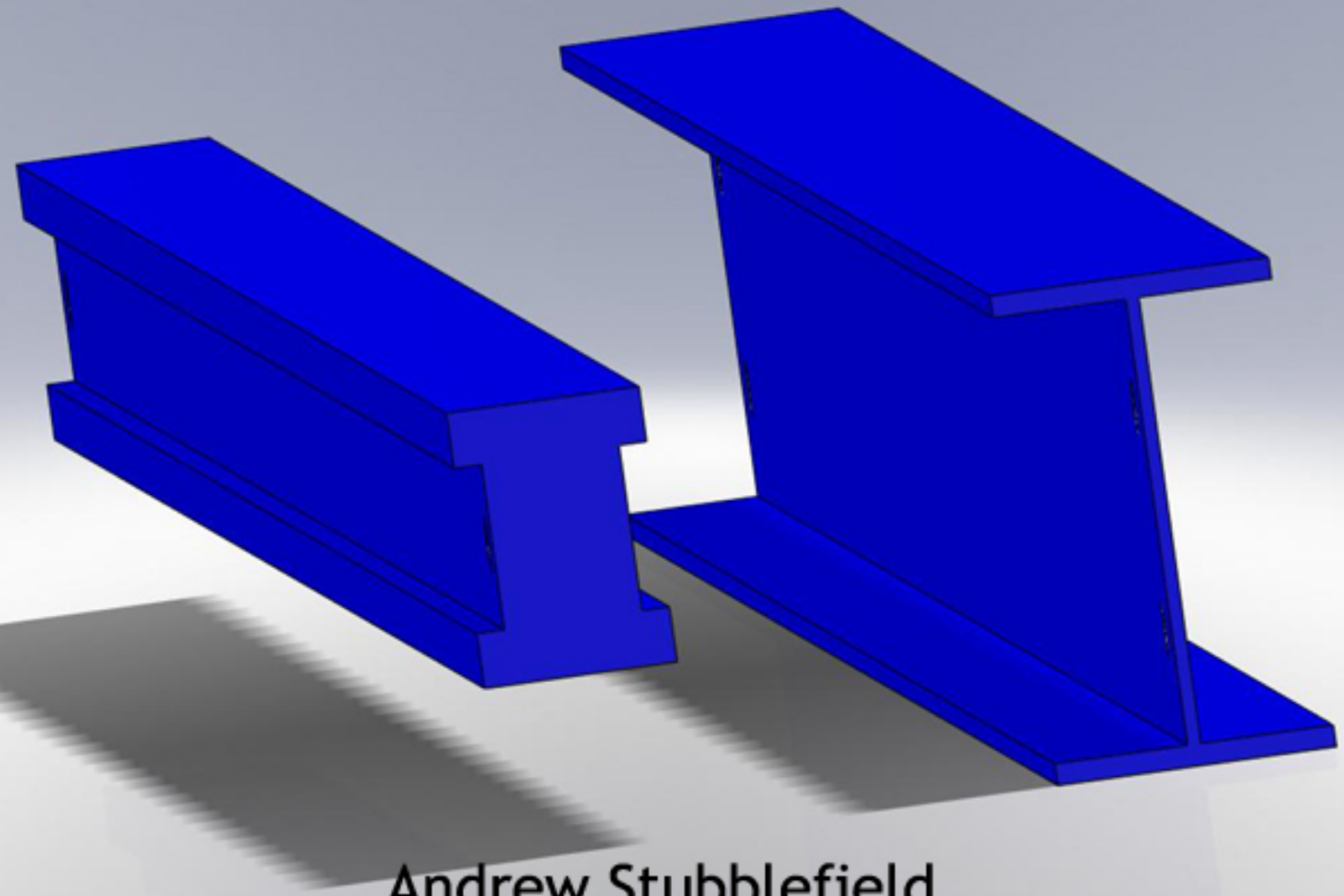
Solid Mechanics & its Applications in Science & Engineering



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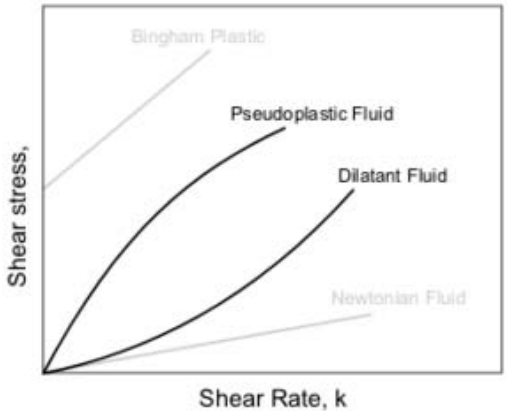
Chapter 1

Viscoelasticity

Viscoelasticity is the property of materials that exhibit both viscous and elastic characteristics when undergoing deformation. Viscous materials, like honey, resist shear flow and strain linearly with time when a stress is applied. Elastic materials strain instantaneously when stretched and just as quickly return to their original state once the stress is removed. Viscoelastic materials have elements of both of these properties and, as such, exhibit time dependent strain. Whereas elasticity is usually the result of bond stretching along crystallographic planes in an ordered solid, viscosity is the result of the diffusion of atoms or molecules inside an amorphous material.

Background

In the nineteenth century, physicists such as Maxwell, Boltzmann, and Kelvin researched and experimented with creep and recovery of glasses, metals, and rubbers. Viscoelasticity was further examined in the late twentieth century when synthetic polymers were engineered and used in a variety of applications. Viscoelasticity calculations depend heavily on the viscosity variable, η. The inverse of η is also known as fluidity, φ. The value of either can be derived as a function of temperature or as a given value (ie for a dashpot).

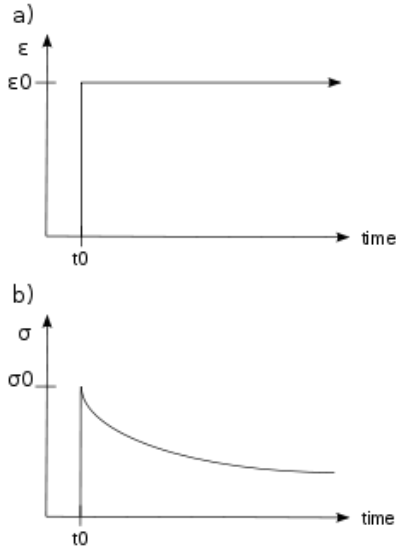


Different types of responses (σ) to a change in strain rate (dℇ/dt)

Depending on the change of strain rate versus stress inside a material the viscosity can be categorized as having a linear, non-linear, or plastic response. When a material exhibits a linear response it is categorized as a Newtonian material. In this case the stress is linearly proportional to the strain rate. If the material exhibits a non-linear response to the strain rate, it is categorized as Non-Newtonian fluid. There is also an interesting case where the viscosity decreases as the shear/strain rate remains constant. A material which exhibits this type of behavior is known as thixotropic. In addition, when the stress is independent of this strain rate, the material exhibits plastic deformation. Many viscoelastic materials exhibit rubber like behavior explained by the thermodynamic theory of polymer elasticity. In reality all materials deviate from Hooke's law in various ways, for example by exhibiting viscous-like as well as elastic characteristics. Viscoelastic materials are those for which the relationship between stress and strain depends on time. Anelastic solids represent a subset of viscoelastic materials: they have a unique equilibrium configuration and ultimately recover fully after removal of a transient load.

Some phenomena in viscoelastic materials are:

- if the stress is held constant, the strain increases with time (creep);
- if the strain is held constant, the stress decreases with time (relaxation);
- the effective stiffness depends on the rate of application of the load;
- if cyclic loading is applied, hysteresis (a phase lag) occurs, leading to a dissipation of mechanical energy;
- acoustic waves experience attenuation;
- rebound of an object following an impact is less than 100%;
- during rolling, frictional resistance occurs.



a) Applied strain and b) induced stress as functions of time for a viscoelastic material.

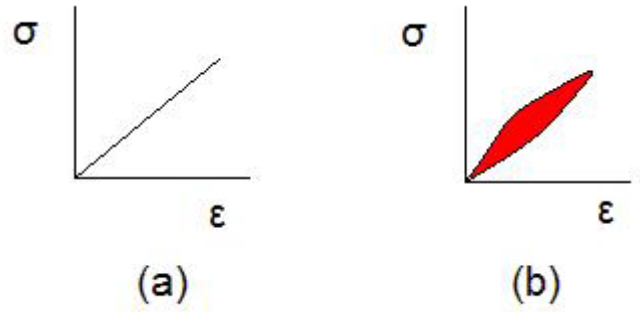
All materials exhibit some viscoelastic response. In common metals such as steel or aluminum, as well as in quartz, at room temperature and at small strain, the behavior does not deviate much from linear elasticity. Synthetic polymers, wood, and human tissue as well as metals at high temperature display significant viscoelastic effects. In some applications, even a small viscoelastic response can be significant. To be complete, an analysis or design involving such materials must incorporate their viscoelastic behavior. Knowledge of the viscoelastic response of a material is based on measurement

Some examples of viscoelastic materials include amorphous polymers, semicrystalline polymers, biopolymers, metals at very high temperatures, and bitumen materials. Cracking occurs when the strain is applied quickly and outside of the elastic limit.

A **viscoelastic** material has the following properties:

- hysteresis is seen in the stress-strain curve.
- stress relaxation occurs: step constant strain causes decreasing stress
- creep occurs: step constant stress causes increasing strain

Elastic behavior versus viscoelastic behavior



Stress-Strain Curves for a purely elastic material (a) and a viscoelastic material (b). The red area is a hysteresis loop and shows the amount of energy lost (as heat) in a loading and unloading cycle. It is equal to $\oint \sigma \, d\varepsilon$, where σ is stress and ε is strain.

Unlike purely elastic substances, a viscoelastic substance has an elastic component and a viscous component. The viscosity of a viscoelastic substance gives the substance a strain rate dependent on time. Purely elastic materials do not dissipate energy (heat) when a load is applied, then removed. However, a viscoelastic substance loses energy when a load is applied, then removed. Hysteresis is observed in the stress-strain curve, with the area of the loop being equal to the energy lost during the loading cycle. Since viscosity is the resistance to thermally activated plastic deformation, a viscous material will lose energy through a loading cycle. Plastic deformation results in lost energy, which is uncharacteristic of a purely elastic material's reaction to a loading cycle.

Specifically, viscoelasticity is a molecular rearrangement. When a stress is applied to a viscoelastic material such as a polymer, parts of the long polymer chain change position. This movement or rearrangement is called Creep. Polymers remain a solid material even when these parts of their chains are rearranging in order to accompany the stress, and as this occurs, it creates a back stress in the material. When the back stress is the same magnitude as the applied stress, the material no longer creeps. When the original stress is taken away, the accumulated back stresses will cause the polymer to return to its original form. The material creeps, which gives the prefix visco-, and the material fully recovers, which gives the suffix -elasticity.

Types of viscoelasticity

Linear viscoelasticity is when the function is separable in both creep response and load. All linear viscoelastic models can be represented by a Volterra equation connecting stress and strain:

$$\epsilon(t) = \frac{\sigma(t)}{E_{\mathbf{inst,creep}}} + \int_0^t K(t-t')\dot{\sigma}(t')dt'$$

or

$$\sigma(t) = E_{\mathbf{inst,relax}}\epsilon(t) + \int_0^t F(t-t')\dot{\epsilon}(t')dt'$$

where

- t is time
- $\sigma(t)$ is stress
- $\varepsilon(t)$ is strain
- $E_{inst,creep}$ and $E_{inst,relax}$ are instantaneous elastic moduli for creep and relaxation
- $K(t)$ is the creep function
- $F(t)$ is the relaxation function

Linear viscoelasticity is usually applicable only for small deformations.

Nonlinear viscoelasticity is when the function is not separable. It is usually happens when the deformations are large or if the material changes its properties under deformations.

An **anelastic** material is a special case of a viscoelastic material: an anelastic material will fully recover to its original state on the removal of load.

Dynamic modulus

Viscoelasticity is studied using dynamic mechanical analysis, applying a small oscillatory strain and measuring the resulting stress.

- Purely elastic materials have stress and strain in phase, so that the response of one caused by the other is immediate.
- In purely viscous materials, strain lags stress by a 90 degree phase lag.
- Viscoelastic materials exhibit behavior somewhere in the middle of these two types of material, exhibiting some lag in strain.

Complex Dynamic modulus G can be used to represent the relations between the oscillating stress and strain:

$$G = G' + iG''$$

where $i^2 = -1$; G' is the *storage modulus* and G'' is the *loss modulus*:

$$G' = \frac{\sigma_0}{\varepsilon_0} \cos \delta$$
$$G'' = \frac{\sigma_0}{\varepsilon_0} \sin \delta$$

where σ_0 and ε_0 are the amplitudes of stress and strain and δ is the phase shift between them.

Constitutive models of linear viscoelasticity

Viscoelastic materials, such as amorphous polymers, semicrystalline polymers, and biopolymers, can be modeled in order to determine their stress or strain interactions as well as their temporal dependencies. These models, which include the Maxwell model, the Kelvin-Voigt model, and the Standard Linear Solid Model, are used to predict a material's response under different loading conditions. Viscoelastic behavior has elastic and viscous components modeled as linear combinations of springs and dashpots, respectively. Each model differs in the arrangement of these elements, and all of these viscoelastic models can be equivalently modeled as electrical circuits. In an equivalent electrical circuit, stress is represented by voltage, and the derivative of strain (velocity) by current. The elastic modulus of a spring is analogous to a circuit's *capacitance* (it stores energy) and the viscosity of a dashpot to a circuit's *resistance* (it dissipates energy).

The elastic components, as previously mentioned, can be modeled as springs of elastic constant E, given the formula:

$$\sigma = E\varepsilon$$

where σ is the stress, E is the elastic modulus of the material, and ε is the strain that occurs under the given stress, similar to Hooke's Law.

The viscous components can be modeled as dashpots such that the stress-strain rate relationship can be given as,

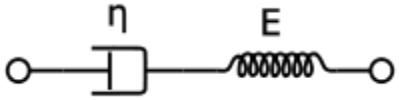
$$\sigma = \eta \frac{d\varepsilon}{dt}$$

where σ is the stress, η is the viscosity of the material, and dε/dt is the time derivative of strain.

The relationship between stress and strain can be simplified for specific stress rates. For high stress states/short time periods, the time derivative components of the stress-strain relationship dominate. A dashpots resists changes in length, and in a high stress state it can be approximated as a rigid rod. Since a rigid rod cannot be stretched past its original length, no strain is added to the system

Conversely, for low stress states/longer time periods, the time derivative components are negligible and the dashpot can be effectively removed from the system - an "open" circuit. As a result, only the spring connected in parallel to the dashpot will contribute to the total strain in the system

Maxwell model



Maxwell model

The Maxwell model can be represented by a purely viscous damper and a purely elastic spring connected in series, as shown in the diagram. The model can be represented by the following equation:

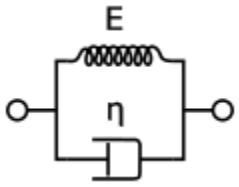
$$\frac{d\epsilon_{Total}}{dt} = \frac{d\epsilon_D}{dt} + \frac{d\epsilon_S}{dt} = \frac{\sigma}{\eta} + \frac{1}{E}\frac{d\sigma}{dt}.$$

Under this model, if the material is put under a constant strain, the stresses gradually relax, When a material is put under a constant stress, the strain has two components. First, an elastic component occurs instantaneously, corresponding to the spring, and relaxes immediately upon release of the stress. The second is a viscous component that grows with time as long as the stress is applied. The Maxwell model predicts that stress decays exponentially with time, which is accurate for most polymers. One limitation of this model is that it does not predict creep accurately. The Maxwell model for creep or

constant-stress conditions postulates that strain will increase linearly with time. However, polymers for the most part show the strain rate to be decreasing with time.

Application to soft solids:thermoplastic polymers in the vicinity of their melting temperature, fresh concrete (neglecting its ageing), numerous metals at a temperature close to their melting point.

Kelvin–Voigt model



Schematic representation of Kelvin–Voigt model.

The Kelvin–Voigt model, also known as the Voigt model, consists of a Newtonian damper and Hookean elastic spring connected in parallel, as shown in the picture. It is used to explain the creep behaviour of polymers.

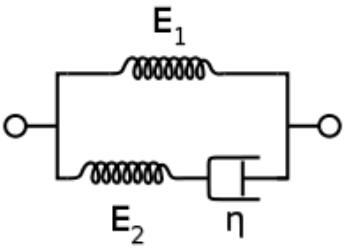
The constitutive relation is expressed as a linear first-order differential equation:

$$\sigma(t) = E\varepsilon(t) + \eta\frac{d\varepsilon(t)}{dt}$$

This model represents a solid undergoing reversible, viscoelastic strain. Upon application of a constant stress, the material deforms at a decreasing rate, asymptotically approaching the steady-state strain. When the stress is released, the material gradually relaxes to its undeformed state. At constant stress (creep), the Model is quite realistic as it predicts strain to tend to σ/E as time continues to infinity. Similar to the Maxwell model, the Kelvin–Voigt model also has limitations. The model is extremely good with modelling creep in materials, but with regards to relaxation the model is much less accurate.

Applications: organic polymers, rubber, wood when the load is not too high.

Standard linear solid model



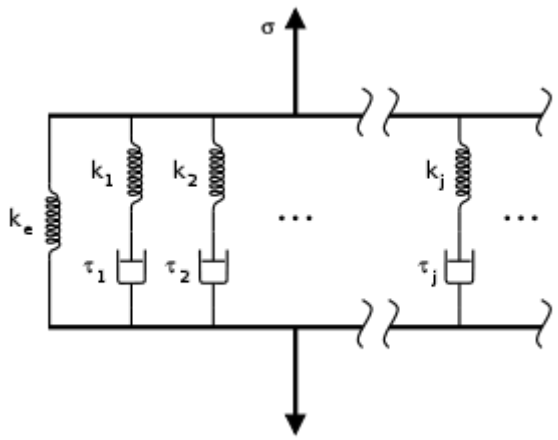
Schematic representation of the Standard Linear Solid model.

The Standard Linear Solid Model effectively combines the Maxwell Model and a Hookean spring in parallel. A viscous material is modeled as a spring and a dashpot in series with each other, both of which are in parallel with a lone spring. For this model, the governing constitutive relation is:

$$\frac{d\varepsilon}{dt} = \frac{\frac{E_2}{\eta}\left(\frac{\eta}{E_2}\frac{d\sigma}{dt} + \sigma - E_1\varepsilon\right)}{E_1 + E_2}$$

Under a constant stress, the modeled material will instantaneously deform to some strain, which is the elastic portion of the strain, and after that it will continue to deform and asymptotically approach a steady-state strain. This last portion is the viscous part of the strain. Although the Standard Linear Solid Model is more accurate than the Maxwell and Kelvin-Voigt models in predicting material responses, mathematically it returns inaccurate results for strain under specific loading conditions and is rather difficult to calculate.

Generalized Maxwell Model



Schematic of Maxwell-Wiechert Model

The Generalized Maxwell also known as the Maxwell-Weichert model (after James Clerk Maxwell and Dieter Weichert) is the most general form of the models described above. It takes into account that relaxation does not occur at a single time, but at a distribution of times. Due to molecular segments of different lengths with shorter ones contributing less than longer ones, there is a varying time distribution. The Weichert model shows this by having as many spring-dashpot Maxwell elements as are necessary to accurately represent the distribution. The Figure on the right represents a possible Wiechert model. Applications: metals and alloys at temperatures lower than one quarter of their absolute melting temperature (expressed in K).

Prony series

In a one-dimensional relaxation test, the material is subjected to a sudden strain that is kept constant over the duration of the test, and the stress is measured over time. The initial stress is due to the elastic response of the material. Then, the stress relaxes over time due to the viscous effects in the material. Typically, either a tensile, compressive, bulk compression, or shear strain is applied. The resulting stress vs. time data can be fitted with a number of equations, called models. Only the notation changes depending of the type of strain applied: tensile-compressive relaxation is denoted *E*, shear is denoted *G*, bulk is denoted *K*. The Prony series for the shear relaxation is

$$G(t) = G_\infty + \Sigma_{i=1}^{N} G_i \exp(-t/\tau_i)$$

where $G_\infty$is the long term modulus once the material is totally relaxed, τ_i are the relaxation times; the higher their values, the longer it takes for the stress to relax. The data is fitted with the equation by using a minimization algorithm that adjust the parameters (G_∞, G_i, τ_i) to minimize the error between the predicted and data values.

An alternative form is obtained noting that the elastic modulus is related to the long term modulus by

$$G(t=0) = G_0 = G_\infty + \Sigma_{i=1}^{N} G_i$$

Therefore,

$$G(t) = G_0 - \Sigma_{i=1}^{N} G_i [1 - \exp(-t/\tau_i)]$$

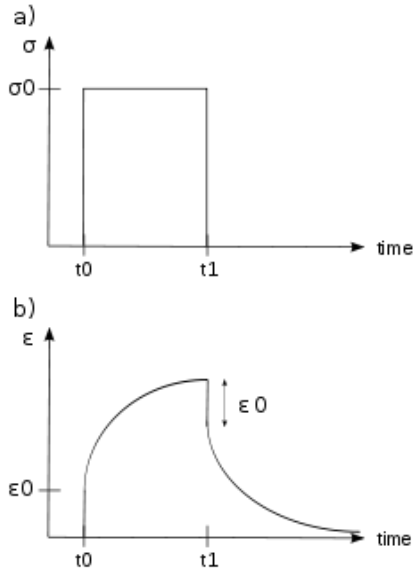
This form is convenient when the elastic shear modulus G_0 is obtained from data independent from the relaxation data, and/or for computer implementation, when it is desired to specify the elastic properties separately from the viscous properties, as in.

A creep experiment is usually easier to perform than a relaxation one, so most data is available as (creep) compliance vs. time. Unfortunately, there is no known closed form for the (creep) compliance in terms of the coefficient of the Prony series. So, if one has creep data, it is not easy to get the coefficients of the (relaxation) Prony series, which are needed for example in. An expedient way to obtain these coefficients is the following. First, fit the creep data with a model that has closed form solutions in both compliance and relaxation; for example the Maxwell-Kelvin model (eq. 7.18-7.19) in or the Standard Solid Model (eq. 7.20-7.21) in (section 7.1.3). Once the parameters of the creep model are known, produce relaxation pseudo-data with the conjugate relaxation model for the same times of the original data. Finally, fit the pseudo data with the Prony series.

Effect of temperature on viscoelastic behavior

The secondary bonds of a polymer constantly break and reform due to thermal motion. Application of a stress favors some conformations over others, so the molecules of the polymer will gradually "flow" into the favored conformations over time. Because thermal motion is one factor contributing to the deformation of polymers, viscoelastic properties change with increasing or decreasing temperature. In most cases, the creep modulus, defined as the ratio of applied stress to the time-dependent strain, decreases with increasing temperature. Generally speaking, an increase in temperature correlates to a logarithmic decrease in the time required to impart equal strain under a constant stress. In other words, it takes less work to stretch a viscoelastic material an equal distance at a higher temperature than it does at a lower temperature.

Viscoelastic creep



a) Applied stress and b) induced strain (b) as functions of time over a short period for a viscoelastic material.

When subjected to a step constant stress, viscoelastic materials experience a time-dependent increase in strain. This phenomenon is known as viscoelastic creep.

At a time t_0, a viscoelastic material is loaded with a constant stress that is maintained for a sufficiently long time period. The material responds to the stress with a strain that increases until the material ultimately fails. When the stress is maintained for a shorter time period, the material undergoes an initial strain until a time t_1, after which the strain immediately decreases (discontinuity) then gradually decreases at times $t > t_1$ to a residual strain.

Viscoelastic creep data can be presented by plotting the creep modulus (constant applied stress divided by total strain at a particular time) as a function of time. Below its critical stress, the viscoelastic creep modulus is independent of stress applied. A family of curves describing strain versus time response to various applied stress may be represented by a single viscoelastic creep modulus versus time curve if the applied stresses are below the material's critical stress value.

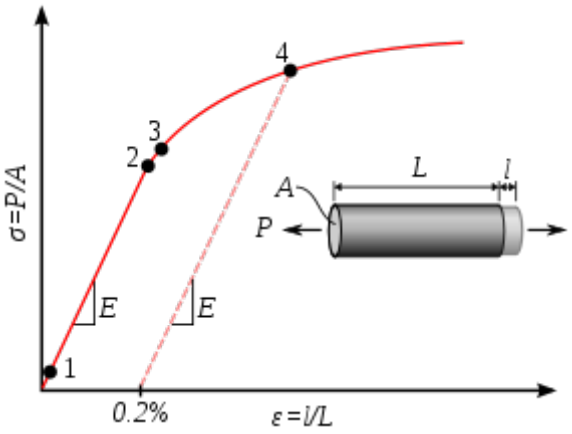
Viscoelastic creep is important when considering long-term structural design. Given loading and temperature conditions, designers can choose materials that best suit component lifetimes.

Measuring viscoelasticity

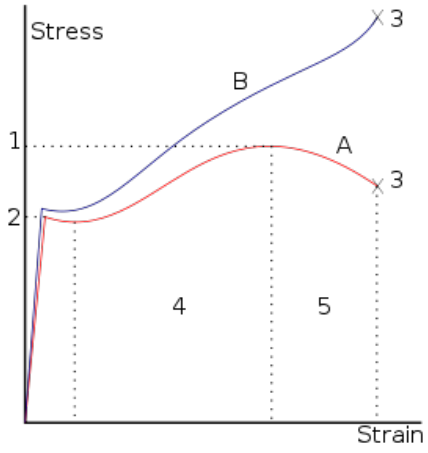
Though there are many instruments that test the mechanical and viscoelastic response of materials, broadband viscoelastic spectroscopy (BVS) and resonant ultrasound spectroscopy (RUS) are more commonly used to test viscoelastic behavior because they can be used above and below ambient temperatures and are more specific to testing viscoelasticity. These two instruments employ a damping mechanism at various frequencies and time ranges with no appeal to time-temperature superposition. Using BVS and RUS to study the mechanical properties of materials is important to understanding how a material exhibiting viscoelasticity will perform.

Chapter 2

Plasticity



Stress-strain curve showing typical yield behavior for nonferrous alloys. Stress (σ) is shown as a function of strain (ε)
1: True elastic limit
2: Proportionality limit
3: Elastic limit
4: Offset yield strength



A stress–strain curve typical of structural steel
1. Ultimate Strength
2. Yield Strength
3. Rupture
4. Strain hardening region
5. Necking region.
A: Apparent stress (F/A_0)
B: Actual stress (F/A)

In physics and materials science, **plasticity** describes the deformation of a material undergoing non-reversible changes of shape in response to applied forces. For example, a solid piece of metal being bent or pounded into a new shape displays plasticity as permanent changes occur within the material itself. In engineering, the transition from elastic behavior to plastic behavior is called yield.

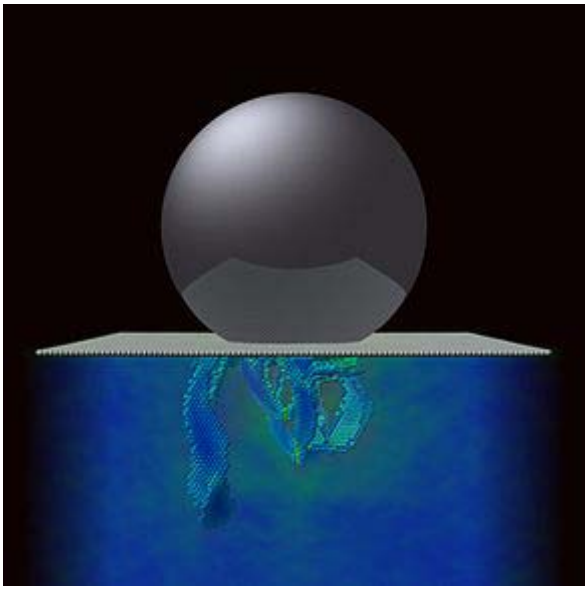
Plastic deformation is observed in most materials including metals, soils, rocks, concrete, foams, bone and skin. However, the physical mechanisms that cause plastic deformation can vary widely. At the crystal scale, plasticity in metals is usually a consequence of dislocations. In most crystalline materials such defects are relatively rare. But there are also materials where defects are numerous and are part of the very crystal structure, in such cases plastic crystallinity can result. In brittle materials such as rock, concrete, and bone, plasticity is caused predominantly by slip at microcracks.

For many ductile metals, tensile loading applied to a sample will cause it to behave in an elastic manner. Each increment of load is accompanied by a proportional increment in extension, and when the load is removed, the piece returns exactly to its original size. However, once the load exceeds some threshold (the yield strength), the extension increases more rapidly than in the elastic region, and when the load is removed, some amount of the extension remains.

However, elastic deformation is an approximation and its quality depends on the considered time frame and loading speed. If the deformation behavior includes elastic deformation as indicated in the adjacent graph it is also often referred to as elastic-plastic or elasto-plastic deformation.

Perfect plasticity is a property of materials to undergo irreversible deformation without any increase in stresses or loads. Plastic materials with hardening necessitate increasingly higher stresses to result in further plastic deformation. Generally plastic deformation is also dependent on the deformation speed, i.e. usually higher stresses have to be applied to increase the rate of deformation and such materials are said to deform visco-plastically.

Physical mechanisms



Plasticity under a spherical Nanoindenter in (111) Copper. All particles in ideal lattice positions are omitted and the color code refers to the von Mises stress field.

Plasticity in metals

Plasticity in a crystal of pure metal is primarily caused by two modes of deformation in the crystal lattice, slip and twinning. Slip is a shear deformation which moves the atoms through many interatomic distances relative to their initial positions. Twinning is the plastic deformation which takes place along two planes due to set of forces applied on a given metal piece.

Slip systems

Crystalline materials contain uniform planes of atoms organized with long-range order. Planes may slip past each other along their close-packed directions, as is shown on the slip systems page. The result is a permanent change of shape within the crystal and plastic deformation. The presence of dislocations increases the likelihood of planes slipping.

Reversible plasticity

On the nano scale the primary plastic deformation in simple fcc metals is reversible, as long as there is no material transport in form of cross-glide.

Shear banding

The presence of other defects within a crystal may entangle dislocations or otherwise prevent them from gliding. When this happens, plasticity is localized to particular regions in the material. For crystals, these regions of localized plasticity are called shear bands.

Plasticity in amorphous materials

Crazing

In amorphous materials, the discussion of “dislocations” is inapplicable, since the entire material lacks long range order. These materials can still undergo plastic deformation. Since amorphous materials, like polymers, are not well-ordered, they contain a large amount of free volume, or wasted space. Pulling these materials in tension opens up these regions and can give materials a hazy appearance. This haziness is the result of *crazing*, where fibrils are formed within the material in regions of high hydrostatic stress. The material may go from an ordered appearance to a "crazy" pattern of strain and stretch marks.

Plasticity in martensitic materials

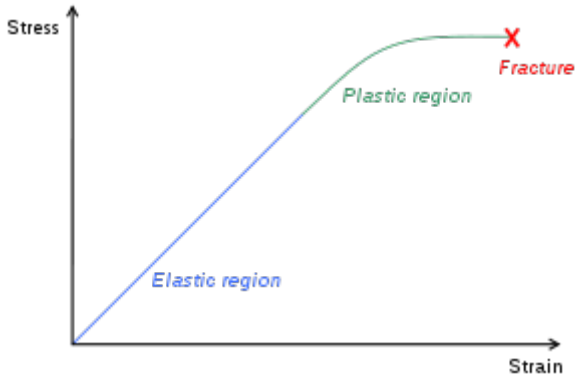
Some materials, especially those prone to Martensitic transformations, deform in ways that are not well described by the classic theories of plasticity and elasticity. One of the best-known examples of this is nitinol, which exhibits pseudoelasticity: deformations which are reversible in the context of mechanical design, but irreversible in terms of thermodynamics.

Plasticity in cellular materials

These materials plastically deform when the bending moment exceeds the fully plastic moment. This applies to open cell foams where the bending moment is exerted on the cell walls. The foams can be made of any material with a plastic yield point which includes rigid polymers and metals. This method of modeling the foam as beams is only valid if the ratio of the density of the foam to the density of the matter is less than 0.3. This is because beams yield axially instead of bending. In closed cell foams, the yield strength is increased if the material is under tension because of the membrane that spans the face of the cells.

Mathematical descriptions of plasticity

Deformation theory



An idealized uniaxial stress-strain curve showing elastic and plastic deformation regimes for the deformation theory of plasticity.

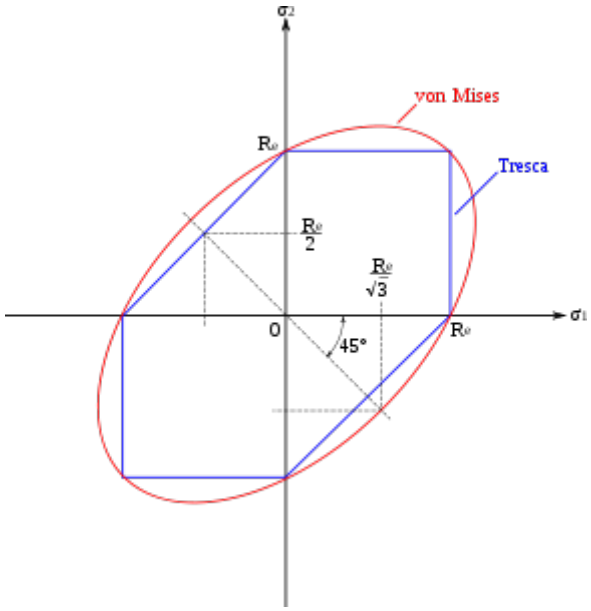
There are several mathematical descriptions of plasticity. One is deformation theory where the stress tensor (of order d in d dimensions) is a function of the strain tensor. Although this description is accurate when a small part of matter is subjected to increasing loading (such as strain loading), this theory cannot account for irreversibility.

Ductile materials can sustain large plastic deformations without fracture. However, even ductile metals will fracture when the strain becomes large enough - this is as a result of work-hardening of the material, which causes it to become brittle. Heat treatment such as annealing can restore the ductility of a worked piece, so that shaping can continue.

Flow plasticity theory

In 1934, Egon Orowan, Michael Polanyi and Geoffrey Ingram Taylor, roughly simultaneously, realized that the plastic deformation of ductile materials could be explained in terms of the theory of dislocations. The more correct mathematical theory of plasticity, flow plasticity theory, uses a set of non-linear, non-integrable equations to describe the set of changes on strain and stress with respect to a previous state and a small increase of deformation.

Yield criteria



Comparison of Tresca criterion to Von Mises criterion.

If the stress exceeds a critical value, as was mentioned above, the material will undergo plastic, or irreversible, deformation. This critical stress can be tensile or compressive. The Tresca and the von Mises criteria are commonly used to determine whether a material has yielded. However, these criteria have proved inadequate for a large range of materials and several other yield criteria are in widespread use.

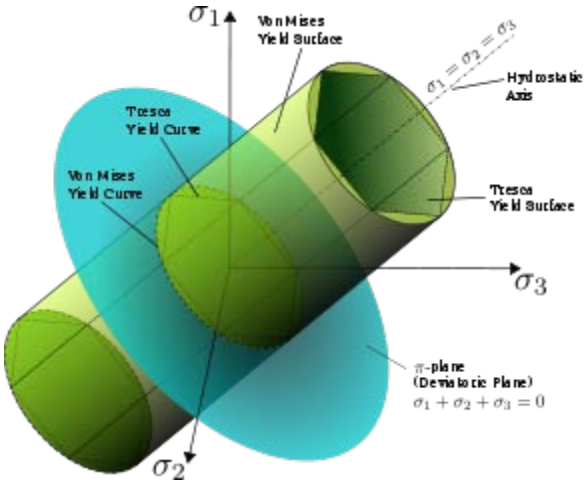
Tresca criterion

This criterion is based on the notion that when a material fails, it does so in shear, which is a relatively good assumption when considering metals. Given the principal stress state, we can use Mohr's circle to solve for the maximum shear stresses our material will experience and conclude that the material will fail if:

$$\sigma_1 - \sigma_3 \geq \sigma_0$$

Where σ_1 is the maximum normal stress, σ_3 is the minimum normal stress, and σ_0 is the stress under which the material fails in uniaxial loading. A yield surface may be constructed, which provides a visual representation of this concept. Inside of the yield surface, deformation is elastic. On the surface, deformation is plastic. It is impossible for a material to have stress states outside its yield surface.

Huber-von Mises criterion



The von Mises yield surfaces in principal stress coordinates circumscribes a cylinder around the hydrostatic axis. Also shown is Tresca's hexagonal yield surface.

This criterion is based on the Tresca criterion but takes into account the assumption that hydrostatic stresses do not contribute to material failure. M.T. Huber was first (1904, Lwów) who proposed the criterion of shear energy. Von Mises solves for an effective stress under uniaxial loading, subtracting out hydrostatic stresses, and claims that all effective stresses greater than that which causes material failure in uniaxial loading will result in plastic deformation.

$$\sigma^2_{\text{effective}} = \tfrac{1}{2}\left[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{11}-\sigma_{33})^2\right] + 3\left(\sigma^2_{12} + \sigma^2_{13} + \sigma^2_{23}\right)$$

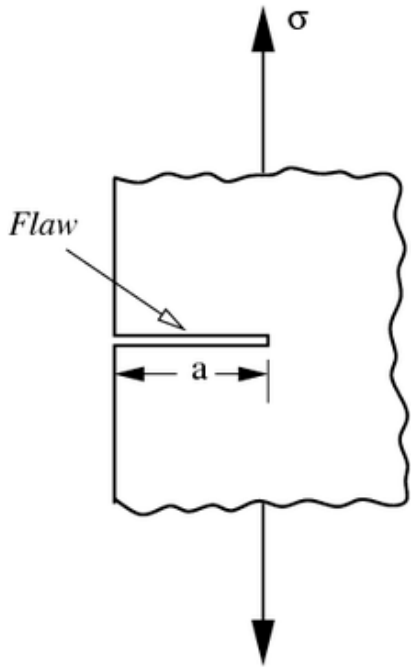
Again, a visual representation of the yield surface may be constructed using the above equation, which takes the shape of an ellipse. Inside the surface, materials undergo elastic deformation. Reaching the surface means the material undergoes plastic deformations. It is physically impossible for a material to go beyond its yield surface.

Chapter 3

Fracture Mechanics

Fracture mechanics is the field of mechanics concerned with the study of the propagation of cracks in materials. It uses methods of analytical solid mechanics to calculate the driving force on a crack and those of experimental solid mechanics to characterize the material's resistance to fracture.

In modern materials science, fracture mechanics is an important tool in improving the mechanical performance of materials and components. It applies the physics of stress and strain, in particular the theories of elasticity and plasticity, to the microscopic crystallographic defects found in real materials in order to predict the macroscopic mechanical failure of bodies. Fractography is widely used with fracture mechanics to understand the causes of failures and also verify the theoretical failure predictions with real life failures.



An edge crack (flaw) of length *a* in a material.

Linear elastic fracture mechanics

Griffith's criterion

Fracture mechanics was developed during World War I by English aeronautical engineer, A. A. Griffith, to explain the failure of brittle materials. Griffith's work was motivated by two contradictory facts:

- The stress needed to fracture bulk glass is around 100 MPa (15,000 psi).
- The theoretical stress needed for breaking atomic bonds is approximately 10,000 MPa (1,500,000 psi).

A theory was needed to reconcile these conflicting observations. Also, experiments on glass fibers that Griffith himself conducted suggested that the fracture stress increases as the fiber diameter decreases. Hence the uniaxial tensile strength, which had been used extensively to predict material failure before Griffith, could not be a specimen-independent material property. Griffith suggested that the low fracture strength observed in experiments, as well as the size-dependence of strength, was due to the presence of microscopic flaws in the bulk material.

To verify the flaw hypothesis, Griffith introduced an artificial flaw in his experimental specimens. The artificial flaw was in the form of a surface crack which was much larger than other flaws in a specimen. The experiments showed that the product of the square root of the flaw length (a) and the stress at fracture (σ_f) was nearly constant, which is expressed by the equation:

$$\sigma_f \sqrt{a} \approx C$$

An explanation of this relation in terms of linear elasticity theory is problematic. Linear elasticity theory predicts that stress (and hence the strain) at the tip of a sharp flaw in a linear elastic material is infinite. To avoid that problem, Griffith developed a thermodynamic approach to explain the relation that he observed.

The growth of a crack requires the creation of two new surfaces and hence an increase in the surface energy. Griffith found an expression for the constant C in terms of the surface energy of the crack by solving the elasticity problem of a finite crack in an elastic plate. Briefly, the approach was:

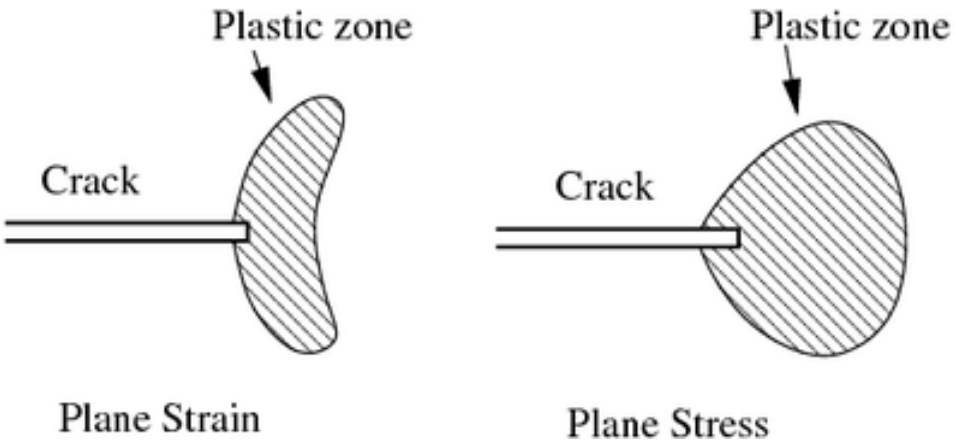
- Compute the potential energy stored in a perfect specimen under an uniaxial tensile load.
- Fix the boundary so that the applied load does no work and then introduce a crack into the specimen. The crack relaxes the stress and hence reduces the elastic energy near the crack faces. On the other hand, the crack increases the total surface energy of the specimen.

- Compute the change in the free energy (surface energy − elastic energy) as a function of the crack length. Failure occurs when the free energy attains a peak value at a critical crack length, beyond which the free energy decreases by increasing the crack length, i.e. by causing fracture. Using this procedure, Griffith found that

$$C = \sqrt{\frac{2E\gamma}{\pi}}$$

where E is the Young's modulus of the material and γ is the surface energy density of the material. Assuming E = 62 GPa and γ = 1 J/m^2 gives excellent agreement of Griffith's predicted fracture stress with experimental results for glass.

Irwin's modification



The plastic zone around crack tip in a ductile material.

Griffith's work was largely ignored by the engineering community until the early 1950s. The reasons for this appear to be (a) in the actual structural materials the level of energy needed to cause fracture is orders of magnitude higher than the corresponding surface energy, and (b) in structural materials there are always some inelastic deformations around the crack front that would make the assumption of linear elastic medium with infinite stresses at the crack tip highly unrealistic. **F. Erdogan (2000)**

Griffith's theory provides excellent agreement with experimental data for brittle materials such as glass. For ductile materials such as steel, though the relation $\sigma_y\sqrt{a} = C$ still holds, the surface energy (γ) predicted by Griffith's theory is usually unrealistically high. A group working under G. R. Irwin at the U.S. Naval Research Laboratory (NRL) during World War II realized that plasticity must play a significant role in the fracture of ductile materials.

In ductile materials (and even in materials that appear to be brittle), a plastic zone develops at the tip of the crack. As the applied load increases, the plastic zone increases in size until the crack grows and the material behind the crack tip unloads. The plastic loading and unloading cycle near the crack tip leads to the dissipation of energy as heat. Hence, a dissipative term has to be added to the energy balance relation devised by Griffith for brittle materials. In physical terms, additional energy is needed for crack growth in ductile materials when compared to brittle materials.

Irwin's strategy was to partition the energy into two parts:

- the stored elastic strain energy which is released as a crack grows. This is the thermodynamic driving force for fracture.
- the dissipated energy which includes plastic dissipation and the surface energy (and any other dissipative forces that may be at work). The dissipated energy provides the thermodynamic resistance to fracture. Then the total energy dissipated is

  $G = 2\gamma + G_p$

where γ is the surface energy and G_p is the plastic dissipation (and dissipation from other sources) per unit area of crack growth.

The modified version of Griffith's energy criterion can then be written as

$$\sigma_f \sqrt{a} = \sqrt{\frac{E\ G}{\pi}}.$$

For brittle materials such as glass, the surface energy term dominates and $G \approx 2\gamma = 2\ J/m^2$. For ductile materials such as steel, the plastic dissipation term dominates and $G \approx G_p = 1000\ J/m^2$. For polymers close to the glass transition temperature, we have intermediate values of $G \approx 2 - 1000\ J/m^2$.

Stress intensity factor

Another significant achievement of Irwin and his colleagues was to find a method of calculating the amount of energy available for fracture in terms of the asymptotic stress and displacement fields around a crack front in a linear elastic solid. This asymptotic expression for the stress field around a crack tip is

$$\sigma_{ij} \approx \left(\frac{K}{\sqrt{2\pi r}} \right) \ f_{ij}(\theta)$$

where σ_{ij} are the Cauchy stresses, r is the distance from the crack tip, θ is the angle with respect to the plane of the crack, and f_{ij} are functions that are independent of the crack geometry and loading conditions. Irwin called the quantity K the *stress intensity factor*. Since the quantity f_{ij} is dimensionless, the stress intensity factor can be expressed in units of $\mathrm{MPa}\text{-}\sqrt{\mathrm{m}}$.

When a rigid line inclusion is considered, a similar asymptotic expression for the stress fields is obtained.

Strain energy release

Irwin was the first to observe that if the size of the plastic zone around a crack is small compared to the size of the crack, the energy required to grow the crack will not be critically dependent on the state of stress at the crack tip. In other words, a purely elastic solution may be used to calculate the amount of energy available for fracture.

The energy release rate for crack growth or *strain energy release rate* may then be calculated as the change in elastic strain energy per unit area of crack growth, i.e.,

$$G := -\left[\frac{\partial U}{\partial a}\right]_P = -\left[\frac{\partial U}{\partial a}\right]_u$$

where U is the elastic energy of the system and a is the crack length. Either the load P or the displacement u can be kept fixed while evaluating the above expressions.

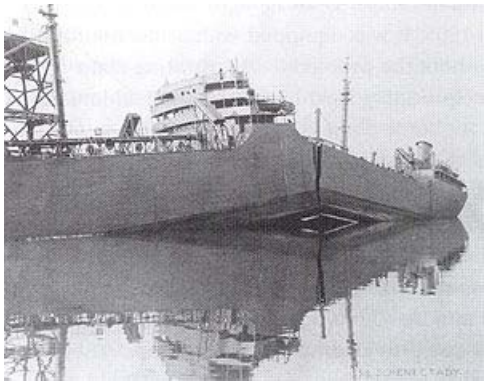
Irwin showed that for a mode I crack (opening mode) the strain energy release rate and the stress intensity factor are related by:

$$G = G_I = \begin{cases} \dfrac{K_I^2}{E} & \text{plane stress} \\ \dfrac{(1-\nu^2)K_I^2}{E} & \text{plane strain} \end{cases}$$

where E is the Young's modulus, v is Poisson's ratio, and K_I is the stress intensity factor in mode I. Irwin also showed that the strain energy release rate of a planar crack in a linear elastic body can be expressed in terms of the mode I, mode II (sliding mode), and mode III (tearing mode) stress intensity factors for the most general loading conditions.

Next, Irwin adopted the additional assumption that the size and shape of the energy dissipation zone remains approximately constant during brittle fracture. This assumption suggests that the energy needed to create a unit fracture surface is a constant that depends only on the material. This new material property was given the name *fracture toughness* and designated G_{Ic}. Today, it is the critical stress intensity factor K_{Ic} which is accepted as the defining property in linear elastic fracture mechanics.

Limitations



The S.S. *Schenectady* split apart by brittle fracture while in harbor (1944)

But a problem arose for the NRL researchers because naval materials, e.g., ship-plate steel, are not perfectly elastic but undergo significant plastic deformation at the tip of a crack. One basic assumption in Irwin's linear elastic fracture mechanics is that the size of the plastic zone is small compared to the crack length. However, this assumption is quite restrictive for certain types of failure in structural steels though such steels can be prone to brittle fracture, which has led to a number of catastrophic failures.

Linear-elastic fracture mechanics is of limited practical use for structural steels for another more practical reason. Fracture toughness testing is very expensive and engineers believe that sufficient information for selection of steels can be obtained from the simpler and cheaper Charpy impact test.

Nonlinear elasticity and plasticity

Vertical stabilizer, which separated from American Airlines Flight 587, leading to a fatal crash

Most engineering materials show some nonlinear elastic and inelastic behavior under operating conditions that involve large loads. In such materials the assumptions of linear elastic fracture mechanics may not hold, that is,

- the plastic zone at a crack tip may have a size of the same order of magnitude as the crack size
- the size and shape of the plastic zone may change as the applied load is increased and also as the crack length increases.

Therefore a more general theory of crack growth is needed for elastic-plastic materials that can account for:

- the local conditions for initial crack growth which include the nucleation, growth, and coalescence of voids or decohesion at a crack tip.
- a global energy balance criterion for further crack growth and unstable fracture.

R-curve

An early attempt in the direction of elastic-plastic fracture mechanics was Irwin's **crack extension resistance curve** or **R-curve**. This curve acknowledges the fact that the resistance to fracture increases with growing crack size in elastic-plastic materials. The R-curve is a plot of the total energy dissipation rate as a function of the crack size and can be used to examine the processes of slow stable crack growth and unstable fracture. However, the R-curve was not widely used in applications until the early 1970s. The main reasons appear to be that the R-curve depends on the geometry of the specimen and the crack driving force may be difficult to calculate.

J-integral

In the mid-1960s James R. Rice (then at Brown University) and G. P. Cherepanov independently developed a new toughness measure to describe the case where there is sufficient crack-tip deformation that the part no longer obeys the linear-elastic approximation. Rice's analysis, which assumes non-linear elastic (or monotonic deformation-theory plastic) deformation ahead of the crack tip, is designated the J integral. This analysis is limited to situations where plastic deformation at the crack tip does not extend to the furthest edge of the loaded part. It also demands that the assumed non-linear elastic behavior of the material is a reasonable approximation in shape and magnitude to the real material's load response. The elastic-plastic failure parameter is designated J_{Ic} and is conventionally converted to K_{Ic} using Equation (3.1) of the Appendix to this. Also note that the J integral approach reduces to the Griffith theory for linear-elastic behavior.

Fully plastic failure

If the material is so tough that the yielded region ahead of the crack extends to the far edge of the specimen before fracture, the crack is no longer an effective stress

concentrator. Instead, the presence of the crack merely serves to reduce the load-bearing area. In this regime the failure stress is conventionally assumed to be the average of the yield and ultimate strengths of the material.

Engineering applications

The following information is needed for a fracture mechanics prediction of failure:

- Applied load
- Residual stress
- Size and shape of the part
- Size, shape, location, and orientation of the crack

Usually not all of this information is available and conservative assumptions have to be made.

Occasionally post-mortem fracture-mechanics analyses are carried out. In the absence of an extreme overload, the causes are either insufficient toughness (K_{Ic}) or an excessively large crack that was not detected during routine inspection.

Short summary

Arising from the manufacturing process, interior and surface flaws are found in all metal structures. Not all such flaws are unstable under service conditions. Fracture mechanics is the analysis of flaws to discover those that are safe (that is, do not grow) and those that are liable to propagate as cracks and so cause failure of the flawed structure. Ensuring safe operation of structure despite these inherent flaws is achieved through damage tolerance analysis. Fracture mechanics as a subject for critical study has barely been around for a century and thus is relatively new. There is a high demand for engineers with fracture mechanics expertise—particularly in this day and age where engineering failure is considered 'shocking' amongst the general public.

Appendix: mathematical relations

Griffith's criterion

For the simple case of a thin rectangular plate with a crack perpendicular to the load Griffith's theory becomes:

$$G = \frac{\pi \sigma^2 a}{E} \qquad (1.1)$$

where G is the strain energy release rate, σ is the applied stress, a is half the crack length, and E is the Young's modulus. The strain energy release rate can otherwise be understood as: *the rate at which energy is absorbed by growth of the crack*.

However, we also have that:

$$G_c = \frac{\pi \sigma_f^2 a}{E} \qquad (1.2)$$

If $G \geq G_c$, this is the criterion for which the crack will begin to propagate.

Irwin's modifications

Eventually a modification of Griffith's solids theory emerged from this work; a term called stress intensity replaced strain energy release rate and a term called fracture toughness replaced surface weakness energy. Both of these terms are simply related to the energy terms that Griffith used:

$$K_I = \sigma \sqrt{\pi a} \qquad (2.1)$$

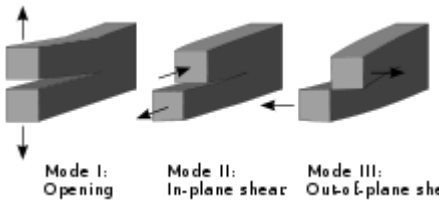
and

$$K_c = \sqrt{E G_c} \text{(for plane stress)} \qquad (2.2)$$

$$K_c = \sqrt{\frac{E G_c}{1 - \nu^2}} \text{(for plane strain)} \qquad (2.3)$$

where K_I is the stress intensity, K_c the fracture toughness, and ν is Poisson's ratio. It is important to recognize the fact that fracture parameter K_c has different values when measured under plane stress and plane strain

Fracture occurs when $K_I \geq K_c$. For the special case of plane strain deformation, K_c becomes K_{Ic} and is considered a material property. The subscript I arises because of the different ways of loading a material to enable a crack to propagate. It refers to so-called "mode I" loading as opposed to mode II or III:



The three fracture modes.

There are three ways of applying a force to enable a crack to propagate:

- **Mode I crack** – Opening mode (a tensile stress normal to the plane of the crack)
- **Mode II crack** – Sliding mode (a shear stress acting parallel to the plane of the crack and perpendicular to the crack front)
- **Mode III crack** – Tearing mode (a shear stress acting parallel to the plane of the crack and parallel to the crack front)

We must note that the expression for K_I in equation 2.1 will be different for geometries other than the center-cracked infinite plate, as discussed in stress intensity. Consequently, it is necessary to introduce a dimensionless correction factor, Y, in order to characterize the geometry. We thus have:

$$K_I = Y\sigma\sqrt{\pi a} \qquad (2.4)$$

where Y is a function of the crack length and width of sheet given by:

$$Y\left(\frac{a}{W}\right) = \sqrt{\sec\left(\frac{\pi a}{W}\right)} \qquad (2.5)$$

for a sheet of finite width W containing a through-thickness crack of length $2a$, or

$$Y\left(\frac{a}{W}\right) = 1.12 - \frac{0.41}{\sqrt{\pi}}\frac{a}{W} + \frac{18.7}{\sqrt{\pi}}\left(\frac{a}{W}\right)^2 - \cdots \qquad (2.6)$$

for a sheet of finite width W containing a through-thickness edge crack of length a

Elasticity and plasticity

Since engineers became accustomed to using K_{Ic} to characterise fracture toughness, a relation has been used to reduce J_{Ic} to it:

$$K_{Ic} = \sqrt{E^* J_{Ic}} \qquad (3.1)$$

where $E^* = E$ for plane stress and $E^* = \frac{E}{1-\nu^2}$ for plane strain

The remainder of the mathematics employed in this approach is interesting, but is probably better summarised in external pages due to its complex nature.

Chapter 4

Bending

Bending of an I-beam

In engineering mechanics, **bending** (also known as **flexure**) characterizes the behavior of a slender structural element subjected to an external load applied perpendicularly to a longitudinal axis of the element. The structural element is assumed to be such that at least one of its dimensions is a small fraction, typically 1/10 or less, of the other two. When the length is considerably longer than the width and the thickness, the element is called a beam. A closet rod sagging under the weight of clothes on clothes hangers is an example of a beam experiencing bending. On the other hand, a shell is a structure of any geometric form where the length and the width are of the same order of magnitude but the thickness of the structure (known as the 'wall') is considerably smaller. A large diameter, but thin-walled, short tube supported at its ends and loaded laterally is an example of a shell experiencing bending.

In the absence of a qualifier, the term *bending* is ambiguous because bending can occur locally in all objects. To make the usage of the term more precise, engineers refer to the *bending of rods*, the *bending of beams*, the *bending of plates*, the *bending of shells* and so on.

Quasistatic bending of beams

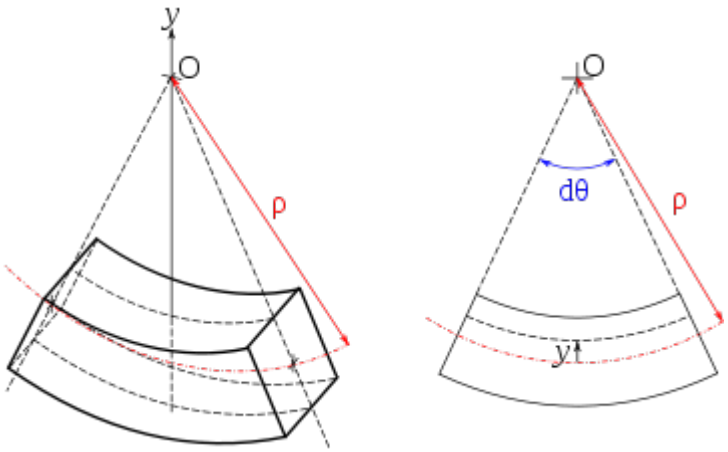
A beam deforms and stresses develop inside it when a transverse load is applied on it. In the quasistatic case, the amount of bending deflection and the stresses that develop are

assumed not to change over time. In a horizontal beam supported at the ends and loaded downwards in the middle, the material at the over-side of the beam is compressed while the material at the underside is stretched. There are two forms of internal stresses caused by lateral loads:

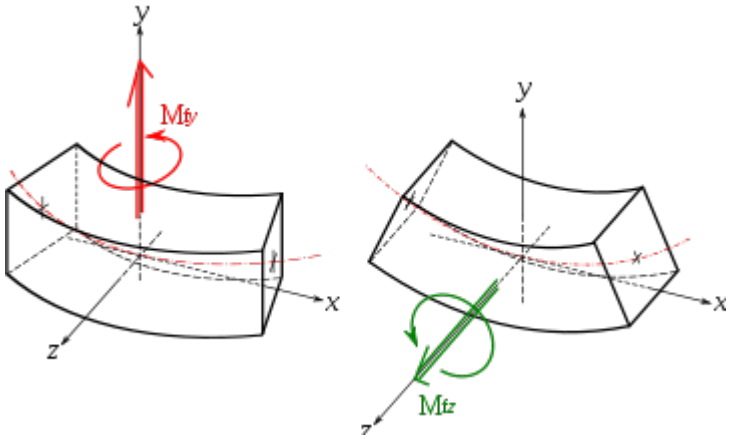
- Shear stress parallel to the lateral loading plus complementary shear stress on planes perpendicular to the load direction;
- Direct compressive stress in the upper region of the beam, and direct tensile stress in the lower region of the beam.

These last two forces form a couple or moment as they are equal in magnitude and opposite in direction. This bending moment resists the sagging deformation characteristic of a beam experiencing bending. The stress distribution in a beam can be predicted quite accurately even when some simplifying assumptions are used.

Euler-Bernoulli bending theory



Element of a bent beam: the fibers form concentric arcs, the top fibers are compressed and bottom fibers stretched.



Bending moments in a beam

In the Euler-Bernoulli theory of slender beams, a major assumption is that 'plane sections remain plane'. In other words, any deformation due to shear across the section is not accounted for (no shear deformation). Also, this linear distribution is only applicable if the maximum stress is less than the yield stress of the material. For stresses that exceed yield, refer to article plastic bending. At yield, the maximum stress experienced in the section (at the furthest points from the neutral axis of the beam) is defined as the flexural strength.

The Euler-Bernoulli equation for the quasistatic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load $q(x)$ is

$$EI \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x)$$

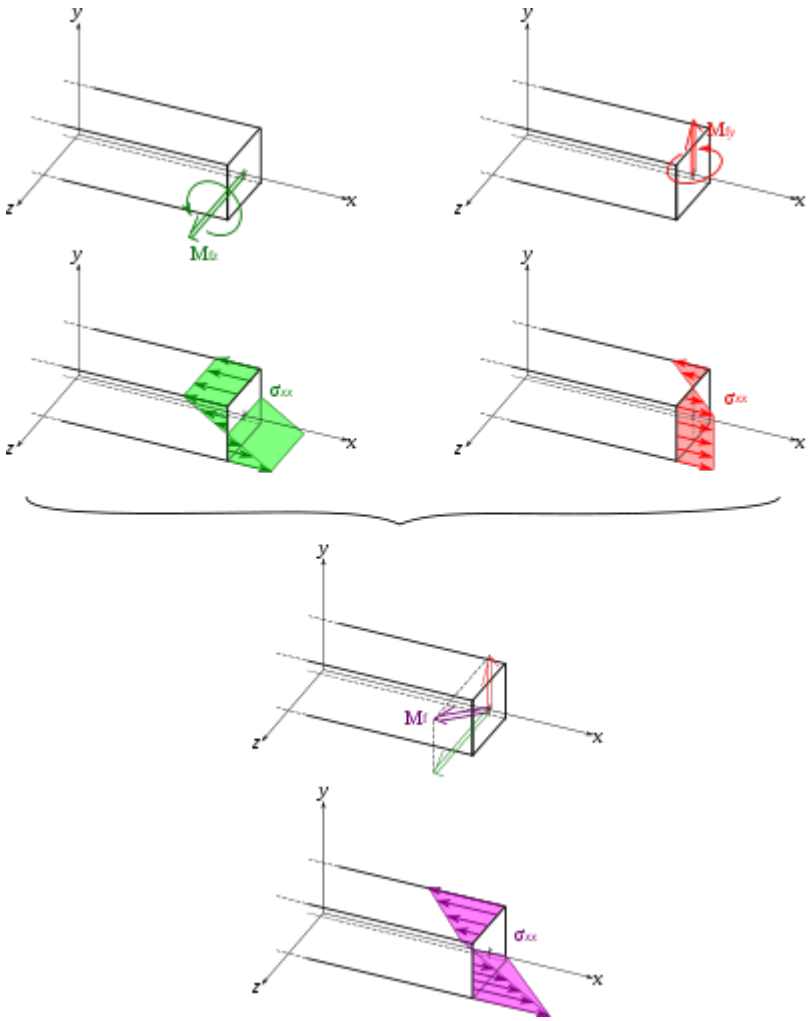
where E is the Young's modulus, I is the area moment of inertia of the cross-section, and $w(x)$ is the deflection of the neutral axis of the beam.

After a solution for the displacement of the beam has been obtained, the bending moment (M) and shear force (Q) in the beam can be calculated using the relations

$$M(x) = -EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}; \quad Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x}$$

Simple beam bending is often analyzed with the Euler-Bernoulli beam equation. The conditions for using simple bending theory are:

1. The beam is subject to pure bending. This means that the shear force is zero, and that no torsional or axial loads are present.
2. The material is isotropic and homogeneous.
3. The material obeys Hooke's law (it is linearly elastic and will not deform plastically).
4. The beam is initially straight with a cross section that is constant throughout the beam length.
5. The beam has an axis of symmetry in the plane of bending.
6. The proportions of the beam are such that it would fail by bending rather than by crushing, wrinkling or sideways buckling.
7. Cross-sections of the beam remain plane during bending.



Deflection of a beam deflected symmetrically and principle of superposition

Compressive and tensile forces develop in the direction of the beam axis under bending loads. These forces induce stresses on the beam. The maximum compressive stress is found at the uppermost edge of the beam while the maximum tensile stress is located at the lower edge of the beam. Since the stresses between these two opposing maxima vary linearly, there therefore exists a point on the linear path between them where there is no bending stress. The locus of these points is the neutral axis. Because of this area with no stress and the adjacent areas with low stress, using uniform cross section beams in bending is not a particularly efficient means of supporting a load as it does not use the full capacity of the beam until it is on the brink of collapse. Wide-flange beams (I-beams) and truss girders effectively address this inefficiency as they minimize the amount of material in this under-stressed region.

The classic formula for determining the bending stress in a beam under simple bending is:

$$\sigma = \frac{My}{I_x}$$

where

- σ is the bending stress
- M - the moment about the neutral axis
- y - the perpendicular distance to the neutral axis
- I_x - the second moment of area about the neutral axis x

Extensions of Euler-Bernoulli beam bending theory

Plastic bending

The equation $\sigma = \frac{My}{I_x}$ is valid only when the stress at the extreme fiber (i.e. the portion of the beam farthest from the neutral axis) is below the yield stress of the material from which it is constructed. At higher loadings the stress distribution becomes non-linear, and ductile materials will eventually enter a *plastic hinge* state where the magnitude of the stress is equal to the yield stress everywhere in the beam, with a discontinuity at the neutral axis where the stress changes from tensile to compressive. This plastic hinge state is typically used as a limit state in the design of steel structures.

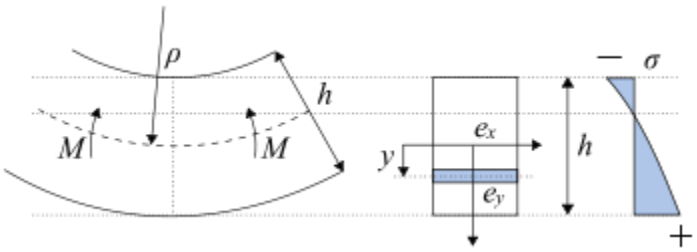
Complex or asymmetrical bending

The equation above is only valid if the cross-section is symmetrical. For homogeneous beams with asymmetrical sections, the axial stress in the beam is given by

$$\sigma_x(y,z) = -\frac{(M_z\ I_y + M_y\ I_{yz})}{I_y\ I_z - I_{yz}^2}y + \frac{(M_y\ I_z + M_z\ I_{yz})}{I_y\ I_z - I_{yz}^2}z$$

where y,z are the coordinates of a point on the cross section at which the stress is to be determined as shown to the right, M_y and M_z are the bending moments about the y and z centroid axes, I_y and I_z are the second moments of area (distinct from moments of inertia) about the y and z axes, and I_{yz} is the product of moments of area. Using this equation it is possible to calculate the bending stress at any point on the beam cross section regardless of moment orientation or cross-sectional shape. Note that M_y,M_z,I_y,I_z,I_{yz} do not change from one point to another on the cross section.

Large bending deformation



For large deformations of the body, the stress in the cross-section is calculated using an extended version of this formula. First the following assumptions must be made:

1. Assumption of flat sections - before and after deformation the considered section of body remains flat (i.e. is not swirled).
2. Shear and normal stresses in this section that are perpendicular to the normal vector of cross section have no influence on normal stresses that are parallel to this section.

Large bending considerations should be implemented when the bending radius ρ is smaller than ten section heights h:

$$\rho < 10h$$

With those assumptions the stress in large bending is calculated as:

$$\sigma = \frac{F}{A} + \frac{M}{\rho A} + \frac{M}{I_x'} y \frac{\rho}{\rho + y}$$

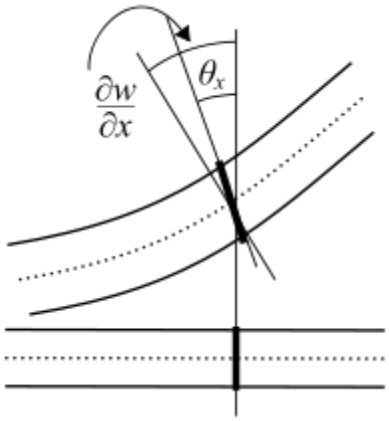
where

F is the normal force
A is the section area
M is the bending moment
ρ is the local bending radius (the radius of bending at the current section)
I_x' is the area moment of inertia along the **x** axis, at the *y* place
y is the position along **y** axis on the section area in which the stress σ is calculated

When bending radius ρ approaches infinity and *y* is near zero, the original formula is back:

$$\sigma = \frac{F}{A} \pm \frac{My}{I}.$$

Timoshenko bending theory



Deformation of a Timoshenko beam. The normal rotates by an amount θ which is not equal to *dw* / *dx*.

In 1921, Timoshenko improved upon the Euler-Bernoulli theory of beams by adding the effect of shear into the beam equation. The kinematic assumptions of the Timoshenko theory are

- normals to the axis of the beam remain straight after deformation
- there is no change in beam thickness after deformation

However, normals to the axis are not required to remain perpendicular to the axis after deformation.

The equation for the quasistatic bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\,\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - \frac{EI}{kAG}\,\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}$$

where *I* is the area moment of inertia of the cross-section, *A* is the cross-sectional area, *G* is the shear modulus, and *k* is a **shear correction factor**. For materials with Poisson's ratios (ν) close to 0.3, the shear correction factor for a rectangular cross-section is approximately

$$k = \frac{5 + 5\nu}{6 + 5\nu}$$

The rotation ($\varphi(x)$) of the normal is described by the equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{q(x)}{kAG}$$

The bending moment (M) and the shear force (Q) are given by

$$M(x) = -EI\ \frac{\mathrm{d}\varphi}{\mathrm{d}x}\ ;\quad Q(x) = kAG\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \varphi\right) = -EI\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{\mathrm{d}M}{\mathrm{d}x}$$

Dynamic bending of beams

The dynamic bending of beams, also known as flexural vibrations of beams, was first investigated by Daniel Bernoulli in the late 18th century. Bernoulli's equation of motion of a vibrating beam tended to overestimate the natural frequencies of beams and was improved marginally by Rayleigh in 1877 by the addition of a mid-plane rotation. In 1921 Stephen Timoshenko improved the theory further by incorporating the effect of shear on the dynamic response of bending beams. This allowed the theory to be used for problems involving high frequencies of vibration where the dynamic Euler-Bernoulli theory is inadequate. The Euler-Bernoulli and Timoshenko theories for the dynamic bending of beams continue to be used widely by engineers.

Euler-Bernoulli theory

The Euler-Bernoulli equation for the dynamic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load $q(x,t)$ is

$$EI\ \frac{\partial^4 w}{\partial x^4} + m\ \frac{\partial^2 w}{\partial t^2} = q(x,t)$$

where E is the Young's modulus, I is the area moment of inertia of the cross-section, $w(x,t)$ is the deflection of the neutral axis of the beam, and m is mass per unit length of the beam.

Free vibrations

For the situation where there is no transverse load on the beam, the bending equation takes the form

$$EI\ \frac{\partial^4 w}{\partial x^4} + m\ \frac{\partial^2 w}{\partial t^2} = 0$$

Free, harmonic vibrations of the beam can then be expressed as

$$w(x,t) = \mathrm{Re}[\hat{w}(x)\, e^{-i\omega t}] \quad \Longrightarrow \quad \frac{\partial^2 w}{\partial t^2} = -\omega^2\, w(x,t)$$

and the bending equation can be written as

$$EI\, \frac{\mathrm{d}^4 \hat{w}}{\mathrm{d}x^4} - m\omega^2 \hat{w} = 0$$

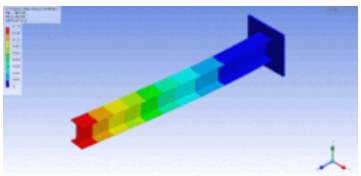
The general solution of the above equation is

$$\hat{w} = A_1 \cosh(\beta x) + A_2 \sinh(\beta x) + A_3 \cos(\beta x) + A_4 \sin(\beta x)$$

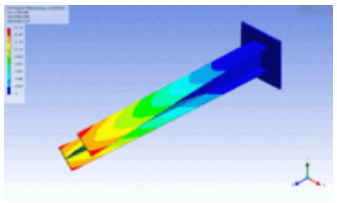
where A_1,A_2,A_3,A_4 are constants and

$$\beta := \left(\frac{m}{EI}\, \omega^2 \right)^{1/4}$$

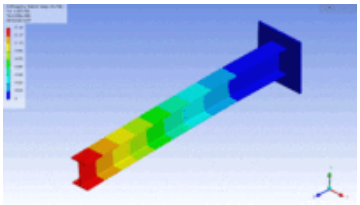
The mode shapes of a cantilevered I-beam



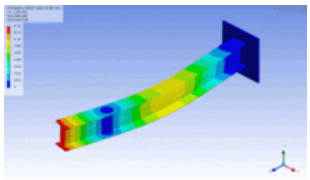
1st lateral bending



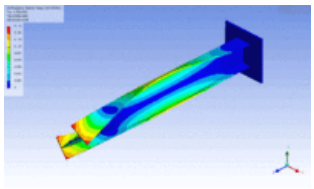
1st torsional



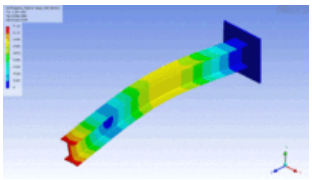
1st vertical bending



2nd lateral bending



2nd torsional



2nd vertical bending

Timoshenko-Rayleigh theory

In 1877, Rayleigh proposed an improvement to the dynamic Euler-Bernoulli beam theory by including the effect of rotational inertia of the cross-section of the beam. Timoshenko improved upon that theory in 1922 by adding the effect of shear into the beam equation. Shear deformations of the normal to the mid-surface of the beam are allowed in the Timoshenko-Rayleigh theory.

The equation for the bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\,\frac{\partial^4 w}{\partial x^4}+m\,\frac{\partial^2 w}{\partial t^2}-\left(J+\frac{EIm}{kAG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2}+\frac{Jm}{kAG}\frac{\partial^4 w}{\partial t^4}=q(x,t)+\frac{J}{kAG}\frac{\partial^2 q}{\partial t^2}-\frac{EI}{kAG}\frac{\partial^2 q}{\partial x^2}$$

where $J=\frac{mI}{A}$ is the polar moment of inertia of the cross-section, $m = \rho A$ is the mass per unit length of the beam, ρ is the density of the beam, A is the cross-sectional area, G is the shear modulus, and k is a **shear correction factor**. For materials with Poisson's ratios (ν) close to 0.3, the shear correction factor are approximately

$$\begin{aligned} k &= \tfrac{5+5\nu}{6+5\nu} \quad \text{rectangular cross-section} \\ &= \tfrac{6+12\nu+6\nu^2}{7+12\nu+4\nu^2} \quad \text{circular cross-section} \end{aligned}$$

Free vibrations

For free, harmonic vibrations the Timoshenko-Rayleigh equations take the form

$$EI\,\frac{\mathrm{d}^4\hat{w}}{\mathrm{d}x^4}+m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}x^2}+m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)\hat{w}=0$$

This equation can be solved by noting that all the derivatives of w must have the same form to cancel out and hence as solution of the form e^{kx} may be expected. This observation leads to the characteristic equation

$$\alpha\,k^4+\beta\,k^2+\gamma=0\,;\;\;\alpha:=EI\,,\;\;\beta:=m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\,,\;\;\gamma:=m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)$$

The solutions of this quartic equation are

$$k_1=+\sqrt{z_+}\,,\quad k_2=-\sqrt{z_+}\,,\quad k_3=+\sqrt{z_-}\,,\quad k_4=-\sqrt{z_-}$$

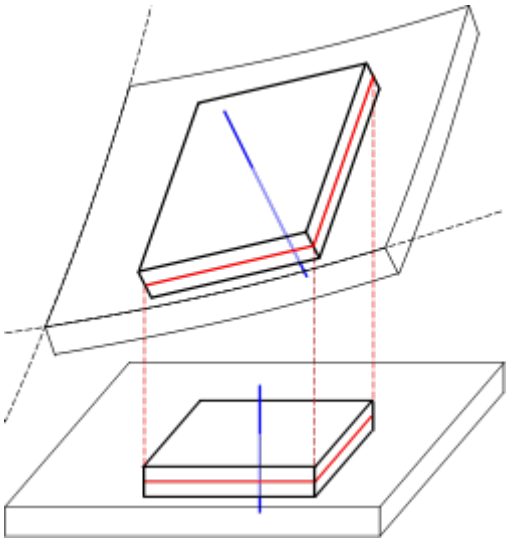
where

$$z_+:=\frac{-\beta+\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}\,,\quad z_-:=\frac{-\beta-\sqrt{\beta^2-4\alpha\gamma}}{2\alpha}$$

The general solution of the Timoshenko-Rayleigh beam equation for free vibrations can then be written as

$$\hat{w}=A_1\,e^{k_1 x}+A_2\,e^{-k_1 x}+A_3\,e^{k_3 x}+A_4\,e^{-k_3 x}$$

Quasistatic bending of plates



Deformation of a thin plate highlighting the displacement, the mid-surface (red) and the normal to the mid-surface (blue)

The defining feature of beams is that one of the dimensions is much *larger* than the other two. A structure is called a plate when it is flat and one of its dimensions is much *smaller* than the other two. There are several theories that attempt to describe the deformation and stress in a plate under applied loads two of which have been used widely. These are

- the Kirchhoff-Love theory of plates (also called classical plate theory)
- the Mindlin-Reissner plate theory (also called the first-order shear theory of plates)

Kirchhoff-Love theory of plates

The assumptions of Kirchhoff-Love theory are

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

These assumptions imply that

$$u_\alpha(\mathbf{x}) = -x_3 \ \frac{\partial w^0}{\partial x_\alpha} = -x_3 \ w^0_{,\alpha} \ ; \quad \alpha = 1,2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

where $\mathbf{u}$ is the displacement of a point in the plate and w^0 is the displacement of the mid-surface.

The strain-displacement relations are

$$\varepsilon_{\alpha\beta} = -x_3 \; w^0_{,\alpha\beta}$$
$$\varepsilon_{\alpha 3} = 0$$
$$\varepsilon_{33} = 0$$

The equilibrium equations are

$$M_{\alpha\beta,\alpha\beta} + q(x) = 0 \; ; \quad M_{\alpha\beta} := \int_{-h}^{h} x_3 \; \sigma_{\alpha\beta} \; dx_3$$

where $q(x)$ is an applied load normal to the surface of the plate.

In terms of displacements, the equilibrium equations for an isotropic, linear elastic plate in the absence of external load can be written as

$$w^0_{,1111} + 2 \; w^0_{,1212} + w^0_{,2222} = 0$$

In direct tensor notation,

$$\nabla^2 \nabla^2 w = 0$$

Mindlin-Reissner theory of plates

The special assumption of this theory is that normals to the mid-surface remain straight and inextensible but not necessarily normal to the mid-surface after deformation. The displacements of the plate are given by

$$u_\alpha(\mathbf{x}) = -x_3 \; \varphi_\alpha \; ; \quad \alpha = 1, 2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

where φ_α are the rotations of the normal.

The strain-displacement relations that result from these assumptions are

$$\varepsilon_{\alpha\beta} = -x_3 \; \varphi_{\alpha,\beta}$$
$$\varepsilon_{\alpha 3} = \frac{1}{2} \kappa \left(w^0_{,\alpha} - \varphi_\alpha \right)$$
$$\varepsilon_{33} = 0$$

where κ is a shear correction factor.

The equilibrium equations are

$$M_{\alpha\beta,\beta} - Q_\alpha = 0$$
$$Q_{\alpha,\alpha} + q = 0$$

where

$$Q_\alpha := \kappa \int_{-h}^{h} \sigma_{\alpha 3}\, dx_3$$

Dynamic bending of plates

Dynamics of thin Kirchhoff plates

The dynamic theory of plates determines the propagation of waves in the plates, and the study of standing waves and vibration modes. The equations that govern the dynamic bending of Kirchhoff plates are

$$M_{\alpha\beta,\alpha\beta} - q(x,t) = J_1\, \ddot{w}^0 - J_3\, \ddot{w}^0_{,\alpha\alpha}$$

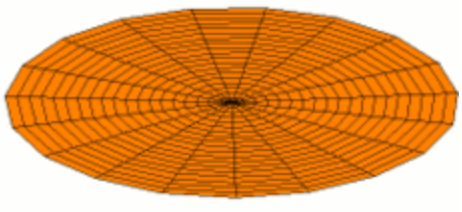
where, for a plate with density ρ = ρ(*x*),

$$J_1 := \int_{-h}^{h} \rho\, dx_3\ ; \quad J_3 := \int_{-h}^{h} x_3^2\, \rho\, dx_3$$

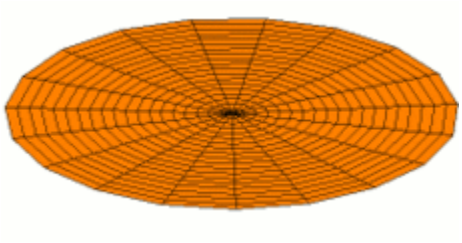
and

$$\ddot{w}^0 = \frac{\partial^2 w^0}{\partial t^2}\ ; \quad \ddot{w}^0_{,\alpha\beta} = \frac{\partial^2 \ddot{w}^0}{\partial x_\alpha \partial x_\beta}$$

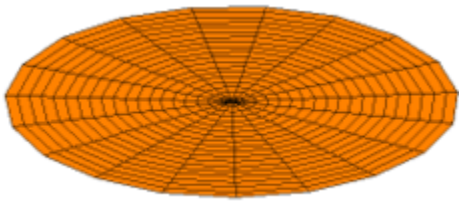
The figures below show some vibrational modes of a circular plate.



mode *k* = 0, *p* = 1



mode $k = 0$, $p = 2$



mode $k = 1$, $p = 2$

Chapter 5

Failure Theory (Material)

Failure theory is the science of predicting the conditions under which solid materials lose their strength under the action of external loads. The failure of a material is usually classified into brittle failure (fracture) or ductile failure (yield). Depending on the conditions (such as temperature, state of stress, loading rate) most materials can fail in a brittle or ductile manner or both. However, for most practical situations, a material may be classified as either brittle or ductile. Though failure theory has been in development for over 200 years, its level of acceptability is yet to reach that of continuum mechanics.

In mathematical terms, failure theory is expressed in the form of various failure criteria which are valid for specific materials. Failure criteria are functions in stress or strain space which separate "failed" states from "unfailed" states. A precise physical definition of a "failed" state is not easily quantified and several working definitions are in use in the engineering community. Quite often, phenomenological failure criteria of the same form are used to predict brittle failure and ductile yield.

Material failure

In materials science, **material failure** is the loss of load carrying capacity of a material unit. This definition *per se* introduces the fact that material failure can be examined in different scales, from microscopic, to macroscopic. In structural problems, where the structural response may be beyond the initiation of nonlinear material behaviour, material failure is of profound importance for the determination of the integrity of the structure. On the other hand, due to the lack of globally accepted fracture criteria, the determination of the structure's damage, due to material failure, is still under intensive research.

Types of material failure

Material failure can be distinguished in two broader categories depending on the scale in which the material is examined:

Microscopic failure

Microscopic material failure is defined in terms of crack propagation and initiation. Such methodologies are useful for gaining insight in the cracking of specimens and simple structures under well defined global load distributions. Microscopic failure considers the initiation and propagation of a crack. Failure criteria in this case are related to microscopic fracture. Some of the most popular failure models in this area are the micromechanical failure models, which combine the advantages of continuum mechanics and classical fracture mechanics. Such models are based on the concept that during plastic deformation, microvoids nucleate and grow until a local plastic neck or fracture of the intervoid matrix occurs, which causes the coalescence of neighbouring voids. Such a model, proposed by Gurson and extended by Tvergaard and Needleman, is known as GTN. Another approach, proposed by Rousselier, is based on continuum damage mechanics (CDM) and thermodynamics. Both models form a modification of the von Mises yield potential by introducing a scalar damage quantity, which represents the void volume fraction of cavities, the porosity *f*.

Macroscopic failure

Macroscopic material failure is defined in terms of load carrying capacity or energy storage capacity, equivalently. Li presents a classification of macroscopic failure criteria in four categories:

- Stress or strain failure
- Energy type failure (S-criterion, T-criterion)
- Damage failure
- Empirical failure.

Five general levels are considered, at which the meaning of deformation and failure is interpreted differently: the structural element scale, the macroscopic scale where macroscopic stress and strain are defined, the mesoscale which is represented by a typical void, the microscale and the atomic scale. The material behaviour at one level is considered as a collective of its behaviour at a sublevel. An efficient deformation and failure model should be consistent at every level.

Brittle material failure criteria

Failure of brittle materials can be determined using several approaches:

- Phenomenological failure criteria
- Linear elastic fracture mechanics
- elastic-plastic fracture mechanics
- Energy-based methods
- Cohesive zone methods

Phenomenological failure criteria

The failure criteria that were developed for brittle solids were the maximum stress/strain criteria. The **maximum stress criterion** assumes that a material fails when the maximum principal stress σ_1 in a material element exceeds the uniaxial tensile strength of the material. Alternatively, the material will fail if the minimum principal stress σ_3 is less than the uniaxial compressive strength of the material. If the uniaxial tensile strength of the material is σ_t and the uniaxial compressive strength is σ_c, then the safe region for the material is assumed to be

$$\sigma_c < \sigma_3 < \sigma_1 < \sigma_t$$

Note that the convention that tension is positive has been used in the above expression.

The **maximum strain criterion** has a similar form except that the principal strains are compared with experimentall determined uniaxial strains at failure, i.e.,

$$\varepsilon_c < \varepsilon_3 < \varepsilon_1 < \varepsilon_t$$

The maximum principal stress and strain criteria continue to be widely used in spite of severe shortcomings.

Numerous other phenomenological failure criteria can be found in the engineering literature. The degree of success of these criteria in predicting failure has been limited. For brittle materials, some popular failure criteria are

- criteria based on invariants of the Cauchy stress tensor
- the Tresca or maximum shear stress failure criterion
- the von Mises or maximum elastic distortional energy criterion
- the Mohr-Coulomb failure criterion for cohesive-frictional solids
- the Drucker-Prager failure criterion for pressure-dependent solids
- the Bresler-Pister failure criterion for concrete
- the Willam-Warnke failure criterion for concrete
- the Hankinson criterion, an empirical failure criterion that is used for orthotropic materials such as wood.
- the Hill yield criteria for anisotropic solids
- the Tsai-Wu failure criterion for anisotropic composites
- the Johnson–Holmquist damage model for high-rate deformations of isotropic solids
- the Hoek-Brown failure criterion for rock masses

Linear elastic fracture mechanics

The approach taken in linear elastic fracture mechanics is to estimate the amount of energy needed to grow a preexisting crack in a brittle material. The earliest fracture mechanics approach for unstable crack growth is Griffiths' theory. When applied to the

mode I opening of a crack, Griffiths' theory predicts that the critical stress (σ) needed to propagate the crack is given by

$$\sigma = \sqrt{\frac{2E\gamma}{\pi\ a}}$$

where *E* is the Young's modulus of the material, γ is the surface energy per unit area of the crack, and 2*a* is the crack length. The quantity $\sigma\sqrt{\pi a}$ is postulated as a material parameter called the '*fracture toughness*. The mode I fracture toughness is defined as

$$K_{Ic} = \sigma_c\sqrt{\pi a}$$

and is determined experimentally. Similar quantities K_{IIc} and K_{IIIc} can be determined for mode II and model III loading conditions.

The state of stress around cracks of various shapes can be expressed in terms of their stress intensity factors. Linear elastic fracture mechanics predicts that a crack will extend when the stress intensity factor at the crack tip is greater than the fracture toughness of the material. Therefore the critical applied stress can also be determined once the stress intensity factor at a crack tip is known.

Energy-based methods

The linear elastic fracture mechanics method is difficult to apply for anisotropic materials (such as composites) or for situations where the loading or the geometry are complex. The **strain energy release rate** approach has proved quite useful for such situations. The strain energy release rate for a mode I crack which runs through the thickness of a plate is defined as

$$G_I := \frac{P}{2t}\frac{du}{da}$$

where *P* is the applied load, *t* is the thickness of the plate, *u* is the displacement at the point of application of the load due to crack growth, and 2*a* is the length of the crack. The crack is expected to propagate when the strain energy release rate exceeds a critical value G_{Ic} - called the **critical strain energy release rate**.

The fracture toughness and the critical stain energy release rate are related by

$$G_{Ic} = \frac{1}{E}K_{Ic}^2$$

where *E* is the Young's modulus. If an initial crack size is known, then a critical stress can be determined using the strain energy release rate criterion.

Ductile material failure criteria

Criteria used to predict the failure of ductile materials are usually called yield criteria. Commonly used failure criteria for ductile materials are:

- the Tresca or maximum shear stress criterion.
- the von Mises yield criterion or distortional strain energy density criterion.
- the Gurson yield criterion for pressure-dependent metals.
- the Hosford yield criterion for metals.
- the Hill yield criteria.
- various criteria based on the invariants of the Cauchy stress tensor.

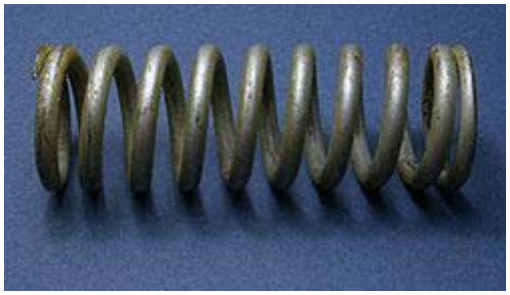
The yield surface of a ductile material usually changes as the material experiences increased deformation. Models for the evolution of the yield surface with increasing strain, temperature, and strain rate are used in conjunction with the above failure criteria for isotropic hardening, kinematic hardening, and viscoplasticity. Some such models are:

- the Johnson-Cook model
- the Steinberg-Guinan model
- the Zerilli-Armstrong model
- the Mechanical threshold stress model
- the Preston-Tonks-Wallace model

There is another important aspect to ductile materials - the prediction of the ultimate failure strength of a ductile material. Several models for predicting the ultimate strength have been used by the engineering community with varying levels of success. For metals, such failure criteria are usually expressed in terms of a combination of porosity and strain to failure or in terms of a damage parameter.

Chapter 6

Hooke's Law



Hooke's law models the properties of springs for small changes in length

In mechanics, and physics, **Hooke's law** of elasticity is an approximation that states that the extension of a spring is in direct proportion with the load applied to it. Many materials obey this law as long as the load does not exceed the material's elastic limit. Materials for which Hooke's law is a useful approximation are known as linear-elastic or "Hookean" materials. Hooke's law in simple terms says that strain is directly proportional to stress.

Mathematically, Hooke's law states that

$$\mathbf{F} = -k\mathbf{x},$$

where

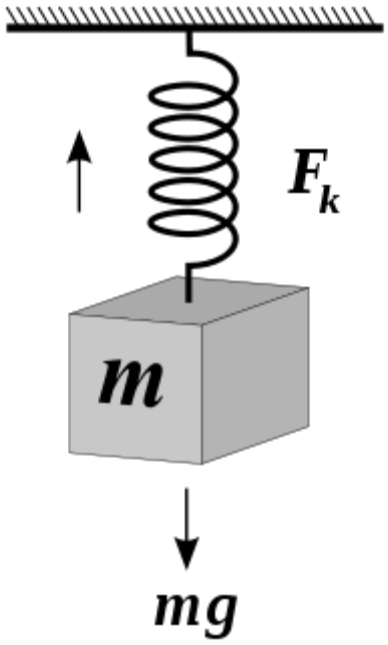
> ***x*** is the displacement of the spring's end from its equilibrium position (a distance, in SI units: meters);
> ***F*** is the restoring force exerted by the spring on that end (in SI units: N or $kg \cdot m \cdot s^{-2}$); and
> *k* is a constant called the *rate* or *spring constant* (in SI units: $N \cdot m^{-1}$ or $kg \cdot s^{-2}$).

When this holds, the behavior is said to be *linear*. If shown on a graph, the line should show a direct variation. There is a negative sign on the right hand side of the equation

because the restoring force always acts in the opposite direction of the displacement (for example, when a spring is stretched to the left, it pulls back to the right).

Hooke's law is named after the 17th century British physicist Robert Hooke. He first stated this law in 1660 as a Latin anagram, whose solution he published in 1678 as *Ut tensio, sic vis*, meaning, "As the extension, so the force".

General application to elastic materials



Hooke's law describes how far the spring will stretch under a specific force

Objects that quickly regain their original shape after being deformed by a force, with the molecules or atoms of their material returning to the initial state of stable equilibrium, often obey Hooke's law.

We may view a rod of any elastic material as a linear spring. The rod has length L and cross-sectional area A. Its extension (strain) is linearly proportional to its tensile stress σ, by a constant factor, the inverse of its modulus of elasticity, E, hence,

$$\sigma = E\varepsilon$$

or

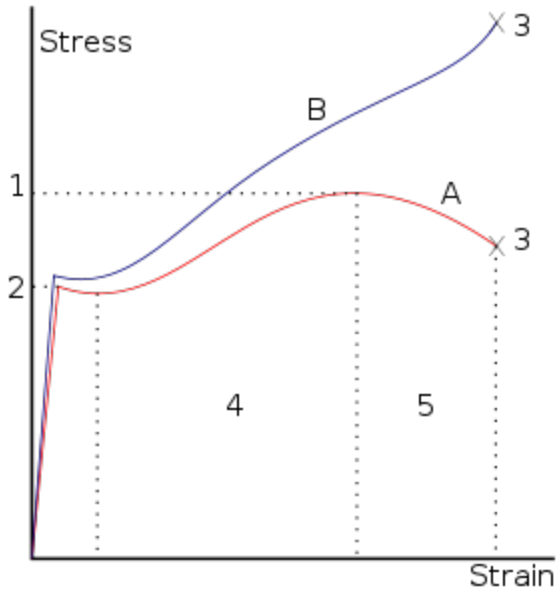
$$\Delta L = \frac{F}{EA}L = \frac{\sigma}{E}L.$$

Hooke's law only holds for some materials under certain loading conditions. Steel exhibits linear-elastic behavior in most engineering applications; Hooke's law is valid for it throughout its **elastic range** (i.e., for stresses below the yield strength). For some other materials, such as aluminium, Hooke's law is only valid for a portion of the elastic range. For these materials a proportional limit stress is defined, below which the errors associated with the linear approximation are negligible.

Rubber is generally regarded as a "non-hookean" material because its elasticity is stress dependent and sensitive to temperature and loading rate.

Applications of the law include spring operated weighing machines, stress analysis and modelling of materials.

The spring equation



Stress–strain curve for low-carbon steel. Hooke's law is only valid for the portion of the curve between the origin and the yield point(2).
1. Ultimate strength
2. Yield strength - corresponds to yield point
3. Rupture
4. Strain hardening region
5. Necking region
A: (F/A_0)
B: True stress (F/A)

The most commonly encountered form of Hooke's law is probably the *spring equation*, which relates the force exerted by a spring to the distance it is stretched by a *spring constant*, *k*, measured in force per length.

$$F = -kx$$

The negative sign indicates that the force exerted by the spring is in direct opposition to the direction of displacement. It is called a "restoring force", as it tends to restore the system to equilibrium. The potential energy stored in a spring is given by

$$PE = \frac{1}{2}kx^2$$

which comes from adding up the energy it takes to incrementally compress the spring. That is, the integral of force over distance. (Note that potential energy of a spring is always non-negative.)

This potential can be visualized as a parabola on the *U*-*x* plane. As the spring is stretched in the positive x-direction, the potential energy increases (the same thing happens as the spring is compressed). The corresponding point on the potential energy curve is higher than that corresponding to the equilibrium position ($x = 0$). The tendency for the spring is to therefore decrease its potential energy by returning to its equilibrium (unstretched) position, just as a ball rolls downhill to decrease its gravitational potential energy.

If a mass *m* is attached to the end of such a spring, the system becomes a harmonic oscillator. It will oscillate with a natural frequency given either as an angular frequency

$$\omega = \sqrt{\frac{k}{m}}$$

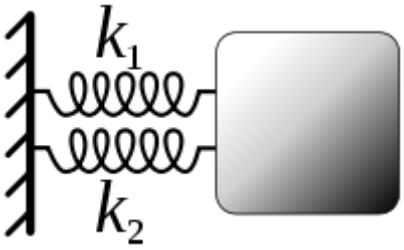
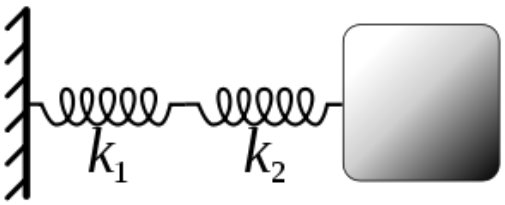
or as a natural frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

This idealized description of spring mechanics works as long as the mass of the spring is very small compared to the mass *m*, there is no significant friction on the system, and the spring is not overextended beyond its natural range (which can deform it permanently).

Multiple springs

When two springs are attached to a mass and compressed, the following table compares values of the springs.

Comparison	In Parallel	In Series
		
Equivalent spring constant	$k_{eq} = k_1 + k_2$	$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$
Compressed distance	$x_1 = x_2$	$\frac{x_1}{x_2} = \frac{k_2}{k_1}$
Energy stored	$\frac{E_1}{E_2} = \frac{k_1}{k_2}$	$\frac{E_1}{E_2} = \frac{k_2}{k_1}$

Derivation

Equivalent Spring Constant (Series)

$$\begin{aligned} F_b &= -k_2 x_2 + k_2 x_1 \\ &= -k_2 x_2 + k_2 \left(\frac{k_2}{k_1 + k_2} x_2 \right) \\ &= -k_2 x_2 \left(\frac{k_1 + k_2}{k_1 + k_2} \right) + \frac{k_2^2}{k_1 + k_2} x_2 \\ &= x_2 \frac{-k_1 k_2 - k_2^2 + k_2^2}{k_1 + k_2} \end{aligned}$$

Equivalent Spring Constant (Parallel)

$$\begin{aligned} F_b &= F_1 + F_2 \\ &= -k_1 x - k_2 x \end{aligned}$$

Compressed Distance

$$|F_1| = |F_2|$$

$$k_1 x_1 = k_2 (x_2 - x_1) .$$

Energy Stored

Tensor expression of Hooke's Law

When working with a three-dimensional stress state, a 4th order tensor $\mathbf{C}$($C_{ijk\ell}$) containing 81 elastic coefficients must be defined to link the stress tensor $\boldsymbol{\sigma}$(σ_{ij}) and the strain tensor $\boldsymbol{\epsilon}$($\epsilon_{k\ell}$).

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\epsilon} .$$

Expressed in terms of components with respect to an orthonormal basis, the generalized form of Hooke's law is written as (using the summation convention)

$$\sigma_{ij} = C_{ijk\ell}\ \epsilon_{k\ell}$$

The tensor $\mathbf{C}$is called the **stiffness tensor** or the **elasticity tensor**. Due to the symmetry of the stress tensor, strain tensor, and stiffness tensor, only 21 elastic coefficients are independent. As stress is measured in units of pressure and strain is dimensionless, the entries of $C_{ijk\ell}$are also in units of pressure.

The expression for generalized Hooke's law can be inverted to get a relation for the strain in terms of stress:

$$\boldsymbol{\epsilon} = \mathbf{S} : \boldsymbol{\sigma} \qquad \text{or} \qquad \epsilon_{ij} = S_{ijk\ell}\ \sigma_{k\ell} .$$

The tensor $\mathbf{S}$is called the **compliance tensor**.

Generalization for the case of large deformations is provided by models of neo-Hookean solids and Mooney-Rivlin solids.

Isotropic materials

Isotropic materials are characterized by properties which are independent of direction in space. Physical equations involving isotropic materials must therefore be independent of the coordinate system chosen to represent them. The strain tensor is a symmetric tensor. Since the trace of any tensor is independent of any coordinate system, the most complete coordinate-free decomposition of a symmetric tensor is to represent it as the sum of a constant tensor and a traceless symmetric tensor.:Ch. 10 Thus:

$$\varepsilon_{ij} = \left(\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + \left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right)$$

where δ_{ij} is the Kronecker delta. In direct tensor notation

$$\boldsymbol{\varepsilon} = \operatorname{vol}(\boldsymbol{\varepsilon}) + \operatorname{dev}(\boldsymbol{\varepsilon}) \ ; \quad \operatorname{vol}(\boldsymbol{\varepsilon}) := \tfrac{1}{3} \operatorname{tr}(\boldsymbol{\varepsilon})\, \mathbf{I} \ ; \quad \operatorname{dev}(\boldsymbol{\varepsilon}) := \boldsymbol{\varepsilon} - \operatorname{vol}(\boldsymbol{\varepsilon})$$

where $\mathbf{I}$ is the second-order identity tensor. The first term on the right is the constant tensor, also known as the **volumetric strain tensor**, and the second term is the traceless symmetric tensor, also known as the **deviatoric strain tensor** or shear tensor.

The most general form of Hooke's law for isotropic materials may now be written as a linear combination of these two tensors:

$$\sigma_{ij} = 3K \left(\tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) + 2G \left(\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{kk}\delta_{ij}\right) \ ; \quad \boldsymbol{\sigma} = 3K \operatorname{vol}(\boldsymbol{\varepsilon}) + 2G \operatorname{dev}(\boldsymbol{\varepsilon})$$

where K is the bulk modulus and G is the shear modulus.

Using the relationships between the elastic moduli, these equations may also be expressed in various other ways. A common form of Hooke's law for isotropic materials, expressed in direct tensor notation, is

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\, \mathbf{I} + 2\mu\, \boldsymbol{\varepsilon} = \mathsf{c} : \boldsymbol{\varepsilon} \ ; \quad \mathsf{c} = \lambda\, \mathbf{I} \otimes \mathbf{I} + 2\mu\, \mathsf{I}$$

where $\lambda := K - 2/3G$ and $\mu := G$ are the Lamé constants, $\mathbf{I}$ is the second-order identity tensor, and I is the symmetric part of the fourth-order identity tensor. In terms of components with respect to a Cartesian basis,

$$\sigma_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij} = c_{ijk\ell}\, \varepsilon_{k\ell} \ ; \quad c_{ijk\ell} = \lambda\, \delta_{ij}\, \delta_{k\ell} + \mu\, (\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk})$$

The inverse relationship is

$$\boldsymbol{\varepsilon} = \tfrac{1}{2\mu}\, \boldsymbol{\sigma} - \tfrac{\lambda}{2\mu(3\lambda+2\mu)} \operatorname{tr}(\boldsymbol{\sigma})\, \mathbf{I} = \tfrac{1}{2G}\, \boldsymbol{\sigma} + \left(\tfrac{1}{9K} - \tfrac{1}{6G}\right) \operatorname{tr}(\boldsymbol{\sigma})\, \mathbf{I}$$

Therefore the compliance tensor in the relation $\boldsymbol{\varepsilon} = \mathsf{s} : \boldsymbol{\sigma}$ is

$$\mathsf{s} = -\tfrac{\lambda}{2\mu(3\lambda+2\mu)}\, \mathbf{I} \otimes \mathbf{I} + \tfrac{1}{2\mu}\, \mathsf{I} = \left(\tfrac{1}{9K} - \tfrac{1}{6G}\right)\, \mathbf{I} \otimes \mathbf{I} + \tfrac{1}{2G}\, \mathsf{I}$$

In terms of Young's modulus and Poisson's ratio, Hooke's law for isotropic materials can then be expressed as

$$\boldsymbol{\varepsilon} = \tfrac{1}{E}\, \boldsymbol{\sigma} - \tfrac{\nu}{E} \left[\operatorname{tr}(\boldsymbol{\sigma})\, \mathbf{I} - \boldsymbol{\sigma}\right]$$

This is the form in which the strain is expressed in terms of the stress tensor in engineering. The expression in expanded form is

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right]$$
$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right]$$
$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right]$$
$$\varepsilon_{12} = \frac{1}{2G}\,\sigma_{12}\ ;\quad \varepsilon_{13} = \frac{1}{2G}\,\sigma_{13}\ ;\quad \varepsilon_{23} = \frac{1}{2G}\,\sigma_{23}$$

where *E* is the modulus of elasticity and ν is Poisson's ratio.

In matrix form, Hooke's law for isotropic materials can be written as

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}$$

where $\gamma_{ij} := 2\varepsilon_{ij}$ is the **engineering shear strain**. The inverse relation may be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}$$

which expression can be simplified thanks to the Lamé constants:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix}$$

Plane stress Hooke's law

Under plane stress conditions $\sigma_{33} = \sigma_{31} = \sigma_{23} = 0$. In that case Hooke's law takes the form

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}$$

The inverse relation is usually written in the reduced form

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{bmatrix}$$

Anisotropic materials

The symmetry of the Cauchy stress tensor ($\sigma_{ij} = \sigma_{ji}$) and the generalized Hooke's laws ($\sigma_{ij} = c_{ijk\ell}\ \epsilon_{k\ell}$) implies that $c_{ijk\ell} = c_{jik\ell}$. Similarly, the symmetry of the infinitesimal strain tensor implies that $c_{ijk\ell} = c_{ij\ell k}$. These symmetries are called the **minor symmetries** of the **stiffness tensor** ($\mathbf{C}$).

If in addition, since the displacement gradient and the Cauchy stress are work conjugate, the stress-strain relation can be derived from a strain energy density functional (U), then

$$\sigma_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \quad \Longrightarrow \quad c_{ijk\ell} = \frac{\partial^2 U}{\partial \epsilon_{ij}\partial \epsilon_{k\ell}}\ .$$

The arbitrariness of the order of differentiation implies that $c_{ijk\ell} = c_{k\ell ij}$. These are called the **major symmetries** of the stiffness tensor. The major and minor symmetries indicate that the stiffness tensor has only 21 independent components.

Matrix representation (stiffness tensor)

It is often useful to express the anisotropic form of Hooke's law in matrix notation, also called Voigt notation. To do this we take advantage of the symmetry of the stress and strain tensors and express them as six-dimensional vectors in an orthonormal coordinate system ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) as

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} \equiv \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \ ; \quad [\boldsymbol{\epsilon}] = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \\ 2\epsilon_{12} \end{bmatrix} \equiv \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

Then the stiffness tensor (C) can be expressed as

$$[\mathsf{C}] = \begin{bmatrix} c_{1111} & c_{1122} & c_{1133} & c_{1123} & c_{1131} & c_{1112} \\ c_{2211} & c_{2222} & c_{2233} & c_{2223} & c_{2231} & c_{2212} \\ c_{3311} & c_{3322} & c_{3333} & c_{3323} & c_{3331} & c_{3312} \\ c_{2311} & c_{2322} & c_{2333} & c_{2323} & c_{2331} & c_{2312} \\ c_{3111} & c_{3122} & c_{3133} & c_{3123} & c_{3131} & c_{3112} \\ c_{1211} & c_{1222} & c_{1233} & c_{1223} & c_{1231} & c_{1212} \end{bmatrix} \equiv \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}$$

and Hooke's law is written as

$$[\boldsymbol{\sigma}] = [\mathsf{C}][\boldsymbol{\epsilon}] \qquad \text{or} \qquad \sigma_i = C_{ij}\epsilon_j \ .$$

Similarly the compliance tensor (S) can be written as

$$[\mathsf{S}] = \begin{bmatrix} s_{1111} & s_{1122} & s_{1133} & 2s_{1123} & 2s_{1131} & 2s_{1112} \\ s_{2211} & s_{2222} & s_{2233} & 2s_{2223} & 2s_{2231} & 2s_{2212} \\ s_{3311} & s_{3322} & s_{3333} & 2s_{3323} & 2s_{3331} & 2s_{3312} \\ 2s_{2311} & 2s_{2322} & 2s_{2333} & 4s_{2323} & 4s_{2331} & 4s_{2312} \\ 2s_{3111} & 2s_{3122} & 2s_{3133} & 4s_{3123} & 4s_{3131} & 4s_{3112} \\ 2s_{1211} & 2s_{1222} & 2s_{1233} & 4s_{1223} & 4s_{1231} & 4s_{1212} \end{bmatrix} \equiv \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix}$$

Change of coordinate system

If a linear elastic material is rotated from a reference configuration to another, then the material is symmetric with respect to the rotation if the components of the stiffness tensor in the rotated configuration are related to the components in the reference configuration by the relation

$$c_{pqrs} = l_{pi}\ l_{qj}\ l_{rk}\ l_{s\ell}\ c_{ijk\ell}$$

where l_{ab} are the components of an orthogonal rotation matrix $[L]$. The same relation also holds for inversions.

In matrix notation, if the transformed basis (rotated or inverted) is related to the reference basis by

$$[\mathbf{e}_i'] = [L][\mathbf{e}_i]$$

then

$$C_{ij}\ \epsilon_i\ \epsilon_j = C'_{ij}\ \epsilon'_i\ \epsilon'_j \ .$$

In addition, if the material is symmetric with respect to the transformation $[L]$ then

$$C_{ij} = C'_{ij} \quad \Longrightarrow \quad C_{ij}\ (\epsilon_i\ \epsilon_j - \epsilon'_i\ \epsilon'_j) = 0\ .$$

Orthotropic materials

Orthotropic materials have three orthogonal planes of symmetry. If the basis vectors ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) are normals to the planes of symmetry then the coordinate transformation relations imply that

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix}$$

The inverse of this relation is commonly written as

$$\begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ 2\epsilon_{yz} \\ 2\epsilon_{zx} \\ 2\epsilon_{xy} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{xz}}{E_x} & 0 & 0 & 0 \\ -\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_y} & 0 & 0 & 0 \\ -\frac{\nu_{zx}}{E_z} & -\frac{\nu_{zy}}{E_z} & \frac{1}{E_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{yz}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{zx}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{bmatrix}$$

where

E_i is the Young's modulus along axis *i*
G_{ij} is the shear modulus in direction *j* on the plane whose normal is in direction *i*
ν_{ij} is the Poisson's ratio that corresponds to a contraction in direction *j* when an extension is applied in direction *i*.

Transversely isotropic materials

A transversely isotropic material is symmetric with respect to a rotation about an axis of symmetry. For such a material, if $\mathbf{e}_3$ is the axis of symmetry, Hooke's law can be expressed as

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0\\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{44} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12})\end{bmatrix}\begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

More frequently, the $x \equiv \mathbf{e}_1$axis is taken to be the axis of symmetry and the inverse Hooke's law is written as

$$\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{zz}\\ 2\epsilon_{yz}\\ 2\epsilon_{zx}\\ 2\epsilon_{xy}\end{bmatrix} = \begin{bmatrix}\frac{1}{E_x} & -\frac{\nu_{xy}}{E_x} & -\frac{\nu_{xy}}{E_x} & 0 & 0 & 0\\ -\frac{\nu_{yx}}{E_y} & \frac{1}{E_y} & -\frac{\nu_{yz}}{E_y} & 0 & 0 & 0\\ -\frac{\nu_{yx}}{E_y} & -\frac{\nu_{zy}}{E_y} & \frac{1}{E_y} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{2(1+\nu_{yz})}{E_y} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G_{xy}} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{xy}}\end{bmatrix}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{zx}\\ \sigma_{xy}\end{bmatrix}$$

Thermodynamic basis of Hooke's law

Linear deformations of elastic materials can be approximated as adiabatic. Under these conditions and for quasistatic processes the first law of thermodynamics for a deformed body can be expressed as

$$\delta W = \delta U$$

where δU is the increase in internal energy and δW is the work done by external forces. The work can be split into two terms

$$\delta W = \delta W_s + \delta W_b$$

where δW_s is the work done by surface forces while δW_b is the work done by body forces. If $\delta\mathbf{u}$is a variation of the displacement field $\mathbf{u}$in the body, then the two external work terms can be expressed as

$$\delta W_s = \int_{\partial\Omega} \mathbf{t}\cdot\delta\mathbf{u}\ \mathrm{dS}\ ; \quad \delta W_b = \int_{\Omega} \mathbf{b}\cdot\delta\mathbf{u}\ \mathrm{dV}$$

where $\mathbf{t}$is the surface traction vector, $\mathbf{b}$is the body force vector, Ωrepresents the body and $\partial\Omega$represents its surface. Using the relation between the Cauchy stress and the surface traction, $\mathbf{t} = \mathbf{n}\cdot\boldsymbol{\sigma}$(where $\mathbf{n}$is the unit outward normal to $\partial\Omega$), we have

$$\delta W = \delta U = \int_{\partial\Omega} (\mathbf{n}\cdot\boldsymbol{\sigma})\cdot\delta\mathbf{u}\ \mathrm{dS} + \int_{\Omega}\mathbf{b}\cdot\delta\mathbf{u}\ \mathrm{dV}$$

Converting the surface integral into a volume integral via the divergence theorem gives

$$\delta U = \int_{\Omega}[\boldsymbol{\nabla}\cdot(\boldsymbol{\sigma}\cdot\delta\mathbf{u}) + \mathbf{b}\cdot\delta\mathbf{u}]\ \mathrm{dV}\ .$$

Using the symmetry of the Cauchy stress and the identity

$$\boldsymbol{\nabla}\cdot(\boldsymbol{A}\cdot\mathbf{b}) = (\boldsymbol{\nabla}\cdot\boldsymbol{A})\cdot\mathbf{b} + \tfrac{1}{2}[\boldsymbol{A}^T : \boldsymbol{\nabla}\mathbf{b} + \boldsymbol{A} : (\boldsymbol{\nabla}\mathbf{b})^T]$$

we have

$$\delta U = \int_{\Omega}[\boldsymbol{\sigma} : \tfrac{1}{2}\{\boldsymbol{\nabla}\delta\mathbf{u} + (\boldsymbol{\nabla}\delta\mathbf{u})^T\} + \{\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{b}\}\cdot\delta\mathbf{u}]\ \mathrm{dV}\ .$$

From the definition of strain and from the equations of equilibrium we have

$$\delta\boldsymbol{\epsilon} = \tfrac{1}{2}[\boldsymbol{\nabla}\delta\mathbf{u} + (\boldsymbol{\nabla}\delta\mathbf{u})^T]\ ;\quad \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}\ .$$

Hence we can write

$$\delta U = \int_{\Omega}\boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}\ \mathrm{dV}$$

and therefore the variation in the internal energy density is given by

$$\delta U_0 = \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}\ .$$

An elastic material is defined as one in which the total internal energy is equal to the potential energy of the internal forces (also called the **elastic strain energy**). Therefore the internal energy density is a function of the strains, $U_0 = U_0(\boldsymbol{\epsilon})$ and the variation of the internal energy can be expressed as

$$\delta U_0 = \frac{\partial U_0}{\partial\boldsymbol{\epsilon}} : \delta\boldsymbol{\epsilon}\ .$$

Since the variation of strain is arbitrary, the stress-strain relation of an elastic material is given by

$$\boldsymbol{\sigma} = \frac{\partial U_0}{\partial \boldsymbol{\epsilon}} \,.$$

For a linear elastic material, the quantity $\partial U_0 / \partial \boldsymbol{\epsilon}$ is a linear function of $\boldsymbol{\epsilon}$, and can therefore be expressed as

$$\boldsymbol{\sigma} = \mathbf{c} : \boldsymbol{\epsilon}$$

where $\mathbf{c}$ is a fourth-order tensor of material constants, also called the **stiffness tensor**.

Chapter 7

Infinitesimal Strain Theory

In continuum mechanics, the **infinitesimal strain theory**, sometimes called **small deformation theory**, **small displacement theory**, or **small displacement-gradient theory**, deals with *infinitesimal deformations* of a continuum body. For an infinitesimal deformation the displacements $\mathbf{u}$and the displacement gradients $\nabla \mathbf{u}$are small compared to unity, i.e., $\|\mathbf{u}\| \ll 1$and $\|\nabla \mathbf{u}\| \ll 1$, allowing for the *geometric linearization* of the Lagrangian finite strain tensor $\mathbf{E}$, and the Eulerian finite strain tensor $\mathbf{e}$, i.e. the non-linear or second-order terms of the finite strain tensor can be neglected. The linearized Lagrangian and Eulerian strain tensors are approximately the same and can be approximated by the **infinitesimal strain tensor** or **Cauchy's strain tensor**, $\boldsymbol{\varepsilon}$. Thus,

$$\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left((\nabla \mathbf{u})^T + \nabla \mathbf{u}\right)$$

or

$$E_{KL} \approx e_{rs} \approx \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

The infinitesimal strain theory is used in the analysis of deformations of materials exhibiting elastic behaviour, such as materials found in mechanical and civil engineering applications, e.g. concrete and steel.

Infinitesimal strain tensor

For *infinitesimal deformations* of a continuum body, in which the displacements and the displacement gradients are small compared to unity, i.e., $\|\mathbf{u}\| \ll 1$and $\|\nabla \mathbf{u}\| \ll 1$, it is possible to perform a *geometric linearization* of the Lagrangian finite strain tensor $\mathbf{E}$, and the Eulerian finite strain tensor $\mathbf{e}$. In such a linearization, the non-linear or second-order terms of the finite strain tensor are neglected. Thus we have

$$\mathbf{E} = \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T + \nabla_{\mathbf{X}}\mathbf{u}(\nabla_{\mathbf{X}}\mathbf{u})^T\right) \approx \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T\right)$$

or

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K}\frac{\partial U_M}{\partial X_L}\right) \approx \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K}\right)$$

and

$$\mathbf{e} = \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T + \nabla_{\mathbf{x}}\mathbf{u}(\nabla_{\mathbf{x}}\mathbf{u})^T\right) \approx \frac{1}{2}\left(\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T\right)$$

or

$$e_{rs} = \frac{1}{2}\left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r} - \frac{\partial u_k}{\partial x_r}\frac{\partial u_k}{\partial x_s}\right) \approx \frac{1}{2}\left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r}\right)$$

This linearization implies that the Lagrangian description and the Eulerian description are approximately the same as there is little difference in the material and spatial coordinates of a given material point in the continuum. Therefore, the material displacement gradient components and the spatial displacement gradient components are approximately equal. Thus we have

$$\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon} = \frac{1}{2}\left((\nabla\mathbf{u})^T + \nabla\mathbf{u}\right) \qquad \text{or} \qquad E_{KL} \approx e_{rs} \approx \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

where ε_{ij} are the components of the *infinitesimal strain tensor* $\boldsymbol{\varepsilon}$, also called *Cauchy's strain tensor*, *linear strain tensor*, or *small strain tensor*.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) & \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

or using different notation:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Furthermore, since the deformation gradient can be expressed as $\boldsymbol{F} = \boldsymbol{\nabla u} + \boldsymbol{I}$ where $\boldsymbol{I}$ is the second-order identity tensor, we have

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\boldsymbol{F}^T + \boldsymbol{F}\right) - \boldsymbol{I}$$

Also, from the general expression for the Lagrangian and Eulerian finite strain tensors we have

$$\mathbf{E}_{(m)} = \frac{1}{2m}(\mathbf{U}^{2m} - \boldsymbol{I}) = \frac{1}{2m}[(\boldsymbol{F}^T\boldsymbol{F})^m - \boldsymbol{I}] \approx \frac{1}{2m}[\{\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^T + \boldsymbol{I}\}^m - \boldsymbol{I}] \approx \boldsymbol{\varepsilon}$$
$$\mathbf{e}_{(m)} = \frac{1}{2m}(\mathbf{V}^{2m} - \boldsymbol{I}) = \frac{1}{2m}[(\boldsymbol{F}\boldsymbol{F}^T)^m - \boldsymbol{I}] \approx \boldsymbol{\varepsilon}$$

Geometric derivation of the infinitesimal strain tensor

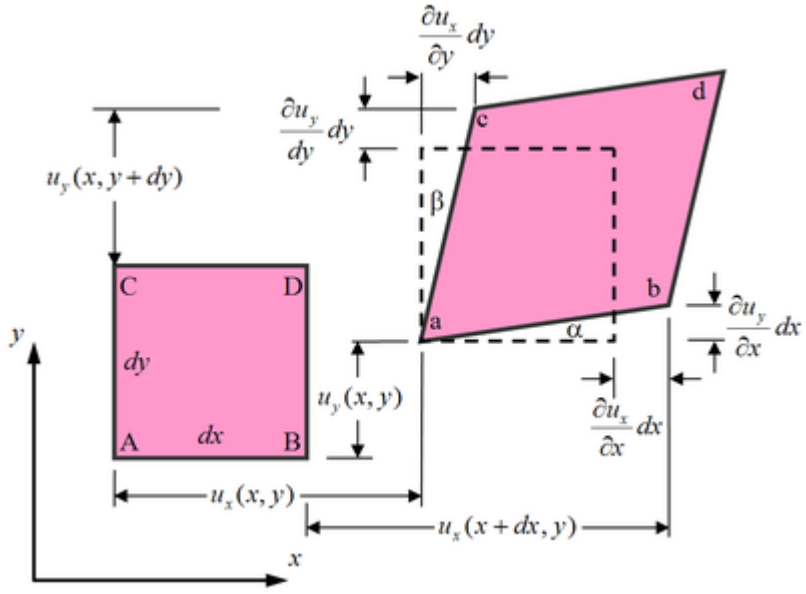


Figure 1. Two-dimensional geometric deformation of an infinitesimal material element.

Considering a two-dimensional deformation of an infinitesimal rectangular material element with dimensions dx by dy (Figure 1), which after deformation, takes the form of a rhombus. From the geometry of Figure 1 we have

$$\overline{ab} = \sqrt{\left(dx + \frac{\partial u_x}{\partial x}dx\right)^2 + \left(\frac{\partial u_y}{\partial x}dx\right)^2}$$
$$= \sqrt{1 + 2\frac{\partial u_x}{\partial x} + \left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2}dx$$

For very small displacement gradients, i.e., $\|\nabla \mathbf{u}\| \ll 1$, we have

$$\overline{ab} \approx dx + \frac{\partial u_x}{\partial x} dx$$

The normal strain in the x-direction of the rectangular element is defined by

$$\varepsilon_x = \frac{\overline{ab} - \overline{AB}}{\overline{AB}}$$

and knowing that $\overline{AB} = dx$, we have

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

Similarly, the normal strain in the y-direction, and z-direction, becomes

$$\varepsilon_y = \frac{\partial u_y}{\partial y} \quad , \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

The engineering shear strain, or the change in angle between two originally orthogonal material lines, in this case line $\overline{AC}$ and $\overline{AB}$, is defined as

$$\gamma_{xy} = \alpha + \beta$$

From the geometry of Figure 1 we have

$$\tan\alpha = \frac{\frac{\partial u_y}{\partial x} dx}{dx + \frac{\partial u_x}{\partial x} dx} = \frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}} \quad , \qquad \tan\beta = \frac{\frac{\partial u_x}{\partial y} dy}{dy + \frac{\partial u_y}{\partial y} dy} = \frac{\frac{\partial u_x}{\partial y}}{1 + \frac{\partial u_y}{\partial y}}$$

For small rotations, i.e. α and β are $\ll 1$ we have

$$\tan\alpha \approx \alpha \quad , \qquad \tan\beta \approx \beta$$

and, again, for small displacement gradients, we have

$$\alpha = \frac{\partial u_y}{\partial x} \quad , \qquad \beta = \frac{\partial u_x}{\partial y}$$

thus

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

By interchanging $x$and $y$and $u_x$and u_y, it can be shown that $\gamma_{xy} = \gamma_{yx}$

Similarly, for the y-z and x-z planes, we have

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \quad , \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

It can be seen that the tensorial shear strain components of the infinitesimal strain tensor can then be expressed using the engineering strain definition, γ, as

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

Physical interpretation of the infinitesimal strain tensor

From finite strain theory we have

$$d\mathbf{x}^2 - d\mathbf{X}^2 = d\mathbf{X} \cdot 2\mathbf{E} \cdot d\mathbf{X} \quad \text{or} \quad (dx)^2 - (dX)^2 = 2E_{KL}\, dX_K\, dX_L$$

For infinitesimal strains then we have

$$d\mathbf{x}^2 - d\mathbf{X}^2 = d\mathbf{X} \cdot 2\boldsymbol{\varepsilon} \cdot d\mathbf{X} \quad \text{or} \quad (dx)^2 - (dX)^2 = 2\varepsilon_{KL}\, dX_K\, dX_L$$

Dividing by $(dX)^2$ we have

$$\frac{dx - dX}{dX}\frac{dx + dX}{dX} = 2\varepsilon_{ij}\frac{dX_i}{dX}\frac{dX_j}{dX}$$

For small deformations we assume that $dx \approx dX$, thus the second term of the left hand side becomes: $\frac{dx + dX}{dX} \approx 2$.

Then we have

$$\frac{dx - dX}{dX} = \varepsilon_{ij} N_i N_j = \mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}$$

where $N_i = \dfrac{dX_i}{dX}$, is the unit vector in the direction of $d\mathbf{X}$, and the left-hand-side expression is the normal strain $e_{(\mathbf{N})}$ in the direction of $\mathbf{N}$. For the particular case of $\mathbf{N}$ in the X_1 direction, i.e. $\mathbf{N} = \mathbf{I}_1$, we have

$$e_{(\mathbf{I}_1)} = \mathbf{I}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{I}_1 = \varepsilon_{11}$$

Similarly, for $\mathbf{N} = \mathbf{I}_2$ and $\mathbf{N} = \mathbf{I}_3$ we can find the normal strains ε_{22} and ε_{33}, respectively. Therefore, the diagonal elements of the infinitesimal strain tensor are the normal strains in the coordinate directions.

Strain transformation rules

If we choose an orthonormal coordinate system ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) we can write the tensor in terms of components with respect to those base vectors as

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \sum_{j=1}^{3} \varepsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

In matrix form,

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

We can easily choose to use another orthonormal coordinate system ($\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$) instead. In that case the components of the tensor are different, say

$$\boldsymbol{\varepsilon} = \sum_{i=1}^{3} \sum_{j=1}^{3} \hat{\varepsilon}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad \Longrightarrow \quad \underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13} \\ \hat{\varepsilon}_{12} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23} \\ \hat{\varepsilon}_{13} & \hat{\varepsilon}_{23} & \hat{\varepsilon}_{33} \end{bmatrix}$$

The components of the strain in the two coordinate systems are related by

$$\hat{\varepsilon}_{ij} = \ell_{ip}\ \ell_{jq}\ \varepsilon_{pq}$$

where the Einstein summation convention for repeated indices has been used and $\ell_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{e}_j$. In matrix form

$$\underline{\underline{\hat{\boldsymbol{\varepsilon}}}} = \underline{\underline{\mathbf{L}}}\ \underline{\underline{\boldsymbol{\varepsilon}}}\ \underline{\underline{\mathbf{L}}}^T$$

or

$$\begin{bmatrix} \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13} \\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23} \\ \hat{\varepsilon}_{31} & \hat{\varepsilon}_{32} & \hat{\varepsilon}_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}^T$$

Strain invariants

Certain operations on the strain tensor give the same result without regard to which orthonormal coordinate system is used to represent the components of strain. The results of these operations are called **strain invariants**. The most commonly used strain invariants are

$$\begin{aligned} I_1 &= \operatorname{tr}(\boldsymbol{\varepsilon}) \\ I_2 &= \tfrac{1}{2}\{\operatorname{tr}(\boldsymbol{\varepsilon}^2) - [\operatorname{tr}(\boldsymbol{\varepsilon})]^2\} \\ I_3 &= \det(\boldsymbol{\varepsilon}) \end{aligned}$$

In terms of components

$$\begin{aligned} I_1 &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ I_2 &= \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2 - \varepsilon_{11}\varepsilon_{22} - \varepsilon_{22}\varepsilon_{33} - \varepsilon_{33}\varepsilon_{11} \\ I_3 &= \varepsilon_{11}(\varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}^2) - \varepsilon_{12}(\varepsilon_{12}\varepsilon_{33} - \varepsilon_{23}\varepsilon_{31}) + \varepsilon_{13}(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{31}) \end{aligned}$$

Principal strains

It can be shown that it is possible to find a coordinate system ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) in which the components of the strain tensor are

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \quad \Longrightarrow \quad \boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \varepsilon_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \varepsilon_3 \mathbf{n}_3 \otimes \mathbf{n}_3$$

The components of the strain tensor in the ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) coordinate system are called the **principal strains** and the directions $\mathbf{n}_i$are called the directions of principal strain. Since there are no shear strain components in this coordinate system, the principal strains represent the maximum and minimum stretches of an elemental volume.

If we are given the components of the strain tensor in an arbitrary orthonormal coordinate system, we can find the principal strains using an eigenvalue decomposition determined by solving the system of equations

$$\left(\underline{\underline{\boldsymbol{\varepsilon}}} - \varepsilon_i \, \underline{\underline{\mathbf{I}}}\right) \mathbf{n}_i = \underline{\underline{\mathbf{0}}}$$

This system of equations is equivalent to finding the vector $\mathbf{n}_i$along which the strain tensor becomes a pure stretch with no shear component.

Volumetric strain

The *dilatation* (the relative variation of the volume) is the trace of the tensor:

$$\delta = \frac{\Delta V}{V_0} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

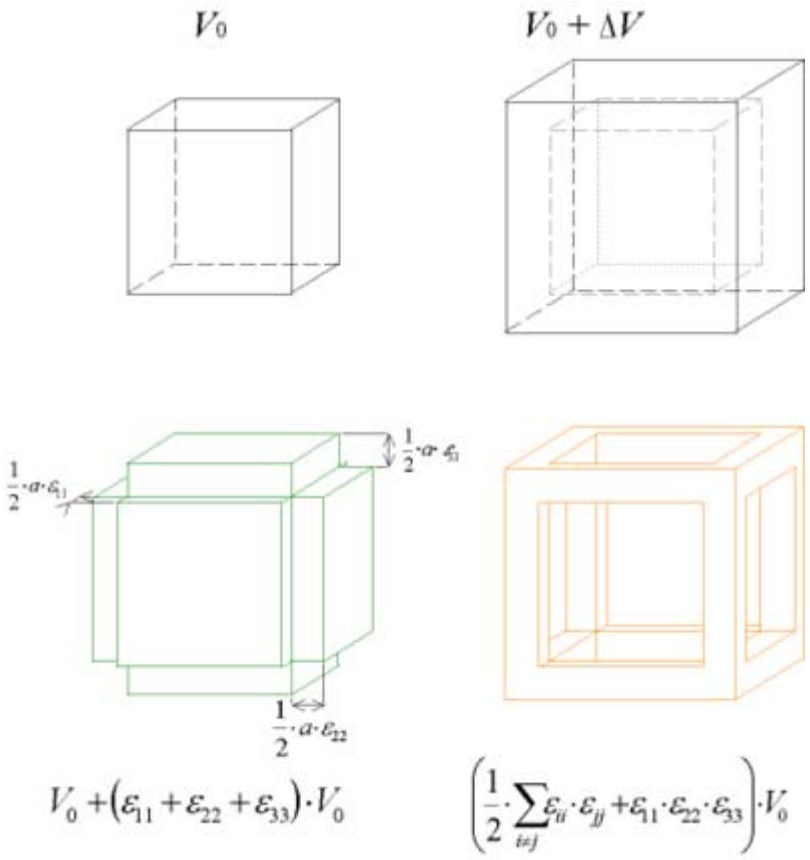
Actually, if we consider a cube with an edge length *a*, it is a quasi-cube after the deformation (the variations of the angles do not change the volume) with the dimensions $a \cdot (1 + \varepsilon_{11}) \times a \cdot (1 + \varepsilon_{22}) \times a \cdot (1 + \varepsilon_{33})$and $V_0 = a^3$, thus

$$\frac{\Delta V}{V_0} = \frac{(1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} + \varepsilon_{11} \cdot \varepsilon_{22} + \varepsilon_{11} \cdot \varepsilon_{33} + \varepsilon_{22} \cdot \varepsilon_{33} + \varepsilon_{11} \cdot \varepsilon_{22} \cdot \varepsilon_{33}) \cdot a^3 - a^3}{a^3}$$

as we consider small deformations,

$$1 \gg \varepsilon_{ii} \gg \varepsilon_{ii} \cdot \varepsilon_{jj} \gg \varepsilon_{11} \cdot \varepsilon_{22} \cdot \varepsilon_{33}$$

therefore the formula.



Real variation of volume (top) and the approximated one (bottom): the green drawing shows the estimated volume and the orange drawing the neglected volume

In case of pure shear, we can see that there is no change of the volume.

Strain deviator tensor

The infinitesimal strain tensor ε_{ij}, similarly to the stress tensor, can be expressed as the sum of two other tensors:

1. a **mean strain tensor** or **volumetric strain tensor** or **spherical strain tensor**, $\varepsilon_M \delta_{ij}$, related to dilation or volume change; and
2. a deviatoric component called the **strain deviator tensor**, ε'_{ij}, related to distortion.

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_M \delta_{ij}$$

where $\varepsilon_M$is the mean strain given by

$$\varepsilon_M = \frac{\varepsilon_{kk}}{3} = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{3} = \tfrac{1}{3} I_1^e$$

The deviatoric strain tensor can be obtained by subtracting the mean strain tensor from the infinitesimal strain tensor:

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{kk}}{3}\delta_{ij}$$

$$\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} - \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_M \end{bmatrix}$$

Octahedral strains

Let ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) be the directions of the three principal strains. An **octahedral plane** is whose normal makes equal angles with the three principal directions. The engineering shear strain on an octahedral plane is called the **octahedral shear strain** and is given by

$$\gamma_{\text{oct}} = \tfrac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$are the principal strains.

The normal strain on an octahedral plane is given by

$$\varepsilon_{\text{oct}} = \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

Equivalent strain

A scalar quantity called the **equivalent strain**, or the von Mises equivalent strain, is often used to describe the state of strain in solids. Several definitions of equivalent strain can be found in the literature. A definition that is commonly used in the literature on plasticity is

$$\varepsilon_{\text{eq}} = \sqrt{\tfrac{2}{3}\boldsymbol{\varepsilon}^{\text{dev}} : \boldsymbol{\varepsilon}^{\text{dev}}} = \sqrt{\tfrac{2}{3}\varepsilon_{ij}^{\text{dev}}\varepsilon_{ij}^{\text{dev}}} \; ; \quad \boldsymbol{\varepsilon}^{\text{dev}} = \boldsymbol{\varepsilon} - \tfrac{1}{3}\text{tr}(\boldsymbol{\varepsilon})\ \mathbf{1}$$

This quantity is work conjugate to the equivalent stress defined as

$$\sigma_{\text{eq}} = \sqrt{\tfrac{3}{2}\boldsymbol{\sigma}^{\text{dev}} : \boldsymbol{\sigma}^{\text{dev}}}$$

Compatibility equations

For prescribed strain components $\varepsilon_{ij}$the strain tensor equation $u_{i,j} + u_{j,i} = 2\varepsilon_{ij}$represents a system of six differential equations for the determination of three displacements components u_i, giving an over-determined system. Thus, a solution does not generally exist for an arbitrary choice of strain components. Therefore, some restrictions, named *compatibility equations*, are imposed upon the strain components. With the addition of the three compatibility equations the number of independent equations is reduced to three, matching the number of unknown displacement components. These constraints on the strain tensor were discovered by Saint-Venant, and are called the "Saint Venant compatibility equations".

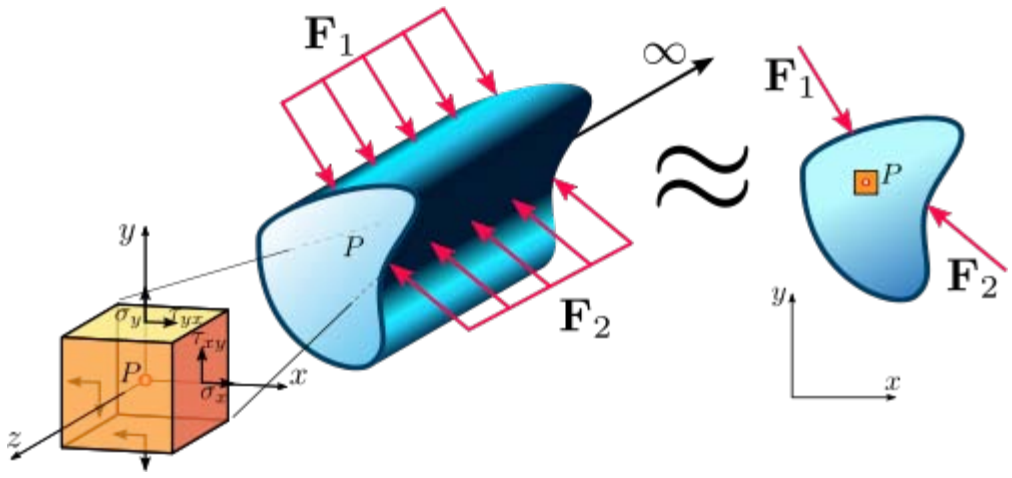
The compatibility functions serve to assure a single-valued continuous displacement function u_i. If the elastic medium is visualized as a set of infinitesimal cubes in the unstrained state, after the medium is strained, an arbitrary strain tensor may not yield a situation in which the distorted cubes still fit together without overlapping.

In index notation, the compatibility equations are expressed as

$$\varepsilon_{ij,km} + \varepsilon_{km,ij} - \varepsilon_{ik,jm} - \varepsilon_{jm,ik} = 0$$

Special cases

Plane strain



Plane strain state in a continuum.

In real engineering components, stress (and strain) are 3-D tensors but in prismatic structures such as a long metal billet, the length of the structure is much greater than the other two dimensions. The strains associated with length, i.e. the normal strain $\varepsilon_{33}$and the shear strains $\varepsilon_{13}$and ε_{23}(if the length is the 3-direction) are constrained by nearby

material and are small compared to the *cross-sectional strains*. The strain tensor can then be approximated by:

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in which the double underline indicates a second order tensor. This strain state is called *plane strain*. The corresponding stress tensor is:

$$\underline{\underline{\boldsymbol{\sigma}}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

in which the non-zero $\sigma_{33}$is needed to maintain the constraint $\epsilon_{33} = 0$. This stress term can be temporarily removed from the analysis to leave only the in-plane terms, effectively reducing the 3-D problem to a much simpler 2-D problem.

Antiplane strain

Antiplane strain is another special state of strain that can occur in a body, for instance in a region close to a screw dislocation. The strain tensor for antiplane strain is given by

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} 0 & 0 & \varepsilon_{13} \\ 0 & 0 & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & 0 \end{bmatrix}$$

Infinitesimal rotation tensor

The infinitesimal strain tensor is defined as

$$\boldsymbol{\varepsilon} = \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T]$$

Therefore the displacement gradient can be expressed as

$$\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$$

where

$$\boldsymbol{\omega} := \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u} - (\boldsymbol{\nabla}\mathbf{u})^T]$$

The quantity $\boldsymbol{\omega}$ is the **infinitesimal rotation tensor**. This tensor is skew symmetric. For infinitesimal deformations the scalar components of $\boldsymbol{\omega}$ satisfy the condition $|\omega_{ij}| \ll 1$. Note that the displacement gradient is small only if **both** the strain tensor and the rotation tensor are infinitesimal.

The axial vector

A skew symmetric second-order tensor has three independent scalar components. These three components are used to define an **axial vector**, $\mathbf{w}$, as follows

$$\omega_{ij} = -e_{ijk}\ w_k\ ;\quad w_i = \tfrac{1}{2}\ e_{ijk}\ \omega_{jk}$$

where e_{ijk} is the permutation symbol. In matrix form

$$\underline{\underline{\boldsymbol{\omega}}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}\ ;\quad \underline{\underline{\mathbf{w}}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

The axial vector is also called the **infinitesimal rotation vector**. The rotation vector is related to the displacement gradient by the relation

$$\mathbf{w} = \tfrac{1}{2}\ \boldsymbol{\nabla} \times \mathbf{u}$$

In index notation

$$w_i = \tfrac{1}{2}\ e_{ijk}\ u_{k,j}$$

If $\|\boldsymbol{\omega}\| \ll 1$ and $\boldsymbol{\varepsilon} = \mathbf{0}$ then the material undergoes an approximate rigid body rotation of magnitude $|\mathbf{w}|$ around the vector $\mathbf{w}$.

Relation between the strain tensor and the rotation vector

Given a continuous, single-valued displacement field $\mathbf{u}$ and the corresponding infinitesimal strain tensor $\boldsymbol{\varepsilon}$, we have

$$\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} = e_{ijk}\ \varepsilon_{lj,i}\ \mathbf{e}_k \otimes \mathbf{e}_l = \tfrac{1}{2}\ e_{ijk}\ \left[u_{l,ji} + u_{j,li}\right]\ \mathbf{e}_k \otimes \mathbf{e}_l$$

Since a change in the order of differentiation does not change the result, $u_{l,ji} = u_{l,ij}$. Therefore

$$e_{ijk} u_{l,ji} = (e_{12k} + e_{21k}) u_{l,12} + (e_{13k} + e_{31k}) u_{l,13} + (e_{23k} + e_{32k}) u_{l,32} = 0$$

Also

$$\tfrac{1}{2}\ e_{ijk}\ u_{j,li} = \left(\tfrac{1}{2}\ e_{ijk}\ u_{j,i}\right)_{,l} = \left(\tfrac{1}{2}\ e_{kij}\ u_{j,i}\right)_{,l} = w_{k,l}$$

Hence

$$\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} = w_{k,l}\ \mathbf{e}_k \otimes \mathbf{e}_l = \boldsymbol{\nabla}\mathbf{w}$$

Relation between rotation tensor and rotation vector

From an important identity regarding the curl of a tensor we know that for a continuous, single-valued displacement field $\mathbf{u}$,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{u}) = \mathbf{0}.$$

Since $\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$we have $\boldsymbol{\nabla} \times \boldsymbol{\omega} = -\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} = -\boldsymbol{\nabla}\mathbf{w}.$

Strain tensor in cylindrical coordinates

In cylindrical polar coordinates (r,θ,z), the displacement vector can be written as

$$\mathbf{u} = u_r\ \mathbf{e}_r + u_\theta\ \mathbf{e}_\theta + u_z\ \mathbf{e}_z$$

The components of the strain tensor in a cylindrical coordinate system are given by

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\varepsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \\
\varepsilon_{zr} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)
\end{aligned}$$

Strain tensor in spherical coordinates

In spherical coordinates (r,θ,φ), the displacement vector can be written as

$$\mathbf{u} = u_r\ \mathbf{e}_r + u_\theta\ \mathbf{e}_\theta + u_\phi\ \mathbf{e}_\phi$$

The components of the strain tensor in a spherical coordinate system are given by

$$\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \\
\varepsilon_{\phi\phi} &= \frac{1}{r\sin\theta}\left(\frac{\partial u_\phi}{\partial \phi} + u_r\sin\theta + u_\theta\cos\theta\right) \\
\varepsilon_{r\theta} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\
\varepsilon_{\theta\phi} &= \frac{1}{2r}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - u_\phi\cot\theta\right) \\
\varepsilon_{\phi r} &= \frac{1}{2}\left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right)
\end{aligned}$$

Chapter 8

Finite Strain Theory

In continuum mechanics, the **finite strain theory**—also called **large strain theory**, or **large deformation theory**—deals with deformations in which both rotations and strains are arbitrarily large, i.e. invalidates the assumptions inherent in infinitesimal strain theory. In this case, the undeformed and deformed configurations of the continuum are significantly different and a clear distinction has to be made between them. This is commonly the case with elastomers, plastically-deforming materials and other fluids and biological soft tissue.

Displacement

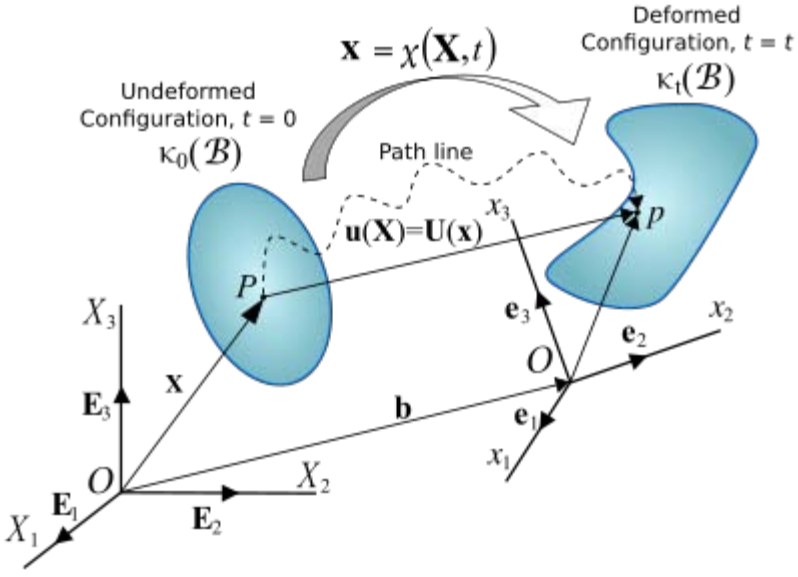


Figure 1. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consists of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change

in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 1).

If after a displacement of the continuum there is a relative displacement between particles, a deformation has occurred. On the other hand, if after displacement of the continuum the relative displacement between particles in the current configuration is zero i.e. the distance between particles remains unchanged, then there is no deformation and a rigid-body displacement is said to have occurred.

The vector joining the positions of a particle P in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X}, t) = u_i \mathbf{e}_i$ in the Lagrangian description, or $\mathbf{U}(\mathbf{x}, t) = U_J \mathbf{E}_J$ in the Eulerian description.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field. In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{b}(\mathbf{X}, t) + \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ} b_J + x_i - \alpha_{iJ} X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}, t) + \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji} x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ} U_J \qquad \text{or} \qquad U_J = \alpha_{Ji} u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ} \mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X}, t) = u_i \mathbf{e}_i = u_i (\alpha_{iJ} \mathbf{E}_J) = U_J \mathbf{E}_J = \mathbf{U}(\mathbf{x}, t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in $\mathbf{b} = 0$, and the direction cosines become Kronecker deltas, i.e.

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}$$

Thus, we have

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = \delta_{Ji}x_i - X_J$$

Displacement gradient tensor

The partial differentiation of the displacement vector with respect to the material coordinates yields the *material displacement gradient tensor* $\nabla_{\mathbf{X}}\mathbf{u}$. Thus we have,

$$\begin{aligned} \mathbf{u}(\mathbf{X},t) &= \mathbf{x}(\mathbf{X},t) - \mathbf{X} & u_i &= x_i - \delta_{iJ}X_J = x_i - X_i \\ \nabla_{\mathbf{X}}\mathbf{u} &= \nabla_{\mathbf{X}}\mathbf{x} - \mathbf{I} \qquad \text{or} & \frac{\partial u_i}{\partial X_K} &= \frac{\partial x_i}{\partial X_K} - \delta_{iK} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \mathbf{F} - \mathbf{I} \end{aligned}$$

where $\mathbf{F}$ is the *deformation gradient tensor*.

Similarly, the partial differentiation of the displacement vector with respect to the spatial coordinates yields the *spatial displacement gradient tensor* $\nabla_{\mathbf{x}}\mathbf{U}$. Thus we have,

$$\begin{aligned} \mathbf{U}(\mathbf{x},t) &= \mathbf{x} - \mathbf{X}(\mathbf{x},t) & U_J &= \delta_{Ji}x_i - X_J = x_J - X_J \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{X} \qquad \text{or} & \frac{\partial U_J}{\partial x_k} &= \delta_{Jk} - \frac{\partial X_J}{\partial x_k} \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \mathbf{F}^{-1} \end{aligned}$$

Deformation gradient tensor

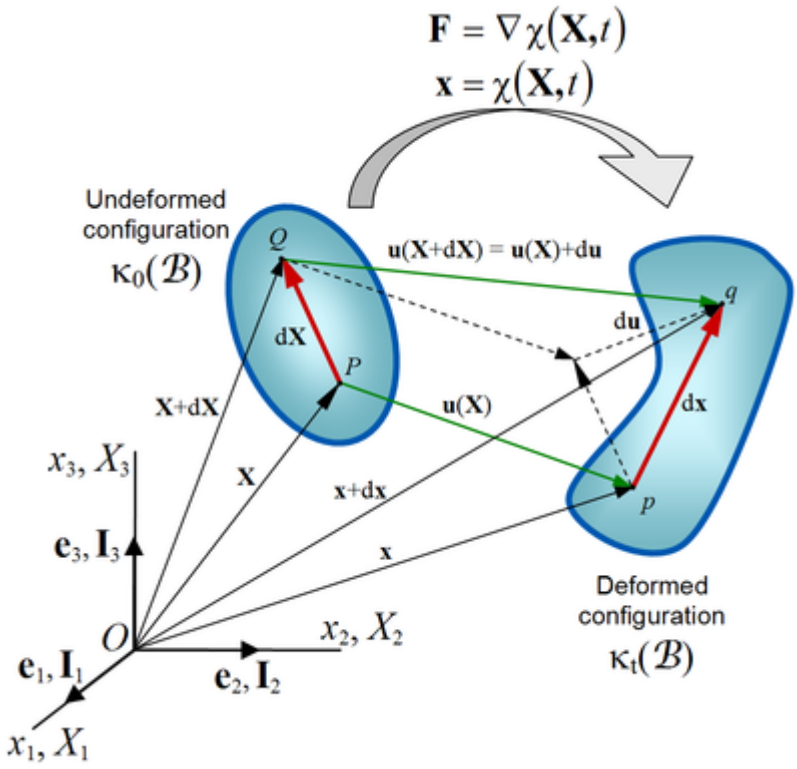


Figure 2. Deformation of a continuum body.

Consider a particle or material point P with position vector $\mathbf{X} = X_I \mathbf{I}_J$ in the undeformed configuration (Figure 2). After a displacement of the body, the new position of the particle indicated by p in the new configuration is given by the vector position $\mathbf{x} = x_i \mathbf{e}_j$. The coordinate systems for the undeformed and deformed configuration can be superimposed for convenience.

Consider now a material point Q neighboring P, with position vector $\mathbf{X} + \Delta\mathbf{X} = (X_I + \Delta X_I)\mathbf{I}_j$. In the deformed configuration this particle has a new position q given by the position vector $\mathbf{x} + \Delta\mathbf{x}$. Assuming that the line segments ΔX and $\Delta\mathbf{x}$ joining the particles P and Q in both the undeformed and deformed configuration, respectively, to be very small, then we can express them as $d\mathbf{X}$ and $d\mathbf{x}$. Thus from Figure 2 we have

$$\begin{aligned}\mathbf{x} + d\mathbf{x} &= \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\ d\mathbf{x} &= \mathbf{X} - \mathbf{x} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\ &= d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) - \mathbf{u}(\mathbf{X}) \\ &= d\mathbf{X} + d\mathbf{u}\end{aligned}$$

where $\mathbf{du}$is the **relative displacement vector**, which represents the relative displacement of $Q$with respect to Pin the deformed configuration.

For an infinitesimal element $d\mathbf{X}$, and assuming continuity on the displacement field, it is possible to use a Taylor series expansion around point P, neglecting higher-order terms, to approximate the components of the relative displacement vector for the neighboring particle Qas

$$\begin{aligned}\mathbf{u}(\mathbf{X} + d\mathbf{X}) &= \mathbf{u}(\mathbf{X}) + d\mathbf{u} \\ &\approx \mathbf{u}(\mathbf{X}) + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X}\end{aligned} \qquad \text{or} \qquad \begin{aligned}u_i^* &= u_i + du_i \\ &\approx u_i + \frac{\partial u_i}{\partial X_J} dX_J\end{aligned}$$

Thus, the previous equation $d\mathbf{x} = d\mathbf{X} + d\mathbf{u}$can be written as

$$\begin{aligned}d\mathbf{x} &= d\mathbf{X} + d\mathbf{u} \\ &= d\mathbf{X} + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X} \\ &= (\mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u})\, d\mathbf{X} \\ &= \mathbf{F} d\mathbf{X}\end{aligned}$$

The *material deformation gradient tensor* $\mathbf{F}(\mathbf{X}, t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$is a second-order tensor that represents the gradient of the mapping function or functional relation $\chi(\mathbf{X}, t)$, which describes the motion of a continuum. The material deformation gradient tensor characterizes the local deformation at a material point with position vector $\mathbf{X}$, i.e. deformation at neighbouring points, by transforming (linear transformation) a material line element emanating from that point from the reference configuration to the current or deformed configuration, assuming continuity in the mapping function $\chi(\mathbf{X}, t)$, i.e. differentiable function of $\mathbf{X}$and time t, which implies that cracks and voids do not open or close during the deformation. Thus we have,

$$\begin{aligned}d\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\, d\mathbf{X} \\ &= \nabla\chi(\mathbf{X}, t)\, d\mathbf{X} \\ &= \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}\end{aligned} \qquad \text{or} \qquad \begin{aligned}dx_j &= \frac{\partial x_j}{\partial X_K}\, dX_K \\ dx_j &= F_{jK}\, dX_K\end{aligned}$$

The deformation gradient tensor $\mathbf{F}(\mathbf{X},t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$ is related to both the reference and current configuration, as seen by the unit vectors $\mathbf{e}_j$ and $\mathbf{I}_K$, therefore it is a *two-point tensor*.

Due to the assumption of continuity of $\chi(\mathbf{X},t)$, $\mathbf{F}$ has the inverse $\mathbf{H} = \mathbf{F}^{-1}$, where $\mathbf{H}$ is the *spatial deformation gradient tensor*. Then, by the implicit function theorem (Lubliner), the Jacobian determinant $J(\mathbf{X},t)$ must be nonsingular, i.e.
$J(\mathbf{X},t) = \det \mathbf{F}(\mathbf{X},t) \neq 0$

Transformation of a surface and volume element

To transform quantities that are defined with respect to areas in a deformed configuration to those relative to areas in a reference configuration, and vice versa, we use the Nanson's relation, expressed as

$$da\ \mathbf{n} = J\ dA\ \mathbf{F}^{-T} \cdot \mathbf{N}$$

where da is an area of a region in the deformed configuration, dA is the same area in the reference configuration, and $\mathbf{n}$ is the outward normal to the area element in the current configuration while $\mathbf{N}$ is the outward normal in the reference configuration, $\mathbf{F}$ is the deformation gradient, and $J = \det \mathbf{F}$.

Polar decomposition of the deformation gradient tensor

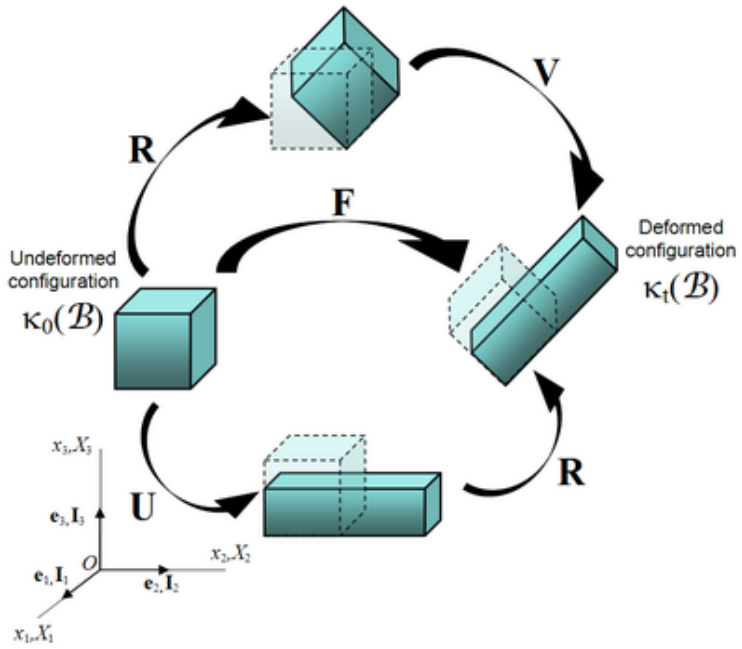


Figure 3. Representation of the polar decomposition of the deformation gradient

The deformation gradient $\mathbf{F}$, like any second-order tensor, can be decomposed, using the polar decomposition theorem, into a product of two second-order tensors (Truesdell and Noll, 1965): an orthogonal tensor and a positive definite symmetric tensor, i.e.

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

where the tensor $\mathbf{R}$is a proper orthogonal tensor, i.e. $\mathbf{R}^{-1} = \mathbf{R}^T$and $\det \mathbf{R} = +1$, representing a rotation; the tensor $\mathbf{U}$is the *right stretch tensor*; and $\mathbf{V}$the *left stretch tensor*. The terms *right* and *left* means that they are to the right and left of the rotation tensor $\mathbf{R}$, respectively. $\mathbf{U}$and $\mathbf{V}$are both positive definite, i.e. $\mathbf{x} \cdot \mathbf{U} \cdot \mathbf{x} \geq 0$and $\mathbf{x} \cdot \mathbf{V} \cdot \mathbf{x} \geq 0$, and symmetric tensors, i.e. $\mathbf{U} = \mathbf{U}^T$and $\mathbf{V} = \mathbf{V}^T$, of second order.

This decomposition implies that the deformation of a line element $d\mathbf{X}$in the undeformed configuration onto $d\mathbf{x}$in the deformed configuration, i.e. $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$, may be obtained either by first stretching the element by $\mathbf{U}$, i.e. $d\mathbf{x}' = \mathbf{U}\, d\mathbf{X}$, followed by a rotation $\mathbf{R}$, i.e. $d\mathbf{x} = \mathbf{R}\, d\mathbf{x}'$; or equivalently, by applying a rigid rotation $\mathbf{R}$first, i.e. $d\mathbf{x}' = \mathbf{R}\, d\mathbf{X}$, followed later by a stretching $\mathbf{V}$, i.e. $d\mathbf{x} = \mathbf{V}\, d\mathbf{x}'$(See Figure 3).

It can be shown that,

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T$$

so that $\mathbf{U}$and $\mathbf{V}$have the same eigenvalues or principal stretches, but different eigenvectors or *principal directions* $\mathbf{N}_i$and $\mathbf{n}_i$, respectively. The principal directions are related by

$$\mathbf{n}_i = \mathbf{R}\mathbf{N}_i.$$

This polar decomposition is unique as $\mathbf{F}$is non-symmetric.

Deformation tensors

Several rotation-independent deformation tensors are used in mechanics. In solid mechanics, the most popular of these are the right and left Cauchy-Green deformation tensors.

Since a pure rotation should not induce any stresses in a deformable body, it is often convenient to use rotation-independent measures of deformation in continuum mechanics. As a rotation followed by its inverse rotation leads to no change ($\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{1}$) we can exclude the rotation by multiplying $\mathbf{F}$by its transpose.

The Right Cauchy-Green deformation tensor

In 1839, George Green introduced a deformation tensor known as the *right Cauchy-Green deformation tensor* or *Green's deformation tensor*, defined as:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \qquad \text{or} \qquad C_{IJ} = F_{kI}\ F_{kJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}.$$

Physically, the Cauchy-Green tensor gives us the square of local change in distances due to deformation, i.e. $d\mathbf{x}^2 = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X}$

Invariants of $\mathbf{C}$are often used in the expressions for strain energy density functions. The most commonly used invariants are

$$I_1^C := \operatorname{tr}(\mathbf{C}) = C_{II} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2^C := \tfrac{1}{2}\left[(\operatorname{tr}\mathbf{C})^2 - \operatorname{tr}(\mathbf{C}^2)\right] = \tfrac{1}{2}\left[(C_{JJ})^2 - C_{IK}C_{KI}\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3^C := \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2.$$

The Finger deformation tensor

The IUPAC recommends that the inverse of the right Cauchy-Green deformation tensor, i. e., $\mathbf{C}^{-1}$, be called the **Finger tensor**. However, that nomenclature is not universally accepted in applied mechanics.

$$\mathbf{f} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T} \qquad \text{or} \qquad f_{IJ} = \frac{\partial X_I}{\partial x_k}\frac{\partial X_J}{\partial x_k}$$

The Left Cauchy-Green or Finger deformation tensor

Reversing the order of multiplication in the formula for the right Green-Cauchy deformation tensor leads to the *left Cauchy-Green deformation tensor* which is defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \qquad \text{or} \qquad B_{ij} = \frac{\partial x_i}{\partial X_K}\frac{\partial x_j}{\partial X_K}$$

The left Cauchy-Green deformation tensor *is often called the* Finger deformation tensor*, named after Josef Finger (1894).*

Invariants of $\mathbf{B}$are also used in the expressions for strain energy density functions. The conventional invariants are defined as

$$I_1 := \mathrm{tr}(\mathbf{B}) = B_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 := \tfrac{1}{2}\left[(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2)\right] = \tfrac{1}{2}\left(B_{ii}^2 - B_{jk}B_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 := \det\mathbf{B} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$$

where $J := \det\mathbf{F}$ is the determinant of the deformation gradient.

For nearly incompressible materials, a slightly different set of invariants is used:

$$(\bar{I}_1 := J^{-2/3}I_1\ ;\quad \bar{I}_2 := J^{-4/3}I_2\ ;\quad J = 1)\ .$$

The Cauchy deformation tensor

Earlier in 1828, Augustin Louis Cauchy introduced a deformation tensor defined as the inverse of the left Cauchy-Green deformation tensor, $\mathbf{B}^{-1}$. This tensor has also been called the **Piola tensor** and the **Finger tensor** in the rheology and fluid dynamics literature.

$$\mathbf{c} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} \qquad \text{or} \qquad c_{ij} = \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}$$

Spectral representation

If there are three distinct *principal stretches* λ_i, the spectral decompositions of $\mathbf{C}$ and $\mathbf{B}$ is given by

$$\mathbf{C} = \sum_{i=1}^{3}\lambda_i^2\mathbf{N}_i\otimes\mathbf{N}_i \qquad \text{and} \qquad \mathbf{B} = \sum_{i=1}^{3}\lambda_i^2\mathbf{n}_i\otimes\mathbf{n}_i$$

Furthermore,

$$\mathbf{U} = \sum_{i=1}^{3}\lambda_i\mathbf{N}_i\otimes\mathbf{N}_i\ ;\quad \mathbf{V} = \sum_{i=1}^{3}\lambda_i\mathbf{n}_i\otimes\mathbf{n}_i$$
$$\mathbf{R} = \sum_{i=1}^{3}\mathbf{n}_i\otimes\mathbf{N}_i\ ;\quad \mathbf{F} = \sum_{i=1}^{3}\lambda_i\mathbf{n}_i\otimes\mathbf{N}_i$$

Observe that

$$\mathbf{V} = \mathbf{R}\ \mathbf{U}\ \mathbf{R}^T = \sum_{i=1}^{3} \lambda_i\ \mathbf{R}\ (\mathbf{N}_i \otimes \mathbf{N}_i)\ \mathbf{R}^T = \sum_{i=1}^{3} \lambda_i\ (\mathbf{R}\ \mathbf{N}_i) \otimes (\mathbf{R}\ \mathbf{N}_i)$$

Therefore the uniqueness of the spectral decomposition also implies that $\mathbf{n}_i = \mathbf{R}\ \mathbf{N}_i$. The left stretch ($\mathbf{V}$) is also called the *spatial stretch tensor* while the right stretch ($\mathbf{U}$) is called the *material stretch tensor*.

The effect of $\mathbf{F}$ acting on $\mathbf{N}_i$ is to stretch the vector by λ_i and to rotate it to the new orientation $\mathbf{n}_i$, i.e.,

$$\mathbf{F}\ \mathbf{N}_i = \lambda_i\ (\mathbf{R}\ \mathbf{N}_i) = \lambda_i\ \mathbf{n}_i$$

In a similar vein,

$$\mathbf{F}^{-T}\ \mathbf{N}_i = \frac{1}{\lambda_i}\ \mathbf{n}_i\ ;\quad \mathbf{F}^T\ \mathbf{n}_i = \lambda_i\ \mathbf{N}_i\ ;\quad \mathbf{F}^{-1}\ \mathbf{n}_i = \frac{1}{\lambda_i}\ \mathbf{N}_i\ .$$

Derivatives of stretch

Derivatives of the stretch with respect to the right Cauchy-Green deformation tensor are used to derive the stress-strain relations of many solids, particularly hyperelastic materials. These derivatives are

$$\frac{\partial \lambda_i}{\partial \mathbf{C}} = \frac{1}{2\lambda_i}\ \mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{2\lambda_i}\ \mathbf{R}^T\ (\mathbf{n}_i \otimes \mathbf{n}_i)\ \mathbf{R}\ ;\quad i = 1, 2, 3$$

and follow from the observations that

$$\mathbf{C} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \lambda_i^2\ ;\qquad \frac{\partial \mathbf{C}}{\partial \mathbf{C}} = \mathsf{I}^{(s)}\ ;\qquad \mathsf{I}^{(s)} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \mathbf{N}_i \otimes \mathbf{N}_i .$$

Physical interpretation of deformation tensors

Let $\mathbf{X} = X^i\ \boldsymbol{E}_i$ be a Cartesian coordinate system defined on the undeformed body and let $\mathbf{x} = x^i\ \boldsymbol{E}_i$ be another system defined on the deformed body. Let a curve $\mathbf{X}(s)$ in the undeformed body be parametrized using $s \in [0, 1]$. Its image in the deformed body is $\mathbf{x}(\mathbf{X}(s))$.

The undeformed length of the curve is given by

$$l_X = \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \frac{d\mathbf{X}}{ds} \right| \, ds = \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \boldsymbol{I} \cdot \frac{d\mathbf{X}}{ds} \right| \, ds$$

After deformation, the length becomes

$$\begin{aligned} l_x &= \int_0^1 \left| \frac{d\mathbf{x}}{ds} \cdot \frac{d\mathbf{x}}{ds} \right| \, ds = \int_0^1 \left| \left(\frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{X}}{ds} \right) \cdot \left(\frac{d\mathbf{x}}{d\mathbf{X}} \cdot \frac{d\mathbf{X}}{ds} \right) \right| \, ds \\ &= \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \left[\left(\frac{d\mathbf{x}}{d\mathbf{X}} \right)^T \cdot \frac{d\mathbf{x}}{d\mathbf{X}} \right] \cdot \frac{d\mathbf{X}}{ds} \right| \, ds \end{aligned}$$

Note that the right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C} := \boldsymbol{F}^T \cdot \boldsymbol{F} = \left(\frac{d\mathbf{x}}{d\mathbf{X}} \right)^T \cdot \frac{d\mathbf{x}}{d\mathbf{X}}$$

Hence,

$$l_x = \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \boldsymbol{C} \cdot \frac{d\mathbf{X}}{ds} \right| \, ds$$

which indicates that changes in length are characterized by $\boldsymbol{C}$.

Finite strain tensors

The concept of *strain* is used to evaluate how much a given displacement differs locally from a rigid body displacement (Ref. Lubliner). One of such strains for large deformations is the *Lagrangian finite strain tensor*, also called the *Green-Lagrangian strain tensor* or *Green - St-Venant strain tensor*, defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \qquad \text{or} \qquad E_{KL} = \frac{1}{2} \left(\frac{\partial x_j}{\partial X_K} \frac{\partial x_j}{\partial X_L} - \delta_{KL} \right)$$

or as a function of the displacement gradient tensor

$$\mathbf{E} = \frac{1}{2} \left[(\nabla_{\mathbf{X}} \mathbf{u})^T + \nabla_{\mathbf{X}} \mathbf{u} + (\nabla_{\mathbf{X}} \mathbf{u})^T \cdot \nabla_{\mathbf{X}} \mathbf{u} \right]$$

or

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K}\frac{\partial U_M}{\partial X_L}\right)$$

The Green-Lagrangian strain tensor is a measure of how much $\mathbf{C}$differs from $\mathbf{I}$. It can be shown that this tensor is a special case of a general formula for Lagrangian strain tensors (Hill 1968):

$$\mathbf{E}_{(m)} = \frac{1}{2m}(\mathbf{U}^{2m} - \mathbf{I})$$

For different values of $m$we have:

$$\mathbf{E}_{(1)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}) \qquad \text{Green-Lagrangian strain tensor}$$
$$\mathbf{E}_{(1/2)} = (\mathbf{U} - \mathbf{I}) \qquad \text{Biot strain tensor}$$
$$\mathbf{E}_{(0)} = \ln \mathbf{U} \qquad \text{Logarithmic strain, Natural strain, True strain, or Hencky strain}$$

The *Eulerian-Almansi finite strain tensor*, referenced to the deformed configuration, i.e. Eulerian description, is defined as

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{c}) \qquad \text{or} \qquad e_{rs} = \frac{1}{2}\left(\delta_{rs} - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\right)$$

or as a function of the displacement gradients we have

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Stretch ratio

The **stretch ratio** is a measure of the extensional or normal strain of a differential line element, which can be defined at either the undeformed configuration or the deformed configuration.

The stretch ratio for the differential element $d\mathbf{X} = dX\mathbf{N}$(Figure) in the direction of the unit vector $\mathbf{N}$at the material point P, in the undeformed configuration, is defined as

$$\Lambda_{(\mathbf{N})} = \frac{dx}{dX}$$

where dx is the deformed magnitude of the differential element $d\mathbf{X}$.

Similarly, the stretch ratio for the differential element $d\mathbf{x} = dx\mathbf{n}$ (Figure), in the direction of the unit vector $\mathbf{n}$ at the material point p, in the deformed configuration, is defined as

$$\frac{1}{\Lambda_{(\mathbf{n})}} = \frac{dX}{dx}.$$

The normal strain $e_{\mathbf{N}}$ in any direction $\mathbf{N}$ can be expressed as a function of the stretch ratio,

$$e_{(\mathbf{N})} = \frac{dx - dX}{dX} = \Lambda_{(\mathbf{N})} - 1.$$

This equation implies that the normal strain is zero, i.e. no deformation, when the stretch is equal to unity. Some materials, such as elastometers can sustain stretch ratios of 3 or 4 before they fail, whereas traditional engineering materials, such as concrete or steel, fail at much lower stretch ratios, perhaps of the order of 1.001 (reference?)

Physical interpretation of the finite strain tensor

The diagonal components E_{KL} of the Lagrangian finite strain tensor are related to the normal strain, e.g.

$$E_{11} = e_{(\mathbf{I}_1)} + \frac{1}{2}e^2_{(\mathbf{I}_1)}$$

where $e_{(\mathbf{I}_1)}$ is the normal strain or engineering strain in the direction $\mathbf{I}_1$.

The off-diagonal components E_{KL} of the Lagrangian finite strain tensor are related to shear strain, e.g.

$$E_{12} = \frac{1}{2}\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}\sin\phi_{12}$$

where ϕ_{12} is the change in the angle between two line elements that were originally perpendicular with directions $\mathbf{I}_1$ and $\mathbf{I}_2$, respectively.

Under certain circumstances, i.e. small displacements and small displacement rates, the components of the Lagrangian finite strain tensor may be approximated by the components of the infinitesimal strain tensor

Deformation tensors in curvilinear coordinates

A representation of deformation tensors in curvilinear coordinates is useful for many problems in continuum mechanics such as nonlinear shell theories and large plastic deformations. Let $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, \xi^3)$ be a given deformation where the space is characterized by the coordinates (ξ^1,ξ^2,ξ^3). The tangent vector to the coordinate curve ξ^i at $\mathbf{x}$ is given by

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$$

The three tangent vectors at $\mathbf{x}$ form a basis. These vectors are related the reciprocal basis vectors by

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

Let us define a field

$$g_{ij} := \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \mathbf{g}_i \cdot \mathbf{g}_j$$

The Christoffel symbols of the first kind can be expressed as

$$\Gamma_{ijk} = \tfrac{1}{2}[(\mathbf{g}_i \cdot \mathbf{g}_k)_{,j} + (\mathbf{g}_j \cdot \mathbf{g}_k)_{,i} - (\mathbf{g}_i \cdot \mathbf{g}_j)_{,k}]$$

To see how the Christoffel symbols are related to the Right Cauchy-Green deformation tensor let us define two sets of bases

$$\mathbf{G}_i := \frac{\partial \mathbf{X}}{\partial \xi^i} \ ; \quad \mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j \ ; \quad \mathbf{g}_i := \frac{\partial \mathbf{x}}{\partial \xi^i} \ ; \quad \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

The deformation gradient in curvilinear coordinates

Using the definition of the gradient of a vector field in curvilinear coordinates, the deformation gradient can be written as

$$\boldsymbol{F} = \boldsymbol{\nabla}_{\mathbf{X}} \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^i} \otimes \mathbf{G}^i = \mathbf{g}_i \otimes \mathbf{G}^i$$

The right Cauchy-Green tensor in curvilinear coordinates

The right Cauchy-Green deformation tensor is given by

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = (\mathbf{G}^i \otimes \mathbf{g}_i) \cdot (\mathbf{g}_i \otimes \mathbf{G}^i) = (\mathbf{g}_i \cdot \mathbf{g}_j)(\mathbf{G}^i \otimes \mathbf{G}^j)$$

If we express $\boldsymbol{C}$ in terms of components with respect to the basis $\{\mathbf{G}^i\}$ we have

$$\boldsymbol{C} = C_{ij}\ \mathbf{G}^i \otimes \mathbf{G}^j$$

Therefore

$$C_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}$$

and the Christoffel symbol of the first kind may be written in the following form.

$\Gamma_{ijk} = \frac{1}{2}[C_{ik,j} + C_{jk,i} - C_{ij,k}] = \frac{1}{2}[(\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,j} + (\mathbf{G}_j \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,i} - (\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_j)_{,k}]$

Some relations between deformation measures and Christoffel symbols

Let us consider a one-to-one mapping from $\mathbf{X} = \{X^1, X^2, X^3\}$to $\mathbf{x} = \{x^1, x^2, x^3\}$and let us assume that there exist two positive definite, symmetric second-order tensor fields $\boldsymbol{G}$and $\boldsymbol{g}$that satisfy

$$G_{ij} = \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial X^\beta}{\partial x^j}\ g_{\alpha\beta}$$

Then,

$$\frac{\partial G_{ij}}{\partial x^k} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\ \frac{\partial X^\beta}{\partial x^j} + \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial^2 X^\beta}{\partial x^j \partial x^k}\right)\ g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial X^\beta}{\partial x^j}\ \frac{\partial g_{\alpha\beta}}{\partial x^k}$$

Noting that

$$\frac{\partial g_{\alpha\beta}}{\partial x^k} = \frac{\partial X^\gamma}{\partial x^k}\ \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

and $g_{\alpha\beta} = g_{\beta\alpha}$ we have

$$\frac{\partial G_{ij}}{\partial x^k} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\ \frac{\partial X^\beta}{\partial x^j} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}\ \frac{\partial X^\beta}{\partial x^i}\right)\ g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial X^\beta}{\partial x^j}\ \frac{\partial X^\gamma}{\partial x^k}\ \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

$$\frac{\partial G_{ik}}{\partial x^j} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\ \frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}\ \frac{\partial X^\beta}{\partial x^i}\right)\ g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial X^\beta}{\partial x^k}\ \frac{\partial X^\gamma}{\partial x^j}\ \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

$$\frac{\partial G_{jk}}{\partial x^i} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\ \frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\ \frac{\partial X^\beta}{\partial x^j}\right)\ g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^j}\ \frac{\partial X^\beta}{\partial x^k}\ \frac{\partial X^\gamma}{\partial x^i}\ \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

Define

$$
\begin{aligned}
{}_{(x)}\Gamma_{ijk} &:= \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right) \\
{}_{(X)}\Gamma_{\alpha\beta\gamma} &:= \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}\right)
\end{aligned}
$$

Hence

$${}_{(x)}\Gamma_{ijk} = \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta}$$

Define

$$[G^{ij}] = [G_{ij}]^{-1} \;;\quad [g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$$

Then

$$G^{ij} = \frac{\partial x^i}{\partial X^\alpha}\,\frac{\partial x^j}{\partial X^\beta}\,g^{\alpha\beta}$$

Define the Christoffel symbols of the second kind as

$${}_{(x)}\Gamma^m_{ij} := G^{mk}\,{}_{(x)}\Gamma_{ijk} \;;\quad {}_{(X)}\Gamma^\nu_{\alpha\beta} := g^{\nu\gamma}\,{}_{(X)}\Gamma_{\alpha\beta\gamma}$$

Then

$$
\begin{aligned}
{}_{(x)}\Gamma^m_{ij} &= G^{mk}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + G^{mk}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial x^k}{\partial X^\rho}\,g^{\nu\rho}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial x^k}{\partial X^\rho}\,g^{\nu\rho}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\delta^\gamma_\rho\,g^{\nu\rho}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,\delta^\beta_\rho\,g^{\nu\rho}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,g^{\nu\gamma}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu}\,g^{\nu\beta}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\nu}\,\delta^\nu_\alpha\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}
\end{aligned}
$$

Therefore

$${}_{(x)}\Gamma^m_{ij} = \frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\alpha}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}$$

The invertibility of the mapping implies that

$$\begin{aligned}\frac{\partial X^\mu}{\partial x^m}\,{}_{(x)}\Gamma^m_{ij} &= \frac{\partial X^\mu}{\partial x^m}\,\frac{\partial x^m}{\partial X^\nu}\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial X^\mu}{\partial x^m}\,\frac{\partial x^m}{\partial X^\alpha}\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\ &= \delta^\mu_\nu\,\frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\nu_{\alpha\beta} + \delta^\mu_\alpha\,\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\ &= \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} + \frac{\partial^2 X^\mu}{\partial x^i \partial x^j}\end{aligned}$$

We can also formulate a similar result in terms of derivatives with respect to *x*. Therefore

$$\begin{aligned}\frac{\partial^2 X^\mu}{\partial x^i \partial x^j} &= \frac{\partial X^\mu}{\partial x^m}\,{}_{(x)}\Gamma^m_{ij} - \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} \\ \frac{\partial^2 x^m}{\partial X^\alpha \partial X^\beta} &= \frac{\partial x^m}{\partial X^\mu}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - \frac{\partial x^i}{\partial X^\alpha}\,\frac{\partial x^j}{\partial X^\beta}\,{}_{(x)}\Gamma^m_{ij}\end{aligned}$$

Compatibility conditions

The problem of compatibility in continuum mechanics involves the determination of allowable single-valued continuous fields on bodies. These allowable conditions leave the body without unphysical gaps or overlaps after a deformation. Most such conditions apply to simply-connected bodies. Additional conditions are required for the internal boundaries of multiply connected bodies.

Compatibility of the deformation gradient

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{F}$ field over a simply connected body are

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$$

Compatibility of the right Cauchy-Green deformation tensor

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{C}$ field over a simply connected body are

$$R^\gamma_{\alpha\beta\rho} := \frac{\partial}{\partial X^\rho}[\,{}_{(X)}\Gamma^\gamma_{\alpha\beta}] - \frac{\partial}{\partial X^\beta}[\,{}_{(X)}\Gamma^\gamma_{\alpha\rho}] + {}_{(X)}\Gamma^\gamma_{\mu\rho}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - {}_{(X)}\Gamma^\gamma_{\mu\beta}\,{}_{(X)}\Gamma^\mu_{\alpha\rho} = 0$$

We can show these are the mixed components of the Riemann-Christoffel curvature tensor. Therefore the necessary conditions for $\boldsymbol{C}$ -compatibility are that the Riemann-Christoffel curvature of the deformation is zero.

Compatibility of the left Cauchy-Green deformation tensor

No general sufficiency conditions are known for the left Cauchy-Green deformation tensor in three-dimensions. Compatibility conditions for two-dimensional $\boldsymbol{B}$fields have been found by Janet Blume.