

Both transfer functions have a single pole located at

$$s = -\frac{R}{L}$$

In addition, the transfer function for the inductor has a zero located at the origin.

Gain and phase angle

The gains across the two components are found by taking the magnitudes of the above expressions:

$$G_L = |H_L(s)| = \left|\frac{V_L(s)}{V_{in}(s)}\right| = \frac{\omega L}{\sqrt{R^2 + (\omega L)^2}}$$

and

$$G_R = |H_R(s)| = \left|\frac{V_R(s)}{V_{in}(s)}\right| = \frac{R}{\sqrt{R^2 + (\omega L)^2}},$$

and the phase angles are:

$$\phi_L = \angle H_L(s) = \tan^{-1}\left(\frac{R}{\omega L}\right)$$

and

$$\phi_R = \angle H_R(s) = \tan^{-1}\left(-\frac{\omega L}{R}\right).$$

Phasor notation

These expressions together may be substituted into the usual expression for the phasor representing the output:

$$V_L = G_L V_{in} e^{j\phi_L}$$
$$V_R = G_R V_{in} e^{j\phi_R}.$$

Impulse Response

The impulse response for each voltage is the inverse Laplace transform of the corresponding transfer function. It represents the response of the circuit to an input voltage consisting of an impulse or Dirac delta function.

The impulse response for the inductor voltage is

$$h_L(t) = \delta(t) - \frac{R}{L} e^{-tR/L} u(t) = \delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t)$$

where $u(t)$ is the Heaviside step function and

$$\tau = \frac{L}{R}$$

is the time constant.

Similarly, the impulse response for the resistor voltage is

$$h_R(t) = \frac{R}{L}e^{-tR/L}u(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

Zero input response (ZIR)

The **Zero input response**, also called the **natural response**, of an RL circuit describes the behavior of the circuit after it has reached constant voltages and currents and is disconnected from any power source. It is called the zero-input response because it requires no input.

The ZIR of an RL circuit is:

$$i(t) = i(0)e^{-(R/L)t} = i(0)e^{-t/\tau}.$$

Frequency domain considerations

These are frequency domain expressions. Analysis of them will show which frequencies the circuits (or filters) pass and reject. This analysis rests on a consideration of what happens to these gains as the frequency becomes very large and very small.

As $\omega \to \infty$:

$$G_L \to 1$$
$$G_R \to 0.$$

As $\omega \to 0$:

$$G_L \to 0$$
$$G_R \to 1.$$

This shows that, if the output is taken across the inductor, high frequencies are passed and low frequencies are attenuated (rejected). Thus, the circuit behaves as a *high-pass filter*. If, though, the output is taken across the resistor, high frequencies are rejected and low frequencies are passed. In this configuration, the circuit behaves as a *low-pass filter*. Compare this with the behaviour of the resistor output in an RC circuit, where the reverse is the case.

The range of frequencies that the filter passes is called its bandwidth. The point at which the filter attenuates the signal to half its unfiltered power is termed its cutoff frequency. This requires that the gain of the circuit be reduced to

$$G_L = G_R = \frac{1}{\sqrt{2}}.$$

Solving the above equation yields

$$\omega_c = \frac{R}{L} \text{ rad/s}$$

or

$$f_c = \frac{R}{2\pi L} \text{ Hz}$$

which is the frequency that the filter will attenuate to half its original power.

Clearly, the phases also depend on frequency, although this effect is less interesting generally than the gain variations.

As $\omega \to 0$:

$$\phi_L \to 90^\circ = \pi/2^c$$
$$\phi_R \to 0.$$

As $\omega \to \infty$:

$$\phi_L \to 0$$
$$\phi_R \to -90^\circ = -\pi/2^c$$

So at DC (0 Hz), the resistor voltage is in phase with the signal voltage while the inductor voltage leads it by 90°. As frequency increases, the resistor voltage comes to have a 90° lag relative to the signal and the inductor voltage comes to be in-phase with the signal.

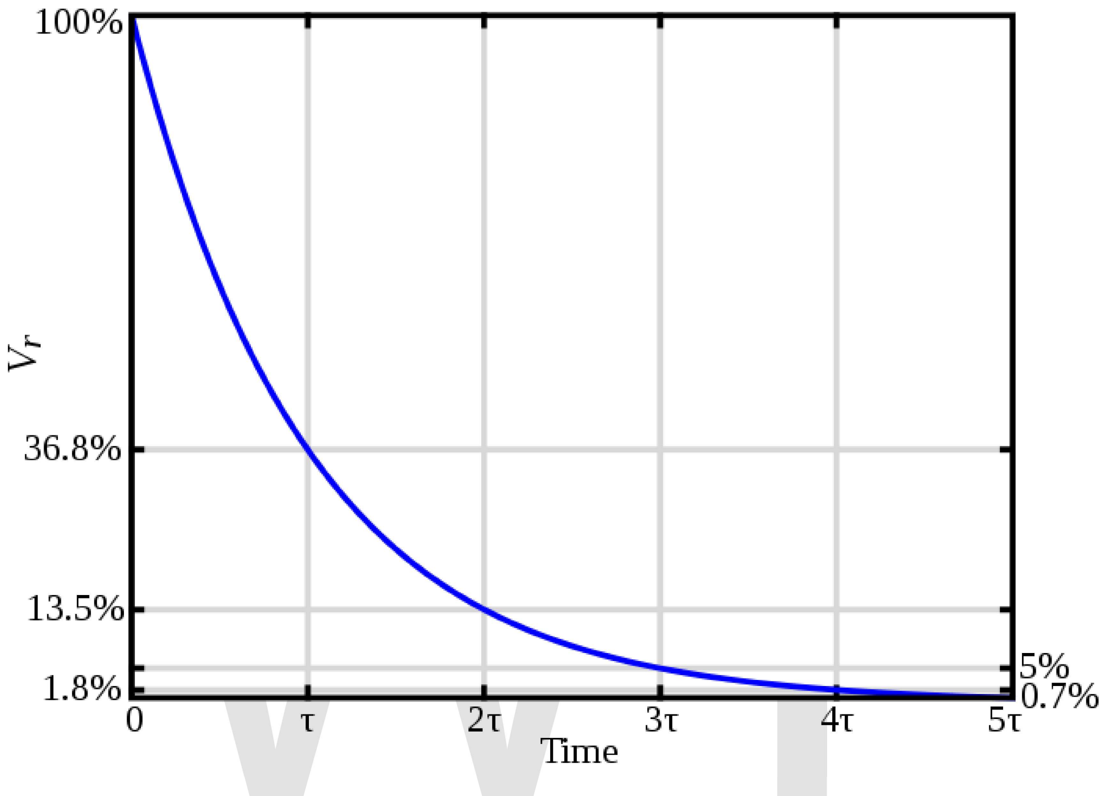
Time domain considerations

The most straightforward way to derive the time domain behaviour is to use the Laplace transforms of the expressions for V_L and V_R given above. This effectively transforms $j\omega \to s$. Assuming a step input (i.e. $V_{in} = 0$ before $t = 0$ and then $V_{in} = V$ afterwards):

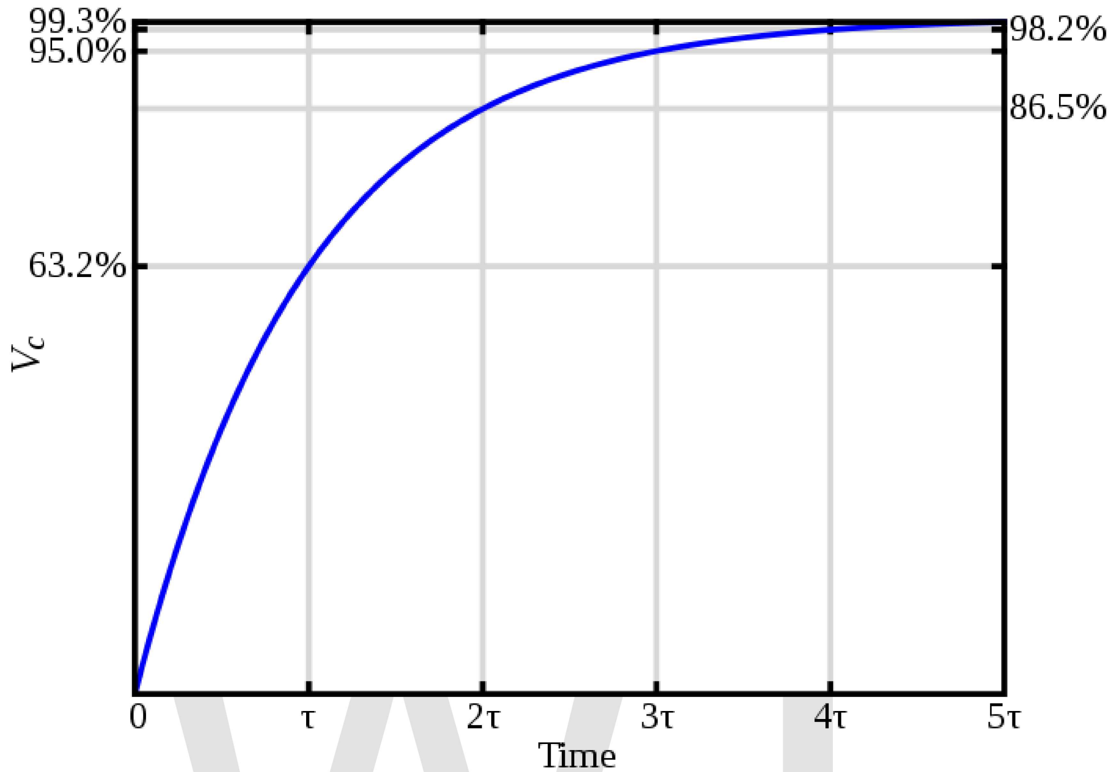
$$V_{in}(s) = V\frac{1}{s}$$
$$V_L(s) = V\frac{sL}{R + sL}\frac{1}{s}$$

and

$$V_R(s) = V\frac{R}{R + sL}\frac{1}{s}.$$



Inductor voltage step-response.



Resistor voltage step-response.

Partial fractions expansions and the inverse Laplace transform yield:

$$V_L(t) = Ve^{-tR/L}$$
$$V_R(t) = V\left(1 - e^{-tR/L}\right).$$

Thus, the voltage across the inductor tends towards 0 as time passes, while the voltage across the resistor tends towards V, as shown in the figures. This is in keeping with the intuitive point that the inductor will only have a voltage across as long as the current in the circuit is changing — as the circuit reaches its steady-state, there is no further current change and ultimately no inductor voltage.

These equations show that a series RL circuit has a time constant, usually denoted $\tau = L / R$ being the time it takes the voltage across the component to either fall (across L) or rise (across R) to within $1 / e$ of its final value. That is, τ is the time it takes V_L to reach $V(1 / e)$ and V_R to reach $V(1 - 1 / e)$.

The rate of change is a *fractional* $\left(1 - \frac{1}{e}\right)$ per τ. Thus, in going from $t = N\tau$ to $t = (N + 1)\tau$, the votage will have moved about 63% of the way from its level at $t = N\tau$ toward its final value. So the voltage across L will have dropped to about 37% after τ, and essentially to zero (0.7%) after about 5τ. Kirchhoff's voltage law implies that the voltage across the resistor will *rise* at the same rate. When the voltage source is then replaced with a short-circuit, the voltage across R drops exponentially with t from V towards 0. R will be discharged to about 37% after τ, and essentially fully discharged (0.7%) after about 5τ. Note that the current, I, in the circuit behaves as the voltage across R does, via Ohm's Law.

The delay in the rise/fall time of the circuit is in this case caused by the back-EMF from the inductor which, as the current flowing through it tries to change, prevents the current (and hence the voltage across the resistor) from rising or falling much faster than the time-constant of the circuit. Since all wires have some self-inductance and resistance, all circuits have a time constant. As a result, when the power supply is switched on, the current does not instantaneously reach its steady-state value, V / R. The rise instead takes several time-constants to complete. If this were not the case, and the current were to reach steady-state immediately, extremely strong inductive electric fields would be generated by the sharp change in the magnetic field — this would lead to breakdown of the air in the circuit and electric arcing, probably damaging components (and users).

These results may also be derived by solving the differential equation describing the circuit:

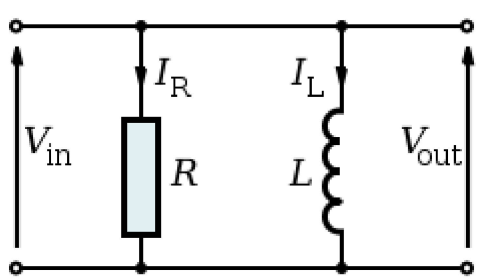
$$V_{in} = IR + L\frac{dI}{dt},$$

and

$$V_R = V_{in} - V_L.$$

The first equation is solved by using an integrating factor and yields the current which must be differentiated to give V_L; the second equation is straightforward. The solutions are exactly the same as those obtained via Laplace transforms.

Parallel circuit



Parallel RL circuit

The parallel RL circuit is generally of less interest than the series circuit unless fed by a current source. This is largely because the output voltage V_{out} is equal to the input voltage V_{in} — as a result, this circuit does not act as a filter for a voltage input signal.

With complex impedances:

$$I_R = \frac{V_{in}}{R}$$

and

$$I_L = \frac{V_{in}}{j\omega L} = -\frac{jV_{in}}{\omega L}.$$

This shows that the inductor lags the resistor (and source) current by 90°.

The parallel circuit is seen on the output of many amplifier circuits, and is used to isolate the amplifier from capacitive loading effects at high frequencies. Because of the phase shift introduced by capacitance, some amplifiers become unstable at very high frequencies, and tend to oscillate. This affects sound quality and component life (especially the transistors), and is to be avoided.

Chapter- 7

RC Filter

A **resistor–capacitor circuit (RC circuit)**, or **RC filter** or **RC network**, is an electric circuit composed of resistors and capacitors driven by a voltage or current source. A first order RC circuit is composed of one resistor and one capacitor and is the simplest type of RC circuit.

RC circuits can be used to filter a signal by blocking certain frequencies and passing others. The four most common RC filters are the high-pass filter, low-pass filter, band-pass filter, and band-stop filter.

Introduction

There are three basic, linear passive lumped analog circuit components: the resistor (R), capacitor (C) and inductor (L). These may be combined in: the RC circuit, the RL circuit, the LC circuit and the RLC circuit with the abbreviations indicating which components are used. These circuits, between them, exhibit a large number of important types of behaviour that are fundamental to much of analog electronics. In particular, they are able to act as passive filters.

Natural response

The simplest RC circuit is a capacitor and a resistor in series. When a circuit consists of only a charged capacitor and a resistor, the capacitor will discharge its stored energy through the resistor. The voltage across the capacitor, which is time dependent, can be found by using Kirchhoff's current law, where the current through the capacitor must equal the current through the resistor. This results in the linear differential equation

$$C\frac{dV}{dt} + \frac{V}{R} = 0.$$

Solving this equation for V yields the formula for exponential decay:

$$V(t) = V_0 e^{-\frac{t}{RC}} ,$$

where V_0 is the capacitor voltage at time $t = 0$.

The time required for the voltage to fall to $\frac{V_0}{e}$ is called the RC time constant and is given by

$$\tau = RC .$$

Complex impedance

The complex impedance, Z_C (in ohms) of a capacitor with capacitance C (in farads) is

$$Z_C = \frac{1}{sC}$$

The complex frequency s is, in general, a complex number,

$$s = \sigma + j\omega$$

where

- j represents the imaginary unit:

  $j^2 = -1$

- σ is the exponential decay constant (in radians per second), and
- ω is the sinusoidal angular frequency (also in radians per second).

Sinusoidal steady state

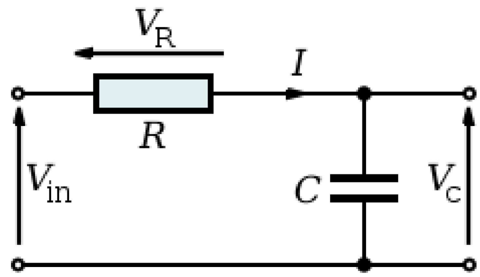
Sinusoidal steady state is a special case in which the input voltage consists of a pure sinusoid (with no exponential decay). As a result,

$$\sigma = 0$$

and the evaluation of s becomes

$$s = j\omega$$

Series circuit



Series RC circuit

By viewing the circuit as a voltage divider, the voltage across the capacitor is:

$$V_C(s) = \frac{1/Cs}{R + 1/Cs} V_{in}(s) = \frac{1}{1 + RCs} V_{in}(s)$$

and the voltage across the resistor is:

$$V_R(s) = \frac{R}{R + 1/Cs} V_{in}(s) = \frac{RCs}{1 + RCs} V_{in}(s).$$

Transfer functions

The transfer function for the capacitor is

$$H_C(s) = \frac{V_C(s)}{V_{in}(s)} = \frac{1}{1 + RCs}.$$

Similarly, the transfer function for the resistor is

$$H_R(s) = \frac{V_R(s)}{V_{in}(s)} = \frac{RCs}{1 + RCs}.$$

Poles and zeros

Both transfer functions have a single pole located at

$$s = -\frac{1}{RC}.$$

In addition, the transfer function for the resistor has a zero located at the origin.

Gain and phase angle

The magnitude of the gains across the two components are:

$$G_C = |H_C(j\omega)| = \left|\frac{V_C(j\omega)}{V_{in}(j\omega)}\right| = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

and

$$G_R = |H_R(j\omega)| = \left|\frac{V_R(j\omega)}{V_{in}(j\omega)}\right| = \frac{\omega RC}{\sqrt{1+(\omega RC)^2}},$$

and the phase angles are:

$$\phi_C = \angle H_C(j\omega) = \tan^{-1}(-\omega RC)$$

and

$$\phi_R = \angle H_R(j\omega) = \tan^{-1}\left(\frac{1}{\omega RC}\right).$$

These expressions together may be substituted into the usual expression for the phasor representing the output:

$$\begin{aligned} V_C &= G_C V_{in} e^{j\phi_C} \\ V_R &= G_R V_{in} e^{j\phi_R}. \end{aligned}$$

Current

The current in the circuit is the same everywhere since the circuit is in series:

$$I(s) = \frac{V_{in}(s)}{R+\frac{1}{Cs}} = \frac{Cs}{1+RCs} V_{in}(s)$$

Impulse response

The impulse response for each voltage is the inverse Laplace transform of the corresponding transfer function. It represents the response of the circuit to an input voltage consisting of an impulse or Dirac delta function.

The impulse response for the capacitor voltage is

$$h_C(t) = \frac{1}{RC}e^{-t/RC}u(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

where $u(t)$ is the Heaviside step function and

$$\tau = RC$$

is the time constant.

Similarly, the impulse response for the resistor voltage is

$$h_R(t) = \delta(t) - \frac{1}{RC}e^{-t/RC}u(t) = \delta(t) - \frac{1}{\tau}e^{-t/\tau}u(t)$$

where $\delta(t)$ is the Dirac delta function

Frequency-domain considerations

These are frequency domain expressions. Analysis of them will show which frequencies the circuits (or filters) pass and reject. This analysis rests on a consideration of what happens to these gains as the frequency becomes very large and very small.

As $\omega \to \infty$:

$$G_C \to 0$$
$$G_R \to 1.$$

As $\omega \to 0$:

$$G_C \to 1$$
$$G_R \to 0.$$

This shows that, if the output is taken across the capacitor, high frequencies are attenuated (rejected) and low frequencies are passed. Thus, the circuit behaves as a *low-pass filter*. If, though, the output is taken across the resistor, high frequencies are passed and low frequencies are rejected. In this configuration, the circuit behaves as a *high-pass filter*.

The range of frequencies that the filter passes is called its bandwidth. The point at which the filter attenuates the signal to half its unfiltered power is termed its cutoff frequency. This requires that the gain of the circuit be reduced to

$$G_C = G_R = \frac{1}{\sqrt{2}}.$$

Solving the above equation yields

$$\omega_c = \frac{1}{RC} \text{ rad/s}$$

or

$$f_c = \frac{1}{2\pi RC} \text{ Hz}$$

which is the frequency that the filter will attenuate to half its original power.

Clearly, the phases also depend on frequency, although this effect is less interesting generally than the gain variations.

As $\omega \to 0$:

$$\begin{array}{l} \phi_C \to 0 \\ \phi_R \to 90^\circ = \pi/2^c \end{array}.$$

As $\omega \to \infty$:

$$\begin{array}{l} \phi_C \to -90^\circ = -\pi/2^c \\ \phi_R \to 0 \end{array}$$

So at DC (0 Hz), the capacitor voltage is in phase with the signal voltage while the resistor voltage leads it by 90°. As frequency increases, the capacitor voltage comes to have a 90° lag relative to the signal and the resistor voltage comes to be in-phase with the signal.

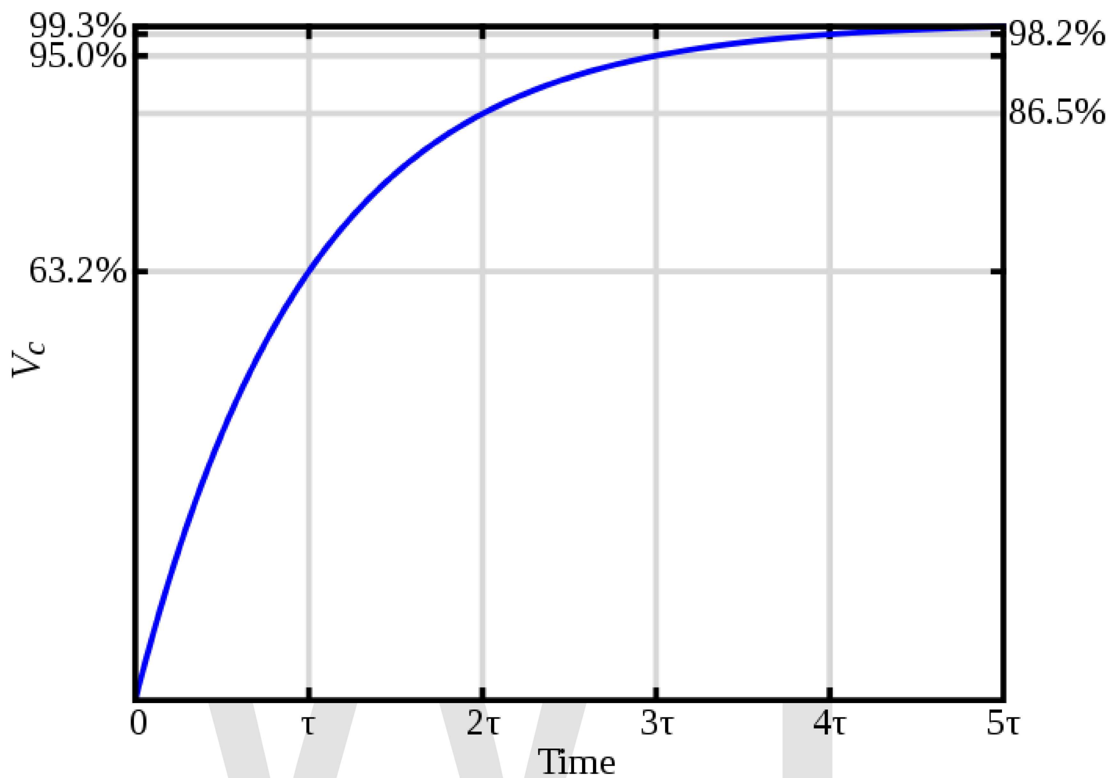
Time-domain considerations

The most straightforward way to derive the time domain behaviour is to use the Laplace transforms of the expressions for V_C and V_R given above. This effectively transforms $j\omega \to s$. Assuming a step input (i.e. $V_{in} = 0$ before $t = 0$ and then $V_{in} = V$ afterwards):

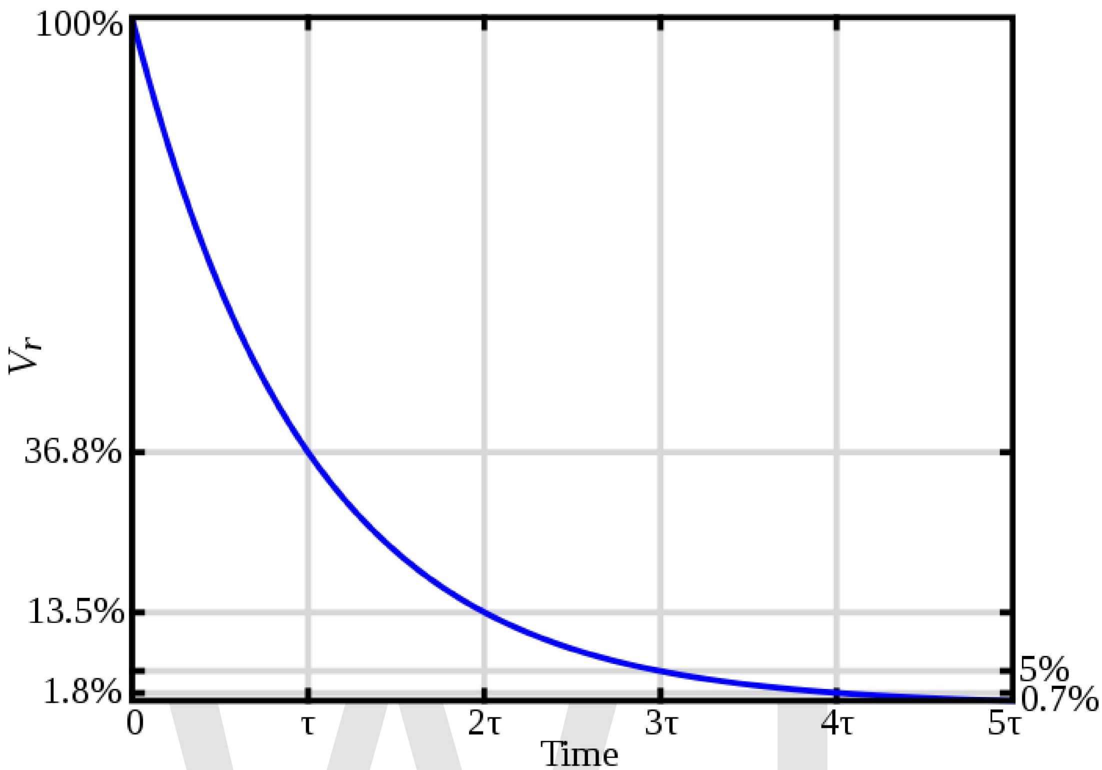
$$V_{in}(s) = V\frac{1}{s}$$
$$V_C(s) = V\frac{1}{1 + sRC}\frac{1}{s}$$

and

$$V_R(s) = V\frac{sRC}{1+sRC}\frac{1}{s}.$$



Capacitor voltage step-response.



Resistor voltage step-response.

Partial fractions expansions and the inverse Laplace transform yield:

$$V_C(t) = V\left(1 - e^{-t/RC}\right)$$
$$V_R(t) = Ve^{-t/RC}.$$

These equations are for calculating the voltage across the capacitor and resistor respectively while the capacitor is charging; for discharging, the equations are vice-versa. These equations can be rewritten in terms of charge and current using the relationships C=Q/V and V=IR.

Thus, the voltage across the capacitor tends towards V as time passes, while the voltage across the resistor tends towards 0, as shown in the figures. This is in keeping with the intuitive point that the capacitor will be charging from the supply voltage as time passes, and will eventually be fully charged and form an open circuit.

These equations show that a series RC circuit has a time constant, usually denoted $\tau = RC$ being the time it takes the voltage across the component to either rise (across C) or fall (across R) to within $1 / e$ of its final value. That is, τ is the time it takes V_C to reach $V(1 - 1 / e)$ and V_R to reach $V(1 / e)$.

The rate of change is a *fractional* $\left(1 - \frac{1}{e}\right)$ per τ. Thus, in going from $t = N\tau$ to $t = (N + 1)\tau$, the voltage will have moved about 63.2 % of the way from its level at $t = N\tau$ toward its final value. So C will be charged to about 63.2 % after τ, and essentially fully charged (99.3 %) after about 5τ. When the voltage source is replaced with a short-circuit, with C fully charged, the voltage across C drops exponentially with *t* from *V* towards 0. C will be discharged to about 36.8 % after τ, and essentially fully discharged (0.7 %) after about 5τ. Note that the current, *I*, in the circuit behaves as the voltage across R does, via Ohm's Law.

These results may also be derived by solving the differential equations describing the circuit:

$$\frac{V_{in} - V_C}{R} = C\frac{dV_C}{dt}$$

and

$$V_R = V_{in} - V_C.$$

The first equation is solved by using an integrating factor and the second follows easily; the solutions are exactly the same as those obtained via Laplace transforms.

Integrator

Consider the output across the capacitor at *high* frequency i.e.

$$\omega \gg \frac{1}{RC}.$$

This means that the capacitor has insufficient time to charge up and so its voltage is very small. Thus the input voltage approximately equals the voltage across the resistor. To see this, consider the expression for *I* given above:

$$I = \frac{V_{in}}{R + 1/j\omega C}$$

but note that the frequency condition described means that

$$\omega C \gg \frac{1}{R}$$

so

$I \approx \frac{V_{in}}{R}$ which is just Ohm's Law.

Now,

$$V_C = \frac{1}{C}\int_0^t I dt$$

so

$$V_C \approx \frac{1}{RC}\int_0^t V_{in} dt,$$

which is an integrator *across the capacitor*.

Differentiator

Consider the output across the resistor at *low* frequency i.e.,

$$\omega \ll \frac{1}{RC}.$$

This means that the capacitor has time to charge up until its voltage is almost equal to the source's voltage. Considering the expression for I again, when

$$R \ll \frac{1}{\omega C},$$

so

$$I \approx \frac{V_{in}}{1/j\omega C}$$
$$V_{in} \approx \frac{I}{j\omega C} \approx V_C$$

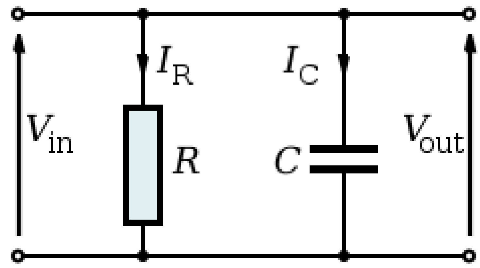
Now,

$$V_R = IR = C\frac{dV_C}{dt}R$$
$$V_R \approx RC\frac{dV_{in}}{dt}$$

which is a differentiator *across the resistor*.

More accurate integration and differentiation can be achieved by placing resistors and capacitors as appropriate on the input and feedback loop of operational amplifiers.

Parallel circuit



Parallel RC circuit

The parallel RC circuit is generally of less interest than the series circuit. This is largely because the output voltage V_{out} is equal to the input voltage V_{in} — as a result, this circuit does not act as a filter on the input signal unless fed by a current source.

With complex impedances:

$$I_R = \frac{V_{in}}{R}$$

and

$$I_C = j\omega C V_{in}.$$

This shows that the capacitor current is 90° out of phase with the resistor (and source) current. Alternatively, the governing differential equations may be used:

$$I_R = \frac{V_{in}}{R}$$

and

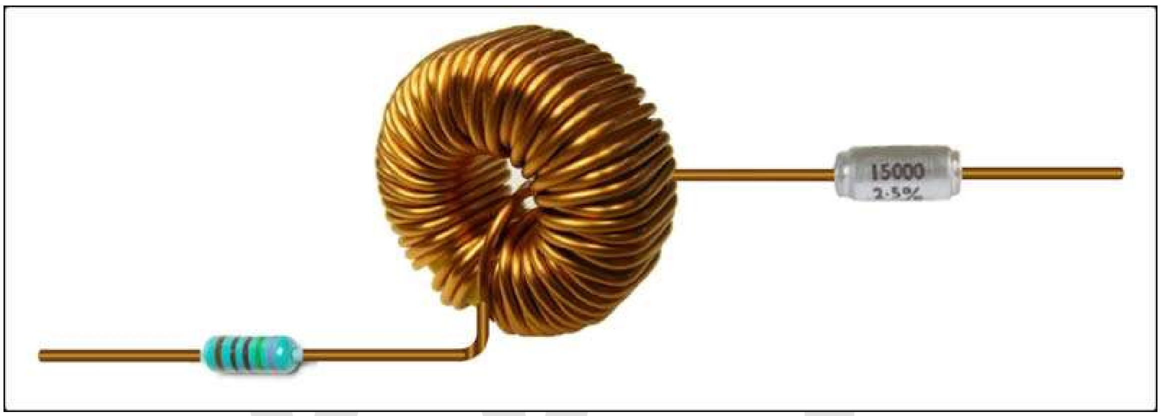
$$I_C = C\frac{dV_{in}}{dt}.$$

When fed by a current source, the transfer function of a parallel RC circuit is

$$\frac{V_{out}}{I_{in}} = \frac{R}{1 + sRC}.$$

Chapter- 8

RLC Circuit



A series RLC circuit: a resistor, inductor, and a capacitor

An **RLC circuit** (or **LCR circuit**) is an electrical circuit consisting of a resistor, an inductor, and a capacitor, connected in series or in parallel. The RLC part of the name is due to those letters being the usual electrical symbols for resistance, inductance and capacitance respectively. The circuit forms a harmonic oscillator for current and will resonate in just the same way as an LC circuit will. The difference that the presence of the resistor makes is that any oscillation induced in the circuit will die away over time if it not kept going by a source. This effect of the resistor is called damping. Some resistance is unavoidable in real circuits, even if a resistor is not specifically included as a component. A pure LC circuit is an ideal which really only exists in theory.

There are many applications for this circuit. They are used in many different types of oscillator circuit. Another important application is for tuning, such as in radio receivers or television sets, where they are used to select a narrow range of frequencies from the ambient radio waves. In this role the circuit is often referred to as a tuned circuit. An RLC circuit can be used as a band-pass filter or a band-stop filter. The tuning application, for instance, is an example of band-pass filtering. The RLC filter is described as a *second-order* circuit, meaning that any voltage or current in the circuit can be described by a second-order differential equation in circuit analysis.

The three circuit elements can be combined in a number of different topologies. All three elements in series or all three elements in parallel are the simplest in concept and the most straightforward to analyse. There are, however, other arrangements, some with practical importance in real circuits. One issue often encountered is the need to take into account inductor resistance. Inductors are typically constructed from coils of wire, the resistance of which is not usually desirable, but it often has a significant effect on the circuit.

Basic concepts

Resonance

An important property of this circuit is its ability to resonate at a specific frequency, the resonance frequency, f_0. Frequencies are measured in units of hertz. Here, however, angular frequency, ω_0, is used which is more mathematically convenient. This is measured in radians per second. They are related to each other by a simple proportion,

$$\omega_0 = 2\pi f_0$$

Resonance occurs because energy is stored in two different ways: in an electric field as the capacitor is charged and in a magnetic field as current flows through the inductor. Energy can be transferred from one to the other within the circuit and this can be oscillatory. A mechanical analogy is a weight suspended on a spring which will oscillate up and down when released. This is no passing metaphor, a weight on a spring is described by exactly the same second order differential equation as an RLC circuit and for all the properties of the one system there will be found an analogous property of the other. The mechanical property answering to the resistor in the circuit is friction in the spring/weight system. Friction will slowly bring any oscillation to a halt if there is no external force driving it. Likewise, the resistance in an RLC circuit will "damp" the oscillation, diminishing it with time if there is no driving AC power source in the circuit.

The resonance frequency is defined as the frequency at which the impedance of the circuit is at a minimum. Equivalently, it can be defined as the frequency at which the impedance is purely real (that is, purely resistive). This occurs because the impedance (reactance) of the inductor and capacitor at resonance are equal but of opposite sign and cancel out. Circuits where L and C are in parallel rather than series actually have a maximum impedance rather than a minimum impedance. For this reason they are often described as antiresonators, it is still usual, however, to name the frequency at which this occurs as the resonance frequency.

Natural frequency

The resonance frequency is defined in terms of the impedance presented to a driving source. It is still possible for the circuit to carry on oscillating (for a time) after the driving source has been removed or it is subjected to a step in voltage (including a step down to zero). This is similar to the way that a tuning fork will carry on ringing after it has been struck, and the effect is often called ringing. This effect is the undriven natural resonance frequency of the circuit and in general is not exactly the same as the driven resonance frequency, although the two will usually be quite close to each other. Various terms are used by different authors to distinguish the two, but resonance frequency unqualified usually means the driven resonance frequency. The driven frequency may be called the undamped resonance frequency or undamped natural frequency and the undriven frequency may be called the damped resonance frequency or the damped natural frequency. The reason for this terminology is that the driven resonance frequency in a series or parallel resonant circuit has the value`

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

This is exactly the same as the resonance frequency of an LC circuit, that is, one with no resistor present, that is, it is the same as a circuit in which there is no damping, hence undamped resonance frequency. The undriven resonance frequency, on the other hand, depends on the value of the resistor and hence is described as the damped resonance frequency. A highly damped circuit will fail to resonate at all when undriven. A circuit with a value of resistor that causes it to be just on the edge of ringing is called critically damped. Either side of critically damped are described as underdamped (ringing happens) and overdamped (ringing is suppressed).

Circuits with topologies more complex than straightforward series or parallel (some examples described later) have a driven resonance frequency that deviates from $\omega_0 = \frac{1}{\sqrt{LC}}$ and for those the undamped resonance frequency, damped resonance frequency and driven resonance frequency can all be different.

Damping

Damping is caused by the resistance in the circuit. It determines whether or not the circuit will resonate naturally (that is, without a driving source). Circuits which will resonate in this way are described as underdamped and those that will not are overdamped. Damping attenuation (symbol α) is measured in nepers per second. However, the unitless damping factor (symbol ζ) is often a more useful measure, which is related to α by

$$\zeta = \frac{\alpha}{\omega_0}$$

The special case of $\zeta = 1$ is called critical damping and represents the case of a circuit that is just on the border of oscillation. It is the minimum damping that can be applied without causing oscillation.

Bandwidth

The resonance effect can be used for filtering, the rapid change in impedance near resonance can be used to pass or block signals close to the resonance frequency. Both band-pass and band-stop filters can be constructed and some filter circuits are shown later. A key parameter in filter design is bandwidth. The bandwidth is measured between the 3dB-points, that is, the frequencies at which the power passed through the circuit has fallen to half the value passed at resonance. There are two of these half-power frequencies, one above, and one below the resonance frequency

$$\Delta\omega = \omega_2 - \omega_1$$

where $\Delta\omega$ is the bandwidth, ω_1 is the lower half-power frequency and ω_2 is the upper half-power frequency. The bandwidth is related to attenuation by,

$$\Delta\omega = 2\alpha$$

when the units are radians per second and nepers per second respectively. Other units may require a conversion factor. A more general measure of bandwidth is the fractional bandwidth, which expresses the bandwidth as a fraction of the resonance frequency and is given by

$$F_{\mathrm{b}} = \frac{\Delta\omega}{\omega_0}$$

The fractional bandwidth is also often stated as a percentage. The damping of filter circuits is adjusted to result in the required bandwidth. A narrow band filter, such as a notch filter, requires low damping. A wide band filter requires high damping.

Q factor

The Q factor is a widespread measure used to characterise resonators. It is defined as the peak energy stored in the circuit divided by the average energy dissipated in it per cycle at resonance. Low *Q* circuits are therefore damped and lossy and high *Q* circuits are underdamped. *Q* is related to bandwidth; low *Q* circuits are wide band and high *Q* circuits are narrow band. In fact, it happens that *Q* is the inverse of fractional bandwidth

$$Q = \frac{1}{F_{\mathrm{b}}} = \frac{\omega_0}{\Delta\omega}$$

Q factor is directly proportional to selectivity, as Q factor depends inversely on bandwidth.

Scaled parameters

The parameters ζ, F_b, and Q are all scaled to ω_0. This means that circuits which have similar parameters share similar characteristics regardless of whether or not they are operating in the same frequency band.

Series RLC circuit

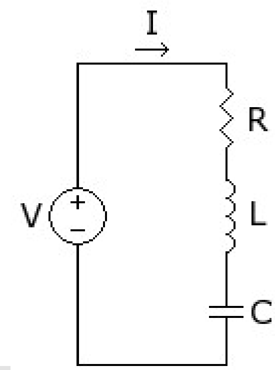


Figure 1. RLC series circuit

V - the voltage of the power source
I - the current in the circuit
R - the resistance of the resistor
L - the inductance of the inductor
C - the capacitance of the capacitor

In this circuit, the three components are all in series with the voltage source. The governing differential equation can be found by substituting into Kirchhoff's voltage law (KVL) the constitutive equation for each of the three elements. From KVL,

$$v_R + v_L + v_C = v(t)$$

where v_R, v_L, v_C are the voltages across R, L and C respectively and $v(t)$ is the time varying voltage from the source. Substituting in the constitutive equations,

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\, d\tau = v(t)$$

For the case where the source is an unchanging voltage, differentiating and dividing by L leads to the second order differential equation:

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

This can usefully be expressed in a more generally applicable form:

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di}{dt} + {\omega_0}^2 i(t) = 0$$

α and ω_0 are both in units of angular frequency. α is called the *neper frequency*, or *attenuation*, and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed. Neper occurs in the name because the units can also be considered to be nepers per second, neper being a unit of attenuation. ω_0 is the angular resonance frequency and is discussed later.

For the case of the series RLC circuit these two parameters are given by:

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

A useful parameter is the *damping factor*, ζ which is defined as the ratio of these two,

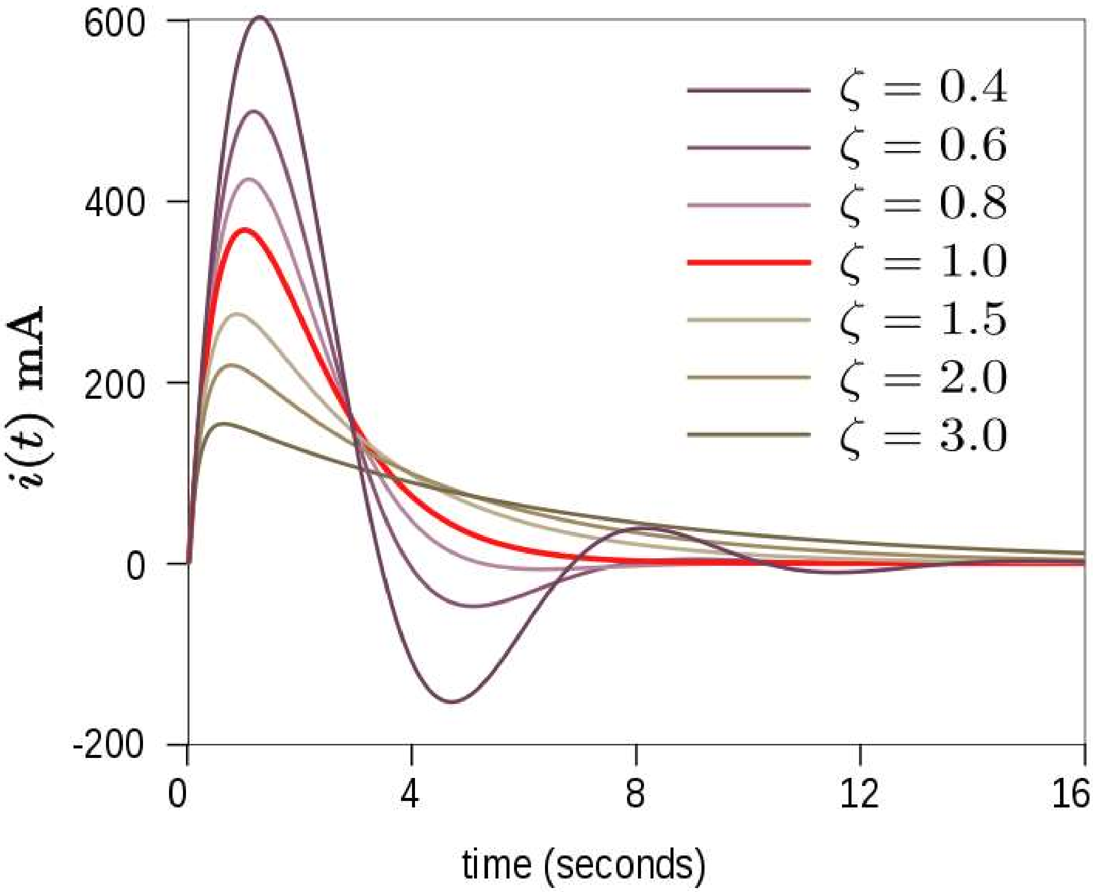
$$\zeta = \frac{\alpha}{\omega_0}$$

In the case of the series RLC circuit, the damping factor is given by,

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$

The value of the damping factor determines the type of transient that the circuit will exhibit. Some authors do not use ζ and call α the damping factor.

Transient response



Plot showing underdamped and overdamped responses of a series RLC circuit. The critical damping plot is the bold red curve. The plots are normalised for L=1, C=1 and $\omega_0=1$

The differential equation for the circuit solves in three different ways depending on the value of ζ. These are underdamped ($\zeta<1$), overdamped ($\zeta>1$) and critically damped ($\zeta=1$). The differential equation has the characteristic equation,

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

The roots of the equation in *s* are,

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$$

The general solution of the differential equation is an exponential in either root or a linear superposition of both,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The coefficients A_1 and A_2 are determined by the boundary conditions of the specific problem being analysed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.

The overdamped response ($\zeta > 1$) is,

$$i(t) = A_1 e^{-\omega_0(\zeta + \sqrt{\zeta^2 - 1})t} + A_2 e^{-\omega_0(\zeta - \sqrt{\zeta^2 - 1})t}$$

The overdamped response is a decay of the transient current without oscillation.

The underdamped response ($\zeta < 1$) is,

$$i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t)$$

By applying standard trigonometric identities the two trigonometric functions may be expressed as a single sinusoid with phase shift,

$$i(t) = B_3 e^{-\alpha t} \sin(\omega_d t + \varphi)$$

The underdamped response is a decaying oscillation at frequency ω_d. The oscillation decays at a rate determined by the attenuation α. The exponential in αdescribes the envelope of the oscillation. B_1 and B_2 (or B_3 and the phase shift φin the second form) are arbitrary constants determined by boundary conditions. The frequency $\omega_d$is given by,

$$\omega_d = \sqrt{\omega_0{}^2 - \alpha^2} = \omega_0 \sqrt{1 - \zeta^2}$$

This is called the damped resonance frequency or the damped natural frequency. It is the frequency the circuit will naturally oscillate at if not driven by an external source. The resonance frequency, ω_0, which is the frequency at which the circuit will resonate when driven by an external oscillation, may often be referred to as the undamped resonance frequency to distinguish it.

The critically damped response ($\zeta = 1$) is,

$$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

The critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation. This consideration is important in control systems where it is required to reach the desired state as quickly as possible without overshooting. D_1 and D_2 are arbitrary constants determined by boundary conditions.

Laplace domain

The series RLC can be analyzed for both transient and steady AC state behavior using the Laplace transform. If the voltage source above produces a waveform with Laplace-transformed *V*(*s*) (where *s* is the complex frequency $s = \sigma + i\omega$), KVL can be applied in the Laplace domain:

$$V(s) = I(s)\left(R + Ls + \frac{1}{Cs}\right)$$

where *I*(*s*) is the Laplace-transformed current through all components. Solving for *I*(*s*):

$$I(s) = \frac{1}{R + Ls + \frac{1}{Cs}} V(s)$$

And rearranging, we have that

$$I(s) = \frac{s}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)} V(s)$$

Laplace admittance

Solving for the Laplace admittance *Y*(*s*):

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right)}$$

Simplifying using parameters α and ω_o defined in the previous section, we have

$$Y(s) = \frac{I(s)}{V(s)} = \frac{s}{L\left(s^2 + 2\alpha s + {\omega_0}^2\right)}$$

Poles and zeros

The zeros of *Y*(*s*) are those values of *s* such that *Y*(*s*) = 0:

$s = 0$ and $|s| \rightarrow \infty$

The poles of *Y*(*s*) are those values of *s* such that $Y(s) \rightarrow \infty$. By the quadratic formula, we find

$$s = -\alpha \pm \sqrt{\alpha^2 - {\omega_0}^2}$$

The poles of $Y(s)$ are identical to the roots s_1 and s_2 of the characteristic polynomial of the differential equation in the section above.

General solution

For an arbitrary E(t), the solution obtained by inverse transform of I(s) is:

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}\left(\cos\omega_d\tau - \frac{\alpha}{\omega_d}\sin\omega_d\tau\right)d\tau$$ in the underdamped case ($\omega_0 > \alpha$)

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}(1-\alpha\tau)d\tau$$ in the critically damped case ($\omega_0 = \alpha$)

$$I(t) = \frac{1}{L}\int_0^t E(t-\tau)e^{-\alpha\tau}\left(\cosh\omega_r\tau - \frac{\alpha}{\omega_r}\sinh\omega_r\tau\right)d\tau$$ in the overdamped case ($\omega_0 < \alpha$)

where $\omega_r = \sqrt{\alpha^2 - {\omega_0}^2}$, and cosh and sinh are the usual hyperbolic functions.

Sinusoidal steady state

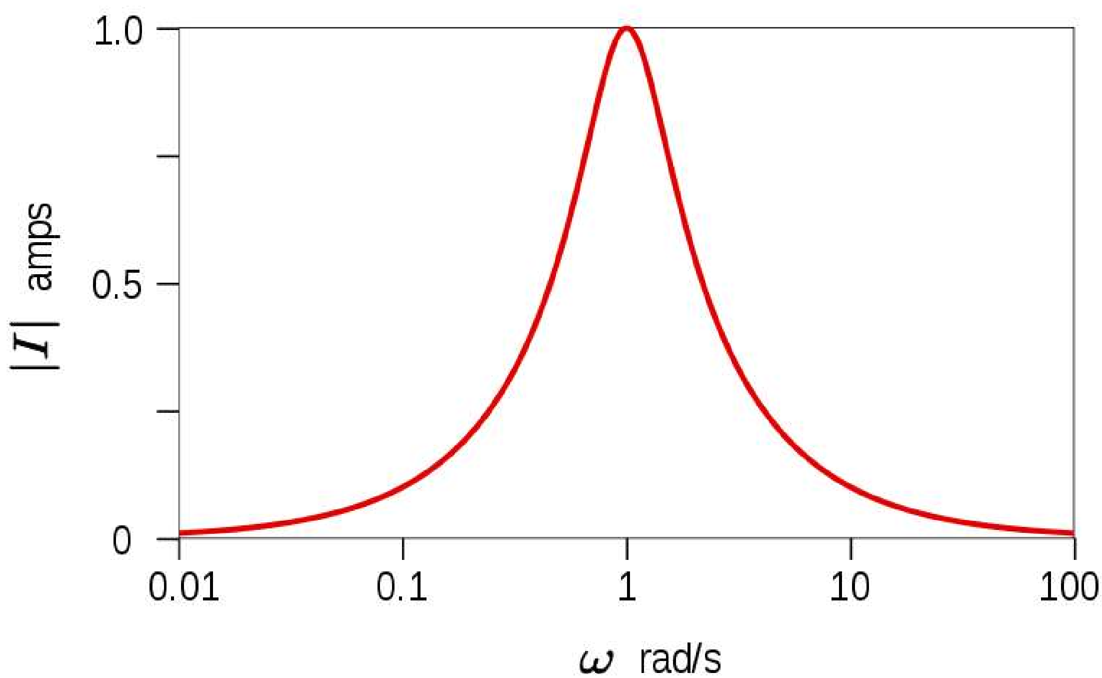


Figure 4. Sinusoidal steady-state analysis

normalised to $R = 1$ ohm, $C = 1$ farad, $L = 1$ henry, and $V = 1.0$ volt

Sinusoidal steady state is represented by letting $s = i\omega$

Taking the magnitude of the above equation with this substitution:

$$|Y(s = i\omega)| = \frac{1}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

and the current as a function of ω can be found from

$$|I(i\omega)| = |Y(i\omega)||V(i\omega)|.$$

Note that there is a peak at $i_{mag}(\omega) = 1$. This is known as the resonance frequency. Solving for this value:

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

Parallel RLC circuit

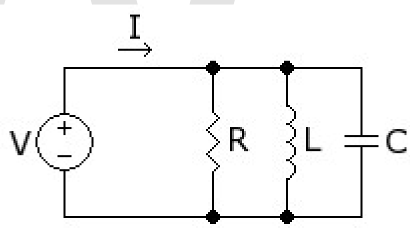


Figure 5. RLC parallel circuit

V - the voltage of the power source
I - the current in the circuit
R - the resistance of the resistor
L - the inductance of the inductor
C - the capacitance of the capacitor

The properties of the parallel RLC circuit can be obtained from the duality relationship of electrical circuits and considering that the parallel RLC is the dual impedance of a series RLC. From this consideration is immediately obtained the result that the differential equations describing this circuit will be identical to the general form of those describing a series RLC.

For the parallel circuit, the attenuation α is given by

$$\alpha = \frac{1}{2RC}$$

and the damping factor is consequently

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$

This is the inverse of the expression for ζ in the series circuit. Likewise, the other scaled parameters, fractional bandwidth and Q are also the inverse of each other. This means that a wide band, low Q circuit in one topology will become a narrow band, high Q circuit in the other topology when constructed from components with identical values. The Q and fractional bandwidth of the parallel circuit are given by

$$Q = R\sqrt{\frac{C}{L}} \text{ and } F_{\mathrm{b}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Frequency domain

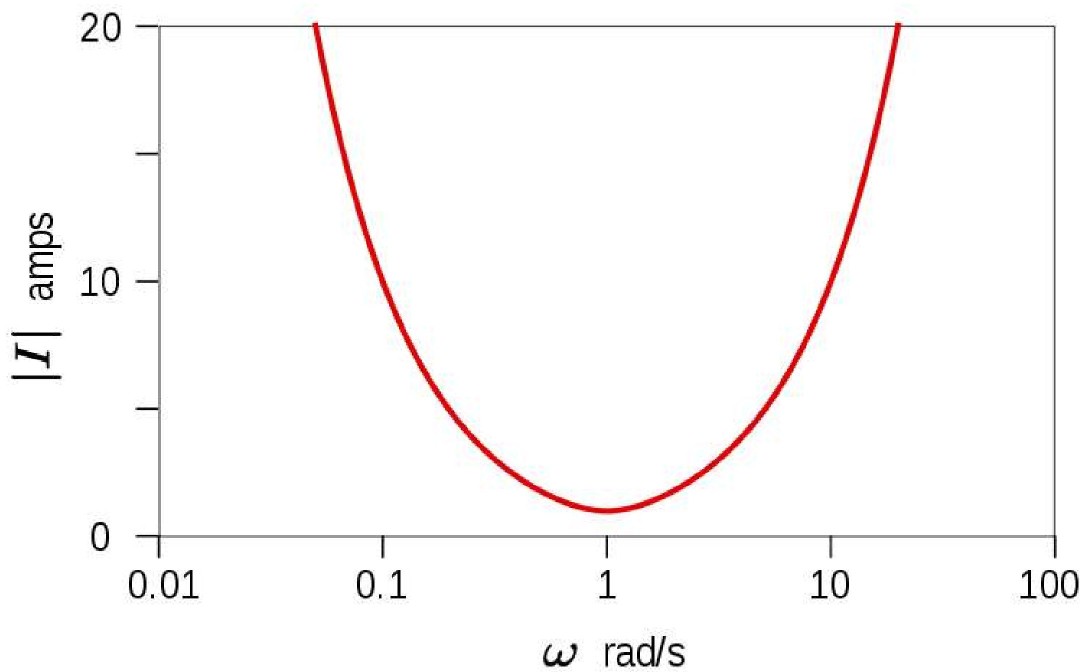


Figure 6. *Sinusoidal steady-state analysis*s

normalised to R = 1 ohm, C = 1 farad, L = 1 henry, and V = 1.0 volt

The complex admittance of this circuit is given by adding up the admittances of the components:

$$\frac{1}{Z} = \frac{1}{Z_L} + \frac{1}{Z_C} + \frac{1}{Z_R} = \frac{1}{j\omega L} + j\omega C + \frac{1}{R}$$

The change from a series arrangement to a parallel arrangement results in the circuit having a peak in impedance at resonance rather than a minimum, so the circuit is an antiresonator.

The graph opposite shows that there is a minimum in the frequency response of the current at the resonance frequency $\omega_0 = \frac{1}{\sqrt{LC}}$ when the circuit is driven by a constant voltage. On the other hand, if driven by a constant current, there would be a maximum in the voltage which would follow the same curve as the current in the series circuit.

Other configurations

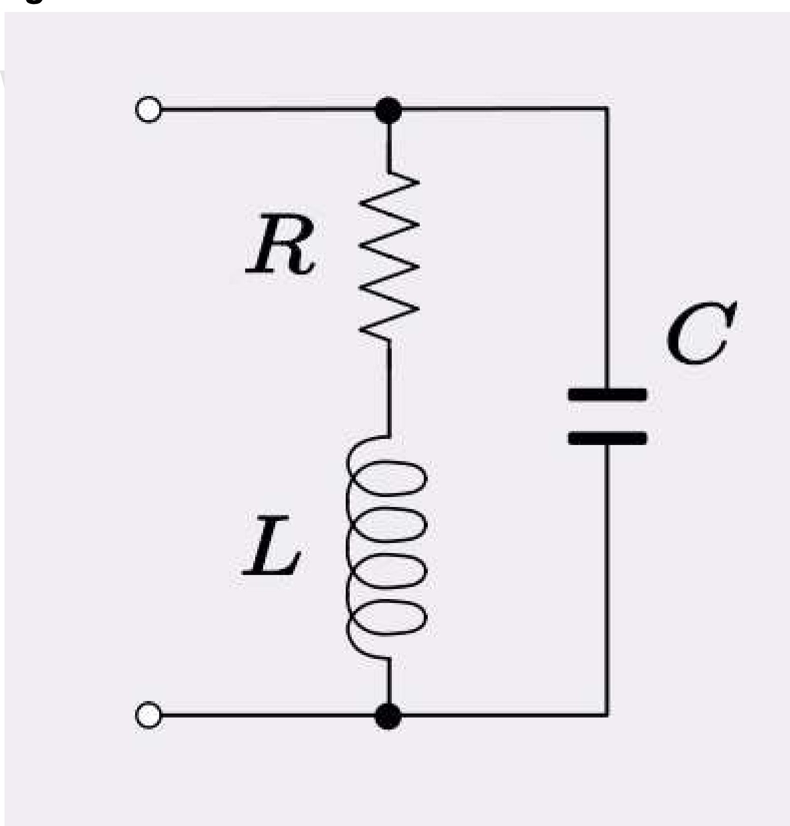


Fig. 7. RLC parallel circuit with resistance in series with the inductor

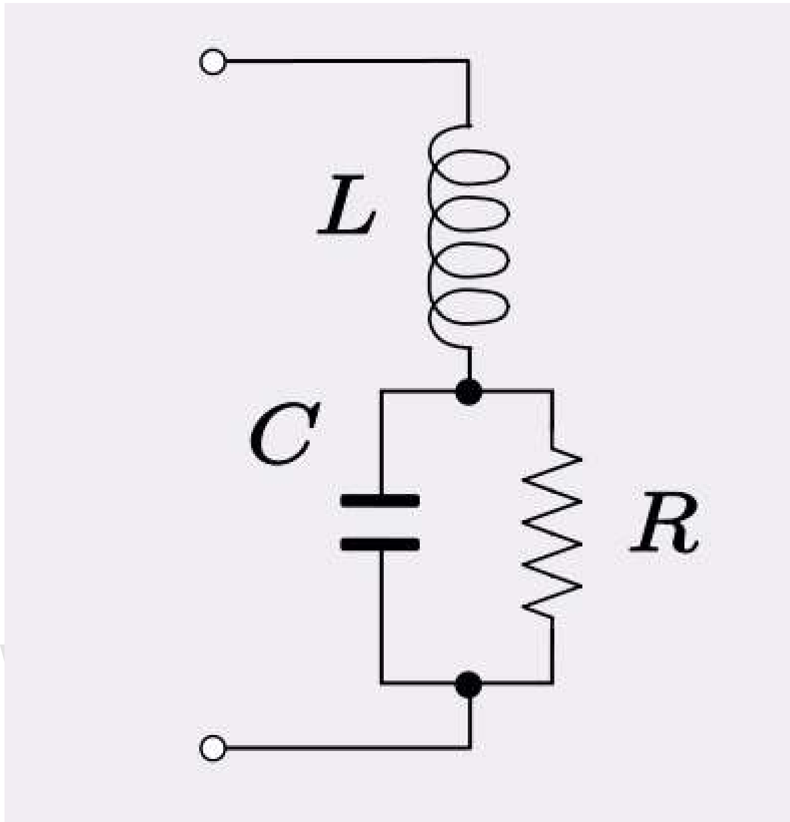


Fig. 8. RLC series circuit with resistance in parallel with the capacitor

A series resistor with the inductor in a parallel LC circuit as shown in figure 7 is a topology commonly encountered where there is a need to take into account the resistance of the coil winding. Parallel LC circuits are frequently used for bandpass filtering and the Q is largely governed by this resistance. The resonant frequency of this circuit is,

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{R}{L}\right)^2}$$

This is the resonant frequency of the circuit defined as the frequency at which the admittance has zero imaginary part. The frequency that appears in the generalised form of the characteristic equation (which is the same for this circuit as previously)

$$s^2 + 2\alpha s + {\omega'_0}^2 = 0$$

is not the same frequency. In this case it is the undamped resonant frequency

$$\omega'_0 = \sqrt{\frac{1}{LC}}$$

In the same vein, a resistor in parallel with the capacitor in a series LC circuit can be used to represent a capacitor with a lossy dielectric. This configuration is shown in figure 8. The resonant frequency in this case is given by

$$\omega_0 = \sqrt{\frac{1}{LC} - \frac{1}{(RC)^2}}$$

Applications

Variable tuned circuits

A very frequent use of these circuits is in the tuning circuits of analogue radios. Adjustable tuning is commonly achieved with a parallel plate variable capacitor which allows the value of C to be changed and tune to stations on different frequencies. For the IF stage in the radio where the tuning is preset in the factory the more usual solution is an adjustable core in the inductor to adjust L. In this design the core (made of a high permeability material that has the effect of increasing inductance) is threaded so that it can be screwed further in, or screwed further out of the inductor winding as required.

Filters

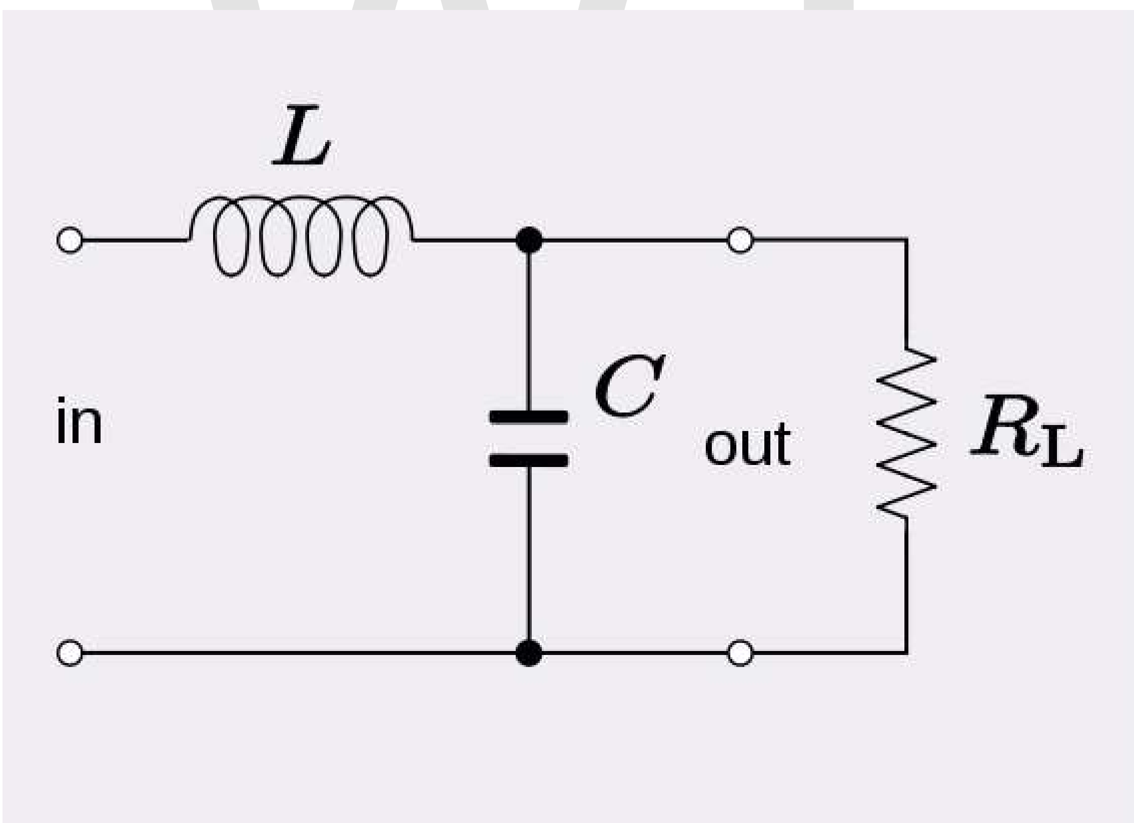


Fig. 9. RLC circuit as a low-pass filter

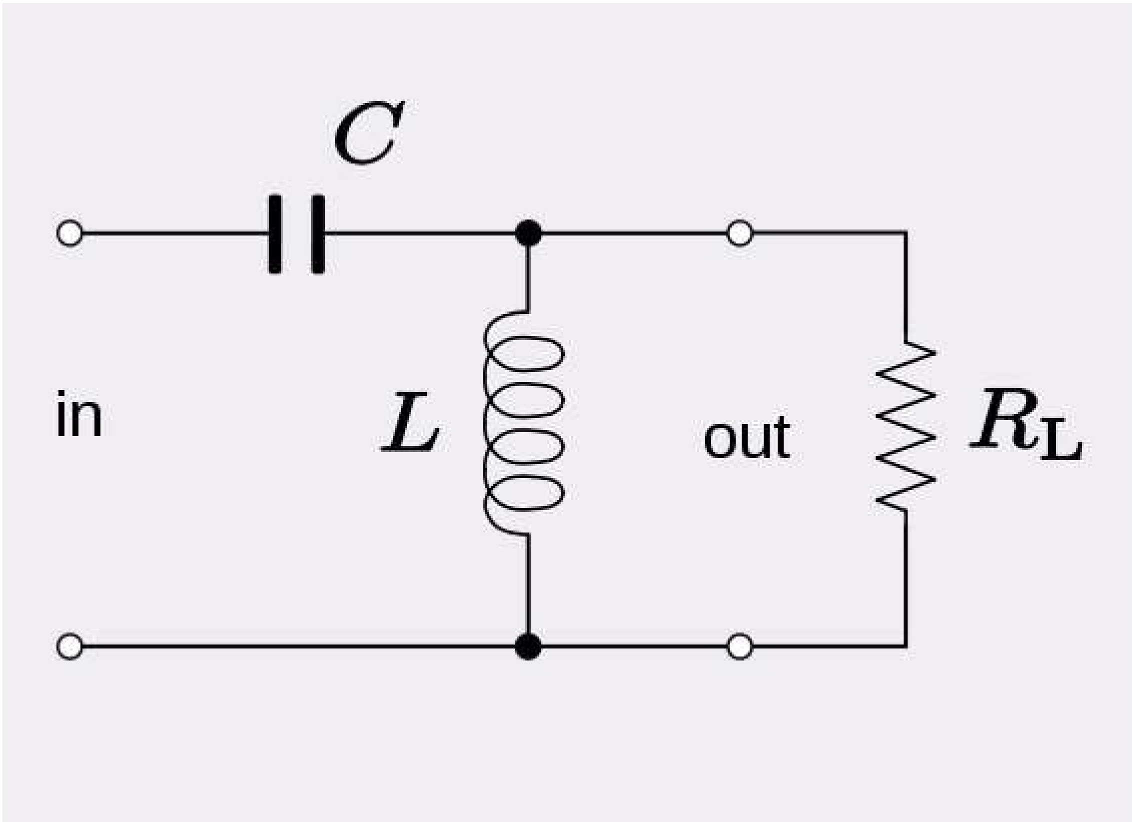


Fig. 10. RLC circuit as a high-pass filter

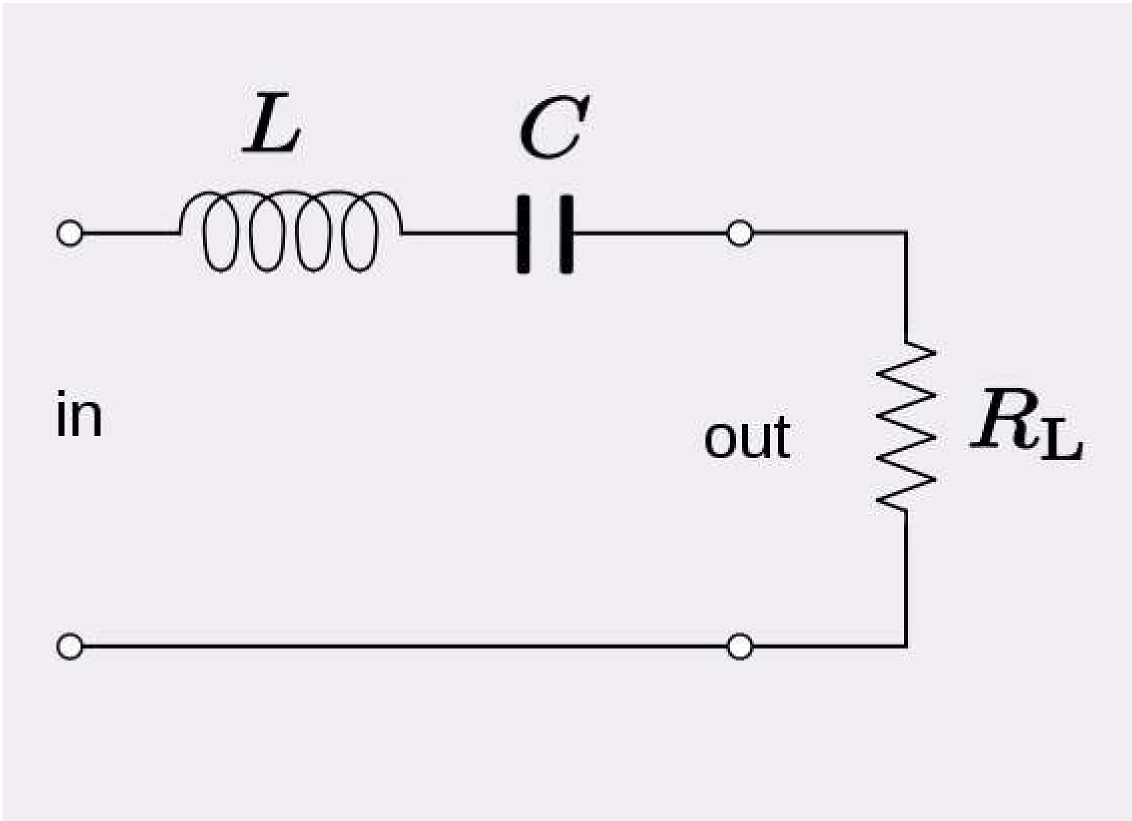


Fig. 11. RLC circuit as a series band-pass filter in series with the line

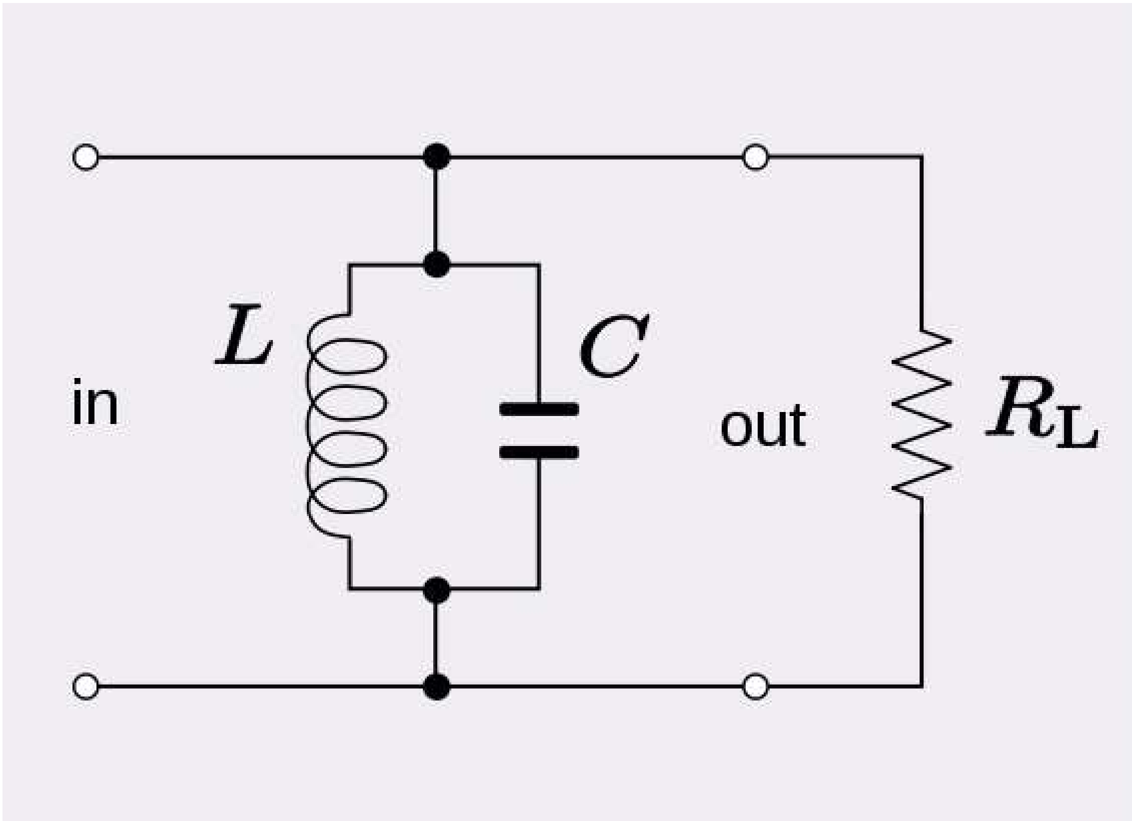


Fig. 12. RLC circuit as a parallel band-pass filter in shunt across the line

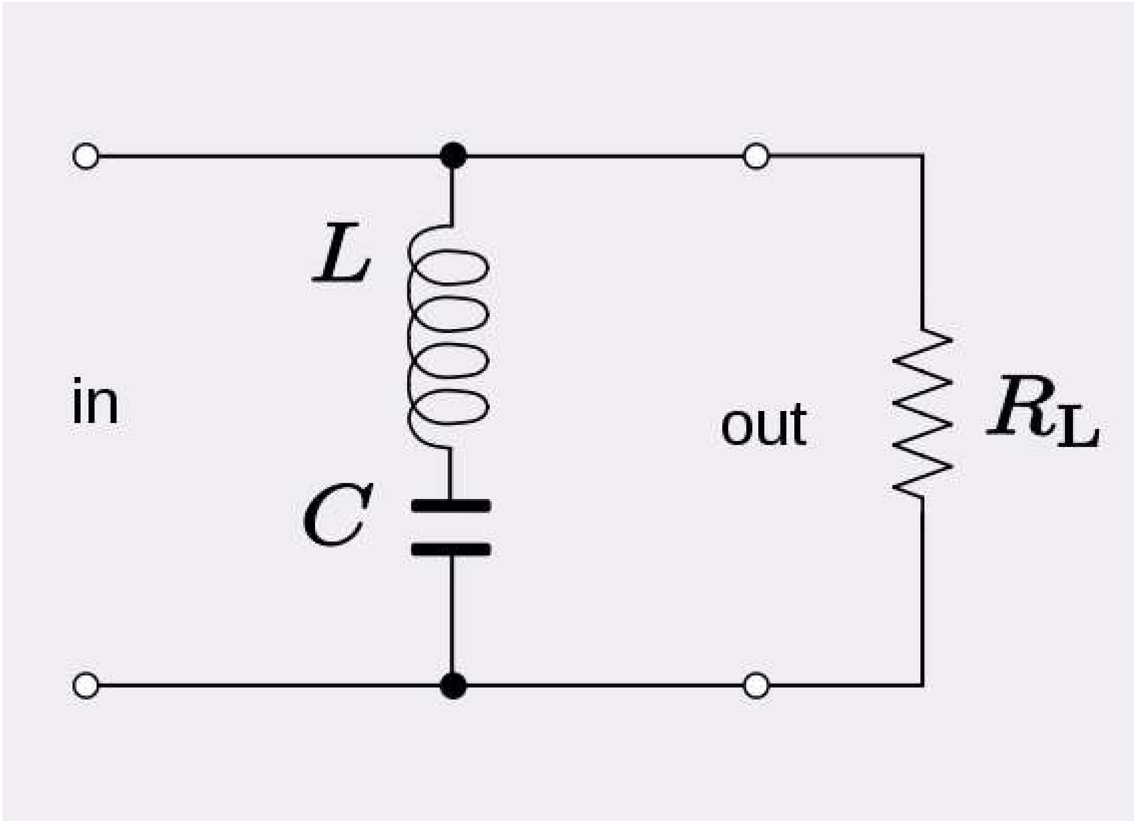


Fig. 13. RLC circuit as a series band-stop filter in shunt across the line

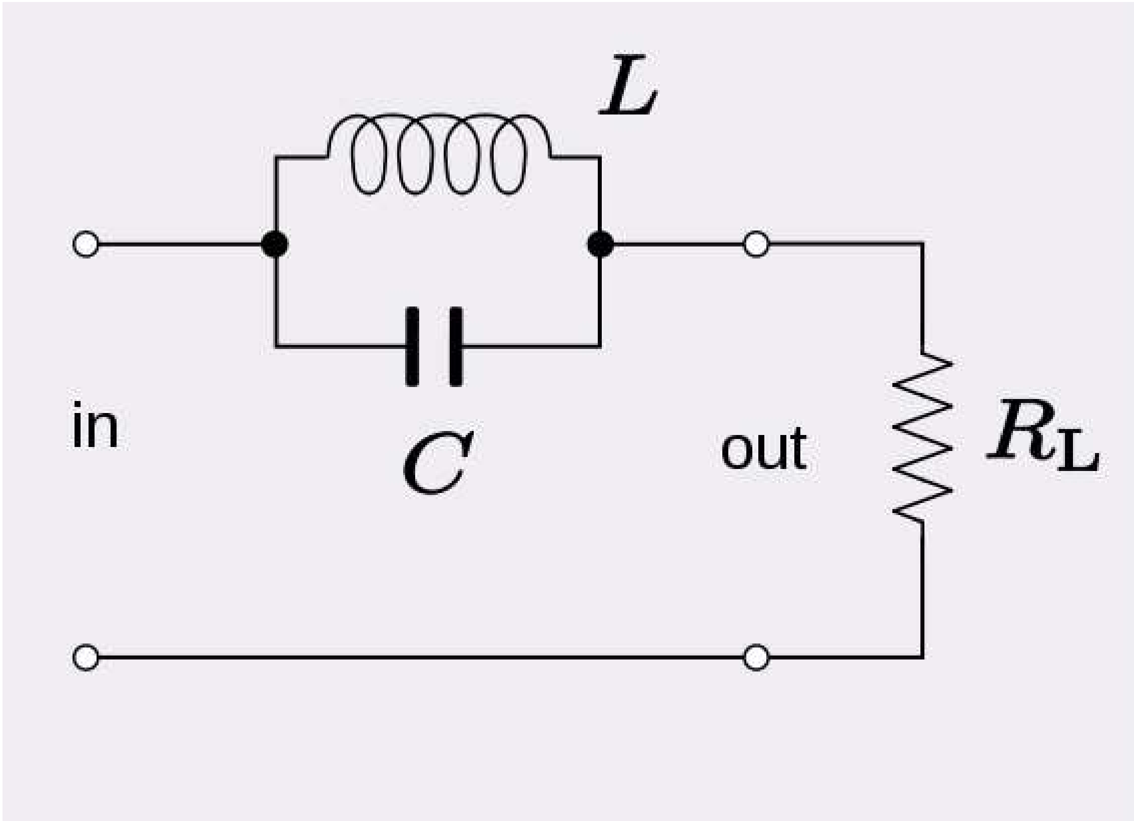


Fig. 14. RLC circuit as a parallel band-stop filter in series with the line

In the filtering application, the resistor *R* becomes the load that the filter is working into. The value of the damping factor is chosen based on the desired bandwidth of the filter. For a wider bandwidth, a larger value of the damping factor is required (and vice versa). The three components give the designer three degrees of freedom. Two of these are required to set the bandwidth and resonant frequency. The designer is still left with one which can be used to scale *R*, *L* and *C* to convenient practical values. Alternatively, *R* may be predetermined by the external circuitry which will use the last degree of freedom.

Low-pass filter

An RLC circuit can be used as a low-pass filter. The circuit configuration is shown in figure 9. The corner frequency, that is, the frequency of the 3dB point, is given by

$$\omega_c = \frac{1}{\sqrt{LC}}$$

This is also the bandwidth of the filter. The damping factor is given by

$$\zeta = \frac{1}{2R_L}\sqrt{\frac{L}{C}}$$

High-pass filter

A high-pass filter is shown in figure 10. The corner frequency is the same as the low-pass filter

$$\omega_c = \frac{1}{\sqrt{LC}}$$

The filter has a stop-band of this width.

Band-pass filter

A band-pass filter can be formed with an RLC circuit by either placing a series LC circuit in series with the load resistor or else by placing a parallel LC circuit in parallel with the load resistor. These arrangements are shown in figures 11 and 12 respectively. The centre frequency is given by

$$\omega_c = \frac{1}{\sqrt{LC}}$$

and the bandwidth for the series circuit is

$$\Delta\omega = \frac{R_L}{L}$$

The shunt version of the circuit is intended to be driven by a high impedance source, that is, a constant current source. Under those conditions the bandwidth is

$$\Delta\omega = \frac{1}{CR_L}$$

Band-stop filter

Figure 13 shows a band-stop filter formed by a series LC circuit in shunt across the load. Figure 14 is a band-stop filter formed by a parallel LC circuit in series with the load. The first case requires a high impedance source so that the current is diverted into the resonator when it becomes low impedance at resonance. The second case requires a low impedance source so that the voltage is dropped across the antiresonator when it becomes high impedance at resonance.

Oscillators

For applications in oscillator circuits, it is generally desirable to make the attenuation (or equivalently, the damping factor) as small as possible. In practice, this objective requires making the circuit's resistance *R* as small as physically possible for a series circuit, or alternatively increasing *R* to as much as possible for a parallel circuit. In either case, the *RLC circuit* becomes a good approximation to an ideal LC circuit. However, for very low attenuation circuits (high Q-factor) circuits, issues such as dielectric losses of coils and capacitors can become important.

In an oscillator circuit

$$\alpha \ll \omega_0.$$

or equivalently

$$\zeta \ll 1.$$

As a result

$$\omega_d \approx \omega_0.$$

Voltage multiplier

In a series RLC circuit at resonance, the current is limited only by the resistance of the circuit

$$I = \frac{V}{R}$$

If R is small, consisting only of the inductor winding resistance say, then this current will be large. It will drop a voltage across the inductor of

$$V_{\mathrm{L}} = \frac{V}{R}\omega_0 L$$

An equal magnitude voltage will also be seen across the capacitor but in antiphase to the inductor. If R can be made sufficiently small, these voltages can be several times the input voltage. The voltage ratio is, in fact, the *Q* of the circuit,

$$\frac{V_{\mathrm{L}}}{V} = Q$$

A similar effect is observed with currents in the parallel circuit. Even though the circuit appears as high impedance to the external source, there is a large current circulating in the internal loop of the parallel inductor and capacitor.

Pulse discharge circuit

An overdamped series RLC circuit can be used as a pulse discharge circuit. Often it is useful to know the values of components that could be used to produce a waveform this is described by the form:

$$I(t) = I_0(e^{-\alpha t} - e^{-\beta t})$$

Such a circuit could consist of an energy storage capacitor, a load in the form of a resistance, some circuit inductance and a switch - all in series. The initial conditions are that the capacitor is at voltage V_0 and there is no current flowing in the inductor. If the inductance L is known, then the remaining parameters are given by the following - Capacitance:

$$C = \frac{1}{L\alpha\beta}$$

Resistance (total of circuit and load):

$$R = L(\alpha + \beta)$$

Initial terminal voltage of capacitor:

$$V_0 = -I_0 L\alpha\beta(\frac{1}{\beta} - \frac{1}{\alpha})$$

Rearranging for the case where R is known - Capacitance:

$$C = \frac{(\alpha + \beta)}{R\alpha\beta}$$

Inductance (total of circuit and load):

$$L = \frac{R}{(\alpha + \beta)}$$

Initial terminal voltage of capacitor:

$$V_0 = \frac{-I_0 R\alpha\beta}{(\alpha + \beta)}(\frac{1}{\beta} - \frac{1}{\alpha})$$

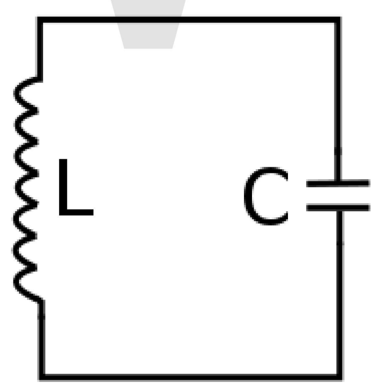
Chapter- 9

LC Circuit

An **LC circuit** is a **resonant circuit** or **tuned circuit** that consists of an inductor, represented by the letter L, and a capacitor, represented by the letter C. When connected together, they can act as an electrical resonator, an electrical analogue of a tuning fork, storing electrical energy oscillating at the circuit's resonant frequency.

LC circuits are used either for generating signals at a particular frequency, or picking out a signal at a particular frequency from a more complex signal. They are key components in many applications such as oscillators, filters, tuners and frequency mixers. An LC circuit is an idealized model since it assumes there is no dissipation of energy due to resistance.

Operation



An LC circuit can store electrical energy vibrating at its resonant frequency. A capacitor stores energy in the electric field between its plates, depending on the voltage across it, and an inductor stores energy in its magnetic field, depending on the current through it.

If a charged capacitor is connected across an inductor, charge will start to flow through the inductor, building up a magnetic field around it, and reducing the voltage on the capacitor. Eventually all the charge on the capacitor will be gone and the voltage across it will reach zero. However, the current will continue, because inductors resist changes in current, and energy to keep it flowing is extracted from the magnetic field, which will begin to decline. The current will begin to charge the capacitor with a voltage of opposite polarity to its original charge. When the magnetic field is completely dissipated the current will stop and the charge will again be stored in the capacitor, with the opposite polarity as before. Then the cycle will begin again, with the current flowing in the opposite direction through the inductor.

The charge flows back and forth between the plates of the capacitor, through the inductor. The energy oscillates back and forth between the capacitor and the inductor until (if not replenished by power from an external circuit) internal resistance makes the oscillations die out. Its action, known mathematically as a harmonic oscillator, is similar to a pendulum swinging back and forth, or water sloshing back and forth in a tank. For this reason the circuit is also called a **tank circuit**. The oscillation frequency is determined by the capacitance and inductance values used. In typical tuned circuits in electronic equipment the oscillations are very fast, thousands to millions of times per second.

Time domain solution

By Kirchhoff's voltage law, the voltage across the capacitor, V_C, must equal the voltage across the inductor, V_L:

$$V_C = V_L.$$

Likewise, by Kirchhoff's current law, the current through the capacitor plus the current through the inductor must equal zero:

$$i_C + i_L = 0.$$

From the constitutive relations for the circuit elements, we also know that

$$V_L(t) = L\frac{di_L}{dt}$$

and

$$i_C(t) = C\frac{dV_C}{dt}.$$

Rearranging and substituting gives the second order differential equation

$$\frac{d^2i(t)}{dt^2} + \frac{1}{LC}i(t) = 0.$$

The parameter ω, the radian frequency, can be defined as: $\omega = (LC)^{-1/2}$. Using this can simplify the differential equation

$$\frac{d^2i(t)}{dt^2} + \omega^2 i(t) = 0.$$

The associated polynomial is $s^2 + \omega^2 = 0$, thus

$$s = +j\omega$$

or

$$s = -j\omega$$

where j is the imaginary unit.

Thus, the complete solution to the differential equation is

$$i(t) = Ae^{+j\omega t} + Be^{-j\omega t}$$

and can be solved for A and B by considering the initial conditions.

Since the exponential is complex, the solution represents a sinusoidal alternating current.

If the initial conditions are such that $A = B$, then we can use Euler's formula to obtain a real sinusoid with amplitude $2A$ and angular frequency $\omega = (LC)^{-1/2}$.

Thus, the resulting solution becomes:

$$i(t) = 2A\cos(\omega t).$$

The initial conditions that would satisfy this result are:

$$i(t = 0) = 2A$$

and

$$\frac{di}{dt}(t = 0) = 0.$$

Resonance effect

The resonance effect occurs when inductive and capacitive reactances are equal in absolute value. (Notice that the LC circuit does not, by itself, resonate. The word resonance refers to a class of phenomena in which a small driving perturbation gives rise to a large effect in the system. The LC circuit must be driven, for example by an AC power supply, for resonance to occur (below).) The frequency at which this equality holds for the particular circuit is called the resonant frequency. The resonant frequency of the LC circuit is

$$\omega = \sqrt{\frac{1}{LC}}$$

where **L** is the inductance in henries, and **C** is the capacitance in farads. The angular frequency ω has units of radians per second.

The equivalent frequency in units of hertz is

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}.$$

LC circuits are often used as filters; the L/C ratio is one of the factors that determines their "Q" and so selectivity. For a series resonant circuit with a given resistance, the higher the inductance and the lower the capacitance, the narrower the filter bandwidth. For a parallel resonant circuit the opposite applies. Positive feedback around the tuned circuit ("regeneration") can also increase selectivity.

Stagger tuning can provide an acceptably wide audio bandwidth, yet good selectivity.

Series LC circuit

Resonance

Here L and C are connected in series to an AC power supply. Inductive reactance magnitude (X_L) increases as frequency increases while capacitive reactance magnitude (X_C) decreases with the increase in frequency. At a particular frequency these two reactances are equal in magnitude but opposite in sign. The frequency at which this happens is the resonant frequency (f_r) for the given circuit.

Hence, at f_r :

$$X_L = -X_C$$
$$\omega L = \frac{1}{\omega C}$$

Converting angular frequency into hertz we get

$$2\pi f L = \frac{1}{2\pi f C}$$

Here *f* is the resonant frequency. Then rearranging,

$$f = \frac{1}{2\pi\sqrt{LC}}$$

In a series AC circuit, X_C leads by 90 degrees while X_L lags by 90. Therefore, they cancel each other out. The only opposition to a current is coil resistance. Hence in series resonance the current is maximum at resonant frequency.

- At f_r, current is maximum. Circuit impedance is minimum. In this state a circuit is called an *acceptor circuit*.
- Below f_r, $X_L \ll (-X_C)$. Hence circuit is capacitive.
- Above f_r, $X_L \gg (-X_C)$. Hence circuit is inductive.

Impedance

First consider the impedance of the series LC circuit. The total impedance is given by the sum of the inductive and capacitive impedances:

$$Z = Z_L + Z_C$$

By writing the inductive impedance as $Z_L = j\omega L$ and capacitive impedance as $Z_C = (j\omega C)^{-1}$ and substituting we have

$$Z = j\omega L + \frac{1}{j\omega C}.$$

Writing this expression under a common denominator gives

$$Z = \frac{(\omega^2 LC - 1)j}{\omega C}.$$

The numerator implies that if $\omega^2 LC = 1$ the total impedance Z will be zero and otherwise non-zero. Therefore the series LC circuit, when connected in series with a load, will act as a band-pass filter having zero impedance at the resonant frequency of the LC circuit.

Parallel LC circuit

Resonance

Here a coil (L) and capacitor (C) are connected in parallel with an AC power supply. Let R be the internal resistance of the coil. When X_L equals X_C, the reactive branch currents are equal and opposite. Hence they cancel out each other to give minimum current in the main line. Since total current is minimum, in this state the total impedance is maximum.

Resonant frequency given by: $f = \frac{1}{2\pi\sqrt{LC}}$.

Note that any reactive branch current is not minimum at resonance, but each is given separately by dividing source voltage (V) by reactance (Z). Hence I=V/Z, as per Ohm's law.

- At f_r, line current is minimum. Total impedance is maximum. In this state a circuit is called a *rejector circuit*.
- Below f_r, circuit is inductive.
- Above f_r,circuit is capacitive.

Impedance

The same analysis may be applied to the parallel LC circuit. The total impedance is then given by:

$$Z = \frac{Z_L Z_C}{Z_L + Z_C}$$

and after substitution of Z_L and Z_C and simplification, gives

$$Z = \frac{-j\omega L}{\omega^2 LC - 1}.$$

Note that

$$\lim_{\omega^2 LC \to 1} Z = \infty$$

but for all other values of $\omega^2 LC$ the impedance is finite (and therefore less than infinity). Hence the parallel LC circuit connected in series with a load will act as band-stop filter having infinite impedance at the resonant frequency of the LC circuit.

Applications

Applications of resonance effect

1. Most common application is **tuning**. For example, when we tune a radio to a particular station, the LC circuits are set at resonance for that particular carrier frequency.
2. A series resonant circuit provides voltage magnification.
3. A parallel resonant circuit provides current magnification.

4. A parallel resonant circuit can be used as load impedance in output circuits of RF amplifiers. Due to high impedance, the gain of amplifier is maximum at resonant frequency.
5. Both parallel and series resonant circuits are used in induction heating.

LC circuits behave as electronic resonators, which are a key component in many applications:

- Oscillators
- Filters
- Tuners
- Mixers
- Foster-Seeley discriminator
- Contactless cards
- Graphics tablets
- Electronic Article Surveillance (Security Tags).