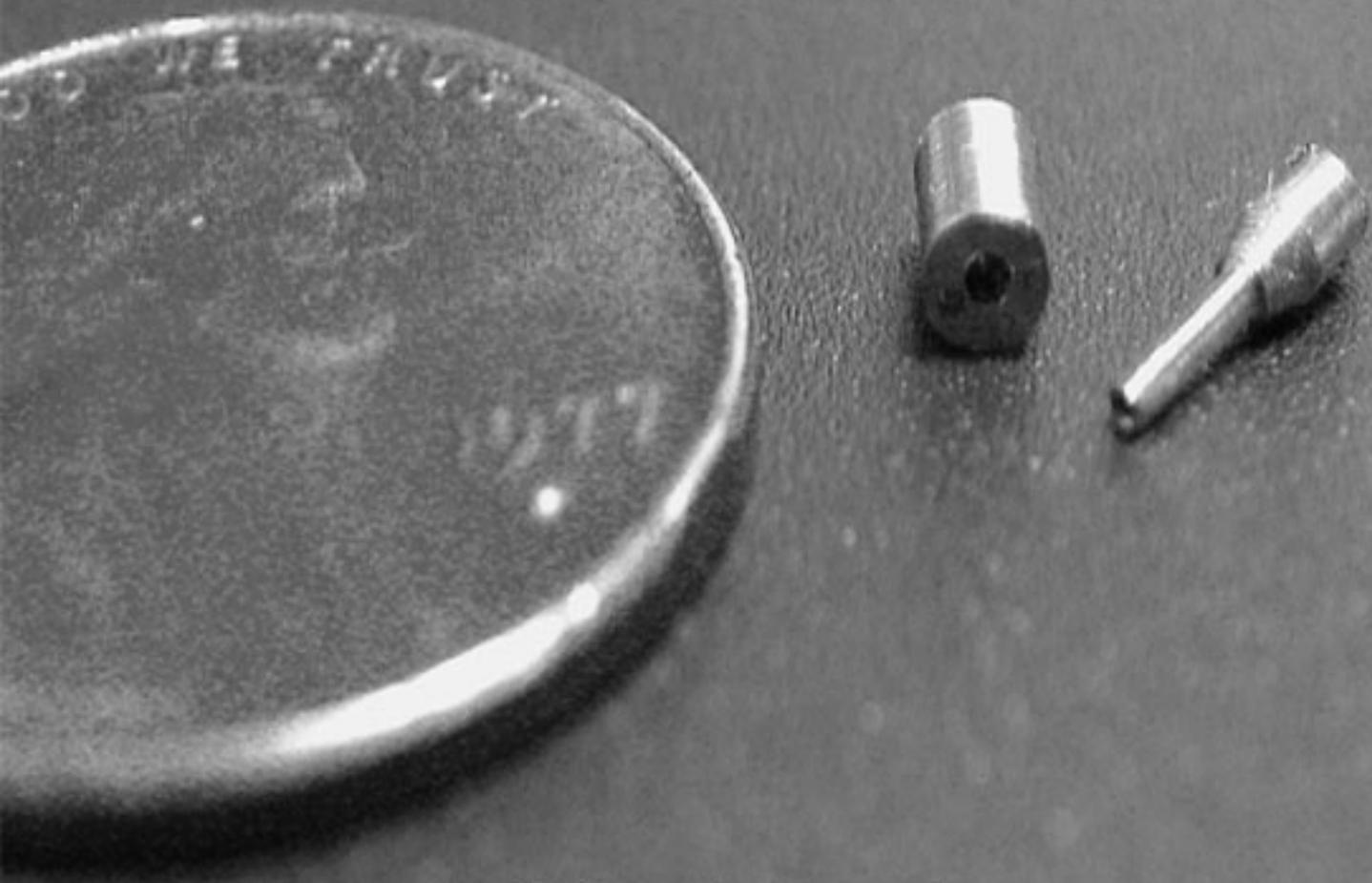


Yield Engineering



Clifford Boudreau

First Edition, 2012

ISBN 978-81-323-4413-1

© All rights reserved.

Published by:

White Word Publications

4735/22 Prakashdeep Bldg,

Ansari Road, Darya Ganj,

Delhi - 110002

Email: info@wtbooks.com

Table of Contents

Chapter 1 - Yield (Engineering)

Chapter 2 - Von Mises Yield Criterion

Chapter 3 - Stress (Mechanics)

Chapter 4 - Stress Concentration

Chapter 5 - Yield Surface

Chapter 6 - Infinitesimal Strain Theory

Chapter 7 - Ultimate Tensile Strength

Chapter 1

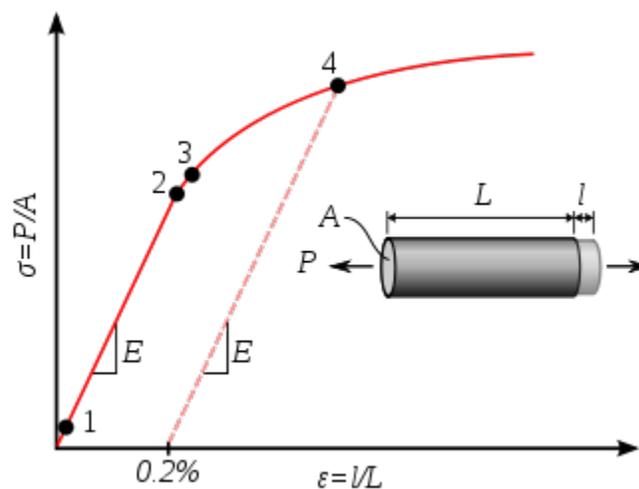
Yield (Engineering)

The **yield strength** or **yield point** of a material is defined in engineering and materials science as the stress at which a material begins to deform plastically. Prior to the yield point the material will deform elastically and will return to its original shape when the applied stress is removed. Once the yield point is passed some fraction of the deformation will be permanent and non-reversible.

In the three-dimensional space of the principal stresses ($\sigma_1, \sigma_2, \sigma_3$), an infinite number of yield points form together a yield surface.

Knowledge of the yield point is vital when designing a component since it generally represents an upper limit to the load that can be applied. It is also important for the control of many materials production techniques such as forging, rolling, or pressing. In structural engineering, this is a soft failure mode which does not normally cause catastrophic failure or ultimate failure unless it accelerates buckling.

Definition



Typical yield behavior for non-ferrous alloys.

1: True elastic limit

- 2: Proportionality limit
- 3: Elastic limit
- 4: Offset yield strength

It is often difficult to precisely define yielding due to the wide variety of stress–strain curves exhibited by real materials. In addition, there are several possible ways to define yielding:

True elastic limit

The lowest stress at which dislocations move. This definition is rarely used, since dislocations move at very low stresses, and detecting such movement is very difficult.

Proportionality limit

Up to this amount of stress, stress is proportional to strain (Hooke's law), so the stress-strain graph is a straight line, and the gradient will be equal to the elastic modulus of the material.

Elastic limit (yield strength)

Beyond the elastic limit, permanent deformation will occur. The lowest stress at which permanent deformation can be measured. This requires a manual load-unload procedure, and the accuracy is critically dependent on equipment and operator skill. For elastomers, such as rubber, the elastic limit is much larger than the proportionality limit. Also, precise strain measurements have shown that plastic strain begins at low stresses.

Yield point

The point in the stress-strain curve at which the curve levels off and plastic deformation begins to occur.

Offset yield point (proof stress)

When a yield point is not easily defined based on the shape of the stress-strain curve an *offset yield point* is arbitrarily defined. The value for this is commonly set at 0.1 or 0.2% of the strain. The offset value is given as a subscript, e.g., $R_{p0.2}=310$ MPa. High strength steel and aluminum alloys do not exhibit a yield point, so this offset yield point is used on these materials.

Upper yield point and lower yield point

Some metals, such as mild steel, reach an upper yield point before dropping rapidly to a lower yield point. The material response is linear up until the upper yield point, but the lower yield point is used in structural engineering as a conservative value. If a metal is only stressed to the upper yield point, and beyond, luders bands can develop.

Yield criterion

A yield criterion, often expressed as yield surface, or yield locus, is a hypothesis concerning the limit of elasticity under any combination of stresses. There are two interpretations of yield criterion: one is purely mathematical in taking a statistical approach while other models attempt to provide a justification based on established physical principles. Since stress and strain are tensor qualities they can be described on

the basis of three principal directions, in the case of stress these are denoted by σ_1 , σ_2 , and σ_3 .

The following represent the most common yield criterion as applied to an isotropic material (uniform properties in all directions). Other equations have been proposed or are used in specialist situations.

Isotropic yield criteria

Maximum Principal Stress Theory - Yield occurs when the largest principal stress exceeds the uniaxial tensile yield strength. Although this criterion allows for a quick and easy comparison with experimental data it is rarely suitable for design purposes.

$$\sigma_1 \leq \sigma_y$$

Maximum Principal Strain Theory - Yield occurs when the maximum principal strain reaches the strain corresponding to the yield point during a simple tensile test. In terms of the principal stresses this is determined by the equation:

$$\sigma_1 - \nu(\sigma_2 + \sigma_3) \leq \sigma_y.$$

Maximum Shear Stress Theory - Also known as the Tresca yield criterion, after the French scientist Henri Tresca. This assumes that yield occurs when the shear stress τ exceeds the shear yield strength τ_y :

$$\tau = \frac{\sigma_1 - \sigma_3}{2} \leq \tau_{ys}.$$

Total Strain Energy Theory - This theory assumes that the stored energy associated with elastic deformation at the point of yield is independent of the specific stress tensor. Thus yield occurs when the strain energy per unit volume is greater than the strain energy at the elastic limit in simple tension. For a 3-dimensional stress state this is given by:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3) \leq \sigma_y^2.$$

Distortion Energy Theory - This theory proposes that the total strain energy can be separated into two components: the *volumetric* (hydrostatic) strain energy and the *shape* (distortion or shear) strain energy. It is proposed that yield occurs when the distortion component exceeds that at the yield point for a simple tensile test. This is generally referred to as the Von Mises yield criterion and is expressed as:

$$\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \leq \sigma_y^2.$$

Based on a different theoretical underpinning this expression is also referred to as **octahedral shear stress theory**.

Other commonly used isotropic yield criteria are the

- Mohr-Coulomb yield criterion
- Drucker-Prager yield criterion
- Bresler-Pister yield criterion
- Willam-Warnke yield criterion

The yield surfaces corresponding to these criteria have a range of forms. However, most isotropic yield criteria correspond to convex yield surfaces.

Anisotropic yield criteria

When a metal is subjected to large plastic deformations the grain sizes and orientations change in the direction of deformation. As a result the plastic yield behavior of the material shows directional dependency. Under such circumstances, the isotropic yield criteria such as the von Mises yield criterion are unable to predict the yield behavior accurately. Several anisotropic yield criteria have been developed to deal with such situations. Some of the more popular anisotropic yield criteria are:

- Hill's quadratic yield criterion.
- Generalized Hill yield criterion.
- Hosford yield criterion.

Factors influencing yield stress

The stress at which yield occurs is dependent on both the rate of deformation (strain rate) and, more significantly, the temperature at which the deformation occurs. Early work by Alder and Philips in 1954 found that the relationship between yield stress and strain rate (at constant temperature) was best described by a power law relationship of the form

$$\sigma_y = C(\dot{\epsilon})^m$$

where C is a constant and m is the strain rate sensitivity. The latter generally increases with temperature, and materials where m reaches a value greater than ~0.5 tend to exhibit super plastic behaviour.

Later, more complex equations were proposed that simultaneously dealt with both temperature and strain rate:

$$\sigma_y = \frac{1}{\alpha} \sinh^{-1} \left[\frac{Z}{A} \right]^{(1/n)}$$

where α and A are constants and Z is the temperature-compensated strain-rate - often described by the Zener-Hollomon parameter:

$$Z = (\dot{\epsilon}) \exp\left(\frac{Q_{HW}}{RT}\right)$$

where Q_{HW} is the activation energy for hot deformation and T is the absolute temperature.

Strengthening mechanisms

There are several ways in which crystalline and amorphous materials can be engineered to increase their yield strength. By altering dislocation density, impurity levels, grain size (in crystalline materials), the yield strength of the material can be fine tuned. This occurs typically by introducing defects such as impurities dislocations in the material. To move this defect (plastically deforming or yielding the material), a larger stress must be applied. This thus causes a higher yield stress in the material. While many material properties depend only on the composition of the bulk material, yield strength is extremely sensitive to the materials processing as well for this reason.

These mechanisms for crystalline materials include

- Work Hardening
- Solid Solution Strengthening
- Particle/Precipitate Strengthening
- Grain boundary strengthening

Work Hardening

Where deforming the material will introduce dislocations, which increases their density in the material. This increases the yield strength of the material, since now more stress must be applied to move these dislocations through a crystal lattice. Dislocations can also interact with each other, becoming entangled.

The governing formula for this mechanism is:

$$\Delta\sigma_y = Gb\sqrt{\rho}$$

where σ_y is the yield stress, G is the shear elastic modulus, b is the magnitude of the Burgers vector, and ρ is the dislocation density.

Solid Solution Strengthening

By alloying the material, impurity atoms in low concentrations will occupy a lattice position directly below a dislocation, such as directly below an extra half plane defect.

This relieves a tensile strain directly below the dislocation by filling that empty lattice space with the impurity atom.

The relationship of this mechanism goes as:

$$\Delta\tau = Gb\sqrt{C_s}\epsilon^{3/2}$$

where τ is the shear stress, related to the yield stress, G and b are the same as in the above example, C_s is the concentration of solute and ϵ is the strain induced in the lattice due to adding the impurity.

Particle/Precipitate Strengthening

Where the presence of a secondary phase will increase yield strength by blocking the motion of dislocations within the crystal. A line defect that, while moving through the matrix, will be forced against a small particle or precipitate of the material. Dislocations can move through this particle either by shearing the particle, or by a process known as bowing or ringing, in which a new ring of dislocations is created around the particle.

The shearing formula goes as:

$$\Delta\tau = \frac{r_{\text{particle}}}{l_{\text{interparticle}}}\gamma_{\text{particle-matrix}}$$

and the bowing/ringing formula:

$$\Delta\tau = \frac{Gb}{l_{\text{interparticle}} - 2r_{\text{particle}}}$$

In these formulas, r_{particle} is the particle radius, $\gamma_{\text{particle-matrix}}$ is the surface tension between the matrix and the particle, $l_{\text{interparticle}}$ is the distance between the particles.

Grain boundary strengthening

Where a buildup of dislocations at a grain boundary causes a repulsive force between dislocations. As grain size decreases, the surface area to volume ratio of the grain increases, allowing more buildup of dislocations at the grain edge. Since it requires a lot of energy to move dislocations to another grain, these dislocations build up along the boundary, and increase the yield stress of the material. Also known as Hall-Petch strengthening, this type of strengthening is governed by the formula:

$$\sigma_y = \sigma_0 + kd^{-1/2}$$

where

σ_0 is the stress required to move dislocations,
 k is a material constant, and
 d is the grain size.

Testing

Yield strength testing involves taking a small sample with a fixed cross-section area, and then pulling it with a controlled, gradually increasing force until the sample changes shape or breaks. Longitudinal and/or transverse strain is recorded using mechanical or optical extensometers.

Indentation hardness correlates linearly with tensile strength for most steels. Hardness testing can therefore be an economical substitute for tensile testing, as well as providing local variations in yield strength due to e.g. welding or forming operations.

Implications for structural engineering

Yielded structures have a lower stiffness, leading to increased deflections and decreased buckling strength. The structure will be permanently deformed when the load is removed, and may have residual stresses. Engineering metals display strain hardening, which implies that the yield stress is increased after unloading from a yield state. Highly optimized structures, such as airplane beams and components, rely on yielding as a fail-safe failure mode. No safety factor is therefore needed when comparing limit loads (the highest loads expected during normal operation) to yield criteria.

Typical yield and ultimate strengths

Note: many of the values depend on manufacturing process and purity/composition.

Material	Yield strength (MPa)	Ultimate strength (MPa)	Density (g/cm³)	free breaking length (km)
ASTM A36 steel	250	400	7.8	3.2
Steel, API 5L X65	448	531	7.8	5.8
Steel, high strength alloy ASTM A514	690	760	7.8	9.0
Steel, prestressing strands	1650	1860	7.8	21.6
Piano wire		2200–2482	7.8	28.7
Carbon Fiber (CF, CFK)		5650	1.75	
High density polyethylene (HDPE)	26-33	37	0.95	2.8
Polypropylene	12-43	19.7-80	0.91	1.3
Stainless steel AISI 302 -	520	860		

Cold-rolled				
Cast iron 4.5% C, ASTM A-48	*	172	7.20	2.4
Titanium alloy (6% Al, 4% V)	830	900	4.51	18.8
Aluminium alloy 2014-T6	400	455	2.7	15.1
Copper 99.9% Cu	70	220	8.92	0.8
Cupronickel 10% Ni, 1.6% Fe, 1% Mn, balance Cu	130	350	8.94	1.4
Brass	approx. 200+	550	5.3	3.8
Spider silk	1150 (??)	1200		50
Silkworm silk	500			25
Aramid (Kevlar or Twaron)	3620		1.44	256.3
UHMWPE	21	46	0.97	240
Bone (limb)	104-121	130		3
Nylon, type 6/6	45	75		2

* Grey cast iron does not have a well defined yield strength because the stress-strain relationship is atypical. The yield strength can vary from 65 to 80% of the tensile strength.

Elements in the annealed state			
	Young's modulus (GPa)	Proof or yield stress (MPa)	Ultimate strength (MPa)
Aluminium	70	15-20	40-50
Copper	130	33	210
Iron	211	80-100	350
Nickel	170	14-35	140-195
Silicon	107	5000-9000	
Tantalum	186	180	200
Tin	47	9-14	15-200
Titanium	120	100-225	240-370
Tungsten	411	550	550-620

Chapter 2

Von Mises Yield Criterion

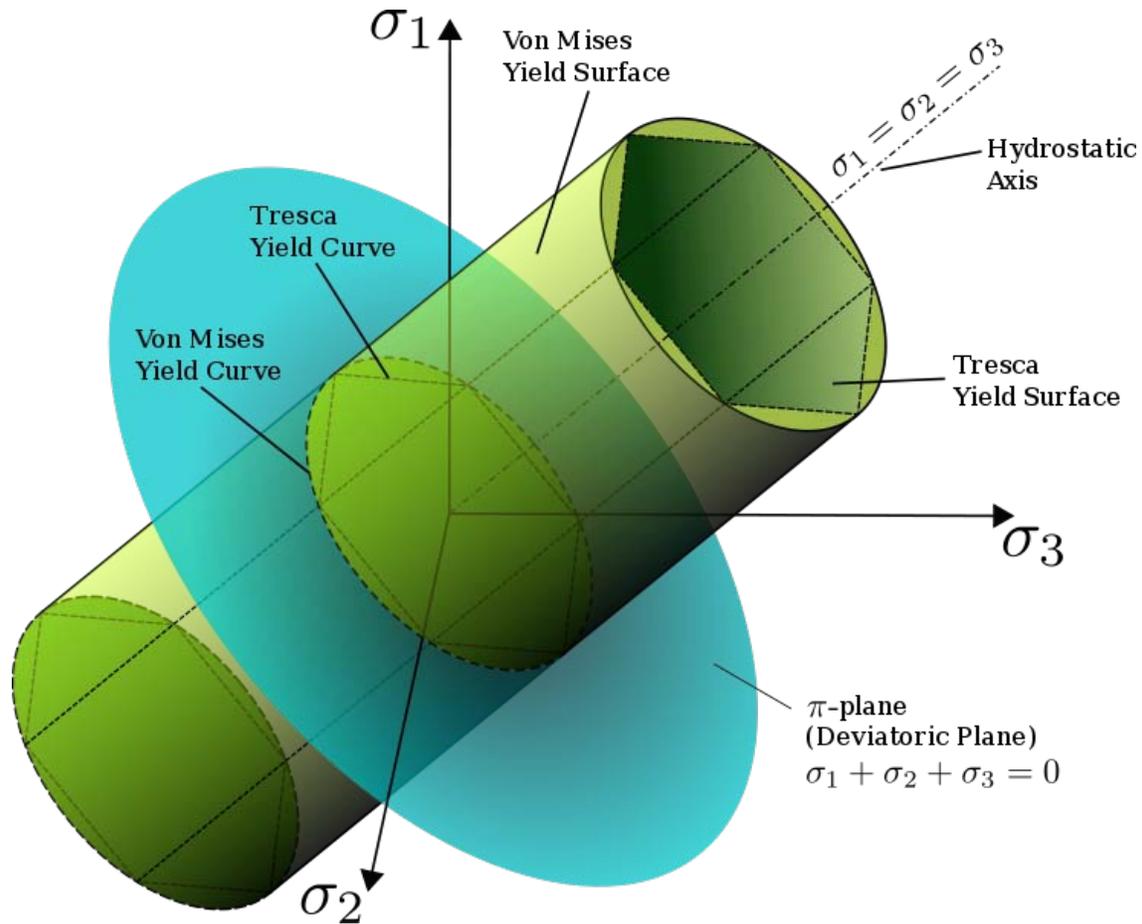
The **Von Mises yield criterion** suggests that the yielding of materials begins when the second deviatoric stress invariant J_2 reaches a critical value k . For this reason, it is sometimes called the *J₂-plasticity* or *J₂ flow theory*. It is part of a plasticity theory that applies best to ductile materials, such as metals. Prior to yield, material response is assumed to be elastic.

In materials science and engineering the von Mises yield criterion can be also formulated in terms of the **von Mises stress** or **equivalent tensile stress**, σ_v , a scalar stress value that can be computed from the stress tensor. In this case, a material is said to start yielding when its von Mises stress reaches a critical value known as the yield strength, σ_y . The von Mises stress is used to predict yielding of materials under any loading condition from results of simple uniaxial tensile tests. The von Mises stress satisfies the property that two stress states with equal distortion energy have equal von Mises stress.

Because the von Mises yield criterion is independent of the first stress invariant, I_1 , it is applicable for the analysis of plastic deformation for ductile materials such as metals, as the onset of yield for these materials does not depend on the hydrostatic component of the stress tensor.

Although formulated by Maxwell in 1865, it is generally attributed to Richard Edler von Mises (1913). Tytus Maksymilian Huber (1904), in a paper in Polish, anticipated to some extent this criterion. This criterion is also referred to as the Maxwell–Huber–Hencky–von Mises theory.

Mathematical formulation



The von Mises yield surfaces in principal stress coordinates circumscribes a cylinder with radius $\sqrt{\frac{2}{3}}\sigma_y$ around the hydrostatic axis. Also shown is Tresca's hexagonal yield surface.

Mathematically the yield function for the von Mises condition is expressed as:

$$f(J_2) = \sqrt{J_2} - k = 0$$

An alternative form is:

$$f(J_2) = J_2 - k^2 = 0$$

where k can be shown to be the yield stress of the material in pure shear. As it will become evident later, at the onset of yielding, the magnitude of the shear yield stress in pure shear is $\frac{1}{\sqrt{3}}$ times lower than the tensile yield stress in the case of simple tension. Thus, we have

$$k = \frac{\sigma_y}{\sqrt{3}}$$

Furthermore, if we define the von Mises stress as $\sigma_v = \sqrt{3J_2}$, the von Mises yield criterion can be expressed as:

$$\begin{aligned} f(J_2) &= \sqrt{3J_2} - \sigma_y \\ &= \sigma_v - \sigma_y = 0 \end{aligned}$$

Substituting J_2 in terms of the principal stresses into the von Mises criterion equation we have

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 6k^2 = 2\sigma_y^2$$

or

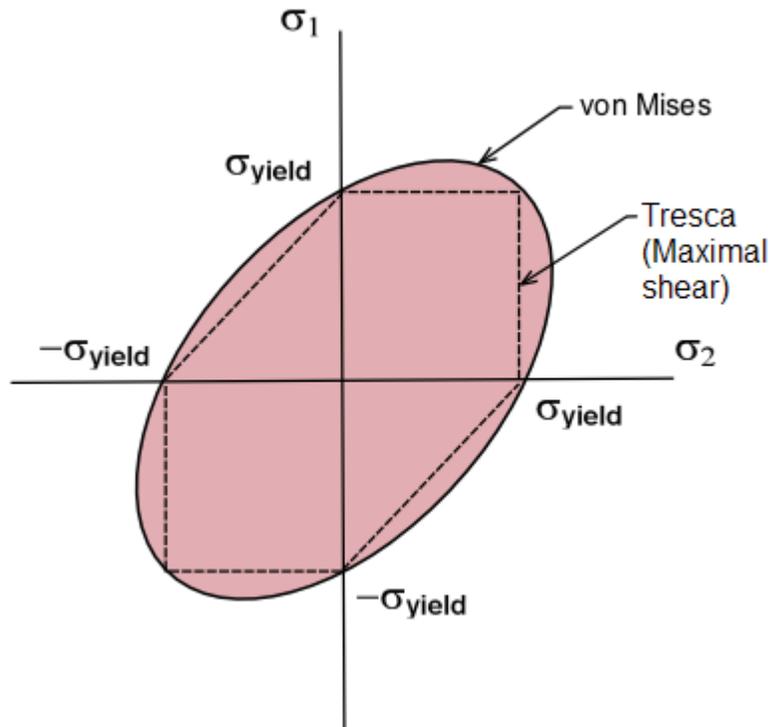
$$(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \sigma_1\sigma_2 - \sigma_2\sigma_3 - \sigma_1\sigma_3 = 3k^2 = \sigma_y^2$$

or as a function of the stress tensor components

$$(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2) = 6k^2 = 2\sigma_y^2$$

This equation defines the yield surface as a circular cylinder whose yield curve, or intersection with the deviatoric plane, is a circle with radius $\sqrt{2}k$, or $\sqrt{\frac{2}{3}}\sigma_y$. This implies that the yield condition is independent of hydrostatic stresses.

Von Mises criterion for different stress conditions



Projection of the von Mises yield criterion into the σ_1, σ_2 plane

In the case of **uniaxial stress** or **simple tension**, $\sigma_1 \neq 0$, $\sigma_3 = \sigma_2 = 0$, the von Mises criterion reduces to

$$\sigma_1 = \sigma_y.$$

Therefore, the material starts to yield, when σ_1 reaches the *yield strength* of the material σ_y , which is a characteristic material property. In practice, this parameter is indeed determined in a tensile test satisfying the uniaxial stress condition.

It is also convenient to define an **Equivalent tensile stress** or **von Mises stress**, σ_v , which is used to predict yielding of materials under **multiaxial loading conditions** using results from simple uniaxial tensile tests. Thus, we define

$$\begin{aligned} \sigma_v &= \sqrt{3J_2} \\ &= \sqrt{\frac{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{11} - \sigma_{33})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2)}{2}} \\ &= \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2}{2}} \\ &= \sqrt{\frac{3}{2} s_{ij}s_{ji}} \end{aligned}$$

where s_{ij} are the components of the stress deviator tensor σ^{dev} :

$$\sigma^{dev} = \sigma - \frac{1}{3}(\sigma \cdot \mathbf{I}) \mathbf{I}$$

In this case, yielding occurs when the equivalent stress, σ_v , reaches the yield strength of the material in simple tension, σ_y . As an example, the stress state of a steel beam in compression differs from the stress state of a steel axle under torsion, even if both specimen are of the same material. In view of the stress tensor, which fully describes the stress state, this difference manifests in six degrees of freedom, because the stress tensor has six independent components. Therefore, it is difficult to tell which of the two specimens is closer to the yield point or has even reached it. However, by means of the von Mises yield criterion, which depends solely on the value of the scalar von Mises stress, i.e., one degree of freedom, this comparison is straightforward: A larger von Mises value implies that the material is closer to the yield point.

In the case of **pure shear stress**, $\sigma_{12} = \sigma_{21} \neq 0$, while all other $\sigma_{ij} = 0$, von Mises criterion becomes:

$$\sigma_{12} = k = \frac{\sigma_y}{\sqrt{3}}$$

This means that, at the onset of yielding, the magnitude of the shear stress in pure shear is $\sqrt{3}$ times lower than the tensile stress in the case of simple tension. The von Mises yield criterion for pure shear stress, expressed in principal stresses, is

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 6\sigma_{12}^2$$

In the case of **plane stress**, $\sigma_3 = 0$, the von Mises criterion becomes:

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = 3k^2 = \sigma_y^2$$

This equation represents an ellipse in the plane $\sigma_1 - \sigma_2$, as shown in the Figure above.

Physical interpretation of the von Mises yield criterion

Hencky (1924) offered a physical interpretation of von Mises criterion suggesting that yielding begins when the elastic energy of distortion reaches a critical value. For this, the von Mises criterion is also known as the **maximum distortion strain energy criterion**. This comes from the relation between J_2 and the elastic strain energy of distortion W_D :

$$W_D = \frac{J_2}{2G} \text{ with the elastic shear modulus } G = \frac{E}{2(1+\nu)}$$

In 1937 Arpad L. Nadai suggested that yielding begins when the octahedral shear stress reaches a critical value, i.e. the octahedral shear stress of the material at yield in simple tension. In this case, the von Mises yield criterion is also known as the **maximum octahedral shear stress criterion** in view of the direct proportionality that exist between J_2 and the octahedral shear stress, τ_{oct} , which by definition is

$$\tau_{oct} = \sqrt{\frac{2}{3}J_2}$$

thus we have

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y$$

Comparison with Tresca yield criterion

Also shown in the figure is Tresca's maximum shear stress criterion (dashed line). Observe that Tresca's yield surface is circumscribed by von Mises'. Therefore, it predicts plastic yielding already for stress states that are still elastic according to the von Mises criterion. As a model for plastic material behavior, Tresca's criterion is therefore more conservative.

Chapter 3

Stress (Mechanics)

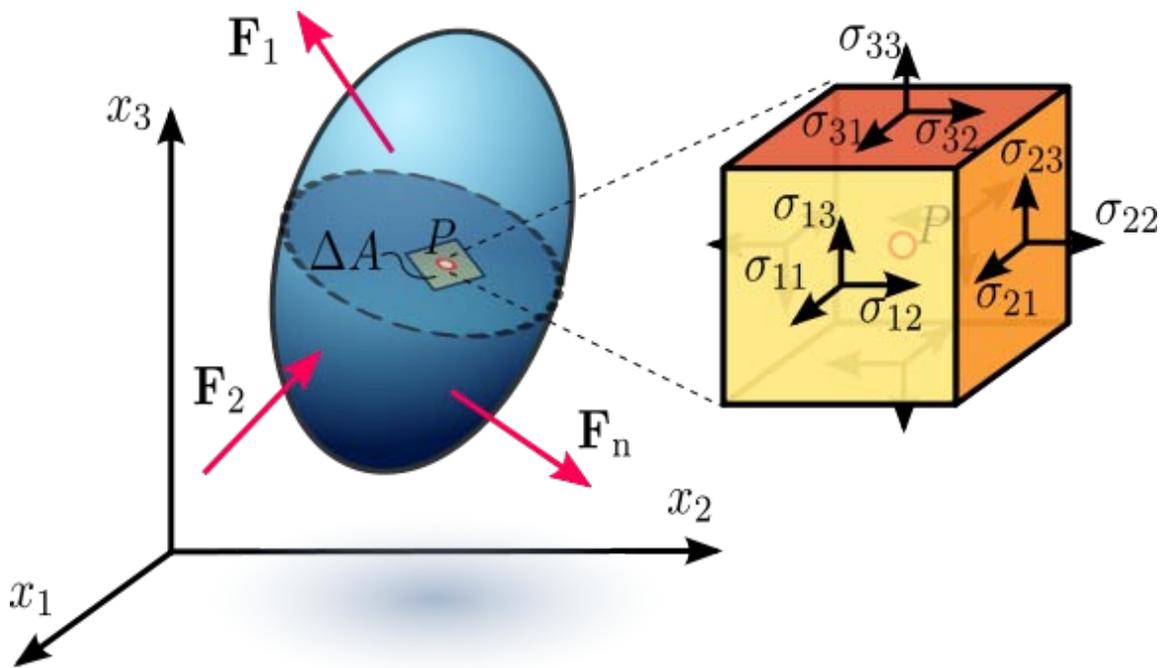


Figure 1.1 Stress in a loaded deformable material body assumed as a continuum.

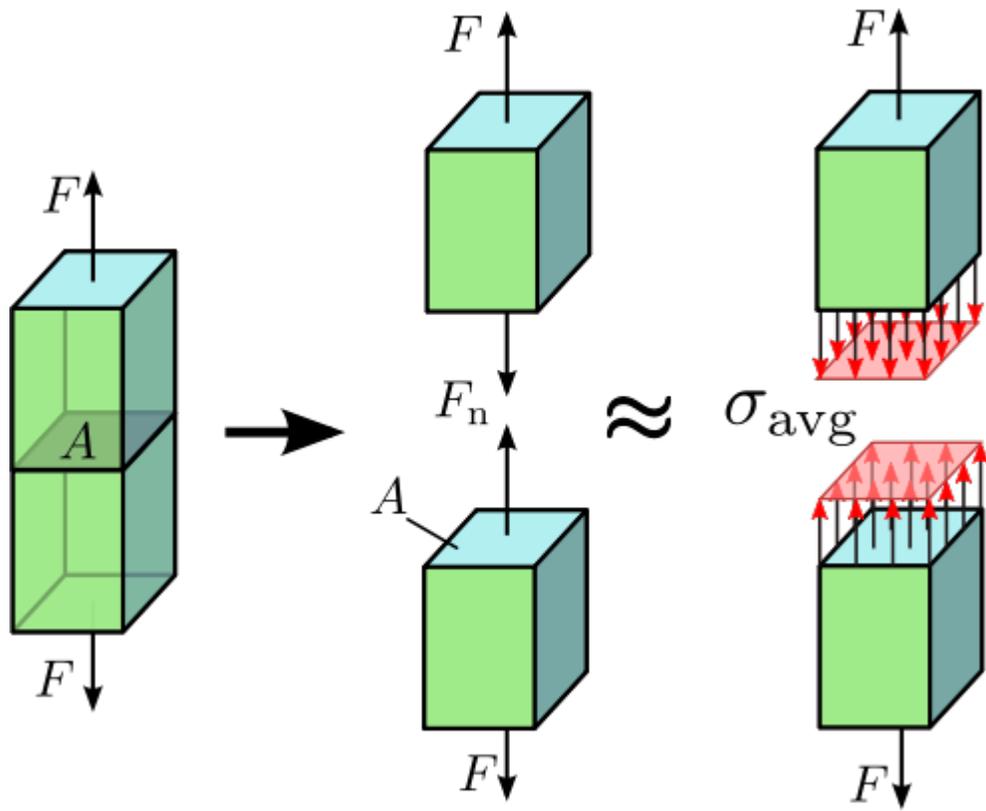


Figure 1.2 Axial stress in a prismatic bar axially loaded

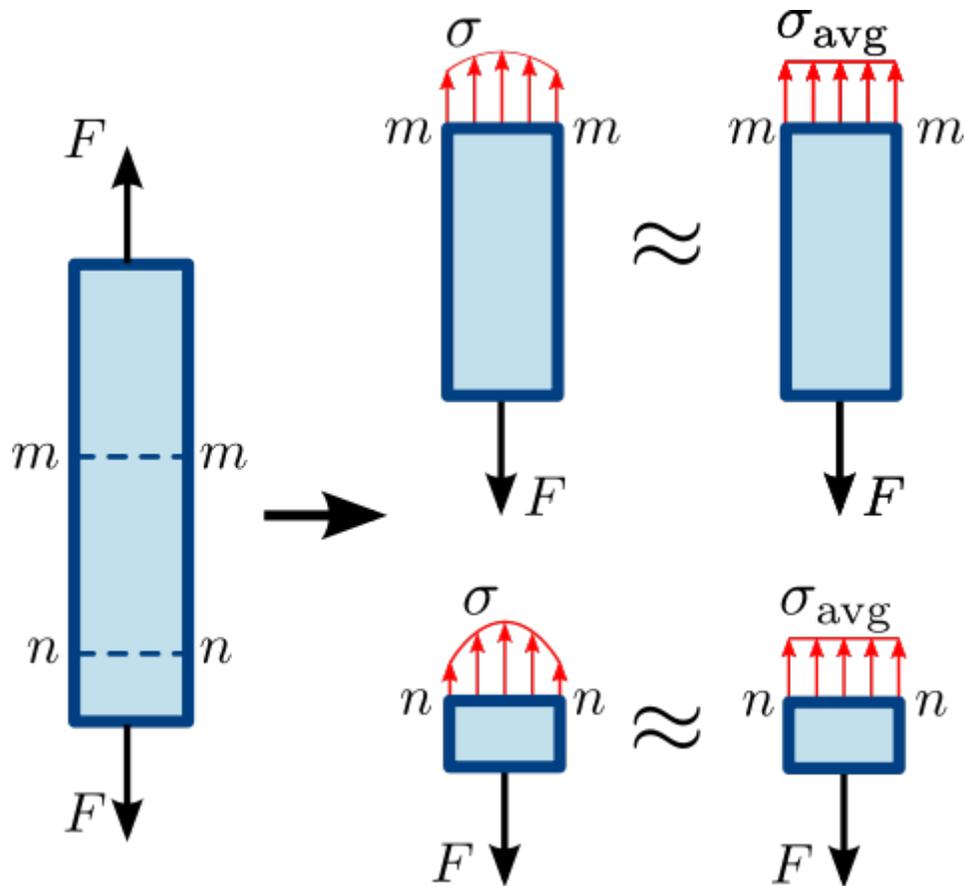


Figure 1.3 Normal stress in a prismatic (straight member of uniform cross-sectional area) bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. However, an average normal stress σ_{avg} can be used

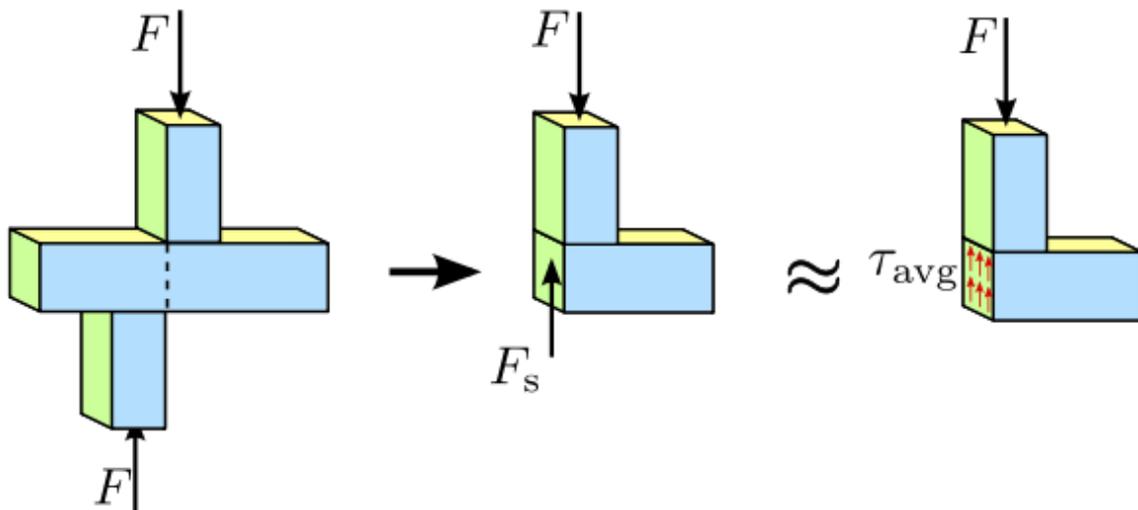


Figure 1.4 Shear stress in a prismatic bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. Nevertheless, an average shear stress τ_{avg} is a reasonable approximation.

In continuum mechanics, **stress** is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, and result in deformation of the body's shape. Beyond certain limits of material strength, this can lead to a permanent change of shape or physical failure.

However, treating physical force as a "one dimensional entity", as it is often done in mechanics, creates a few problems. Any model of continuum mechanics which explicitly expresses force as a variable generally fails to merge and describe deformation of matter and solid bodies, because the attributes of matter and solids are three dimensional. Classical models of continuum mechanics assume an average force and fail to properly incorporate "geometrical factors", which are important to describe stress distribution and accumulation of energy during the continuum.

The dimension of stress is that of pressure, and therefore the SI unit for stress is the pascal (symbol Pa), which is equivalent to one newton (force) per square meter (unit area), that is N/m^2 . In Imperial units, stress is measured in pound-force per square inch, which is abbreviated as psi.

Introduction

Stress is a measure of the average force per unit area of a surface within a deformable body on which internal forces act. It is a measure of the intensity of the internal forces acting between particles of a deformable body across imaginary internal surfaces. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. External forces are either surface forces or body forces. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, *i.e.* the stress distribution in the body is expressed as a piecewise continuous function of space coordinates and time.

Normal , shear stresses and virial stresses

For the simple case of a body axially loaded, e.g., a prismatic bar subjected to tension or compression by a force passing through its centroid (Figures 1.2 and 1.3) the stress σ , or intensity of internal forces, can be obtained by dividing the total *normal force* F_n , determined from the equilibrium of forces, by the cross-sectional area A of the prism it is acting upon. The normal force can be a *tensile force* if acting outward from the plane, or *compressive force* if acting inward to the plane. In the case of a prismatic bar axially loaded, the stress σ is represented by a scalar called *engineering stress* or *nominal stress* that represents an average stress (σ_{avg}) over the area, meaning that the stress in the cross section is uniformly distributed. Thus, we have

$$\sigma_{\text{avg}} = \frac{F_n}{A} \approx \sigma$$

A different type of stress is obtained when transverse forces F are applied to the prismatic bar as shown in Figure 1.4. Considering the same cross-section as before, from static equilibrium the internal force has a magnitude equal to F_s and in opposite direction parallel to the cross-section. F_s is called the *shear force*. Dividing the shear force F_s by the area A of the cross section we obtain the *shear stress*. In this case the shear stress τ is a scalar quantity representing an average shear stress (τ_{avg}) in the section, *i.e.* the stress in the cross-section is uniformly distributed. In materials science and in engineering aspects the average of the "scalar" shear force (τ_{avg}) are true for crystallized materials during brittle fracture and operates through the fractured cross-section or stress plane.

$$\tau_{\text{avg}} = \frac{F_s}{A} \approx \tau$$

In Figure 1.3, the normal stress is observed in two planes $m - m$ and $n - n$ of the axially loaded prismatic bar. The stress on plane $n - n$, which is closer to the point of application of the load F , varies more across the cross-section than that of plane $m - m$. However, if the cross-sectional area of the bar is very small, *i.e.* the bar is slender, the variation of stress across the area is small and the normal stress can be approximated by σ_{avg} . On the other hand, the variation of shear stress across the section of a prismatic bar cannot be assumed to be uniform.

Virial stress is a measure of stress on an atomic scale. It is given by

$$\tau_{ij} = \frac{1}{\Omega} \sum_{k \in \Omega} \left(-m^{(k)} (u_i^{(k)} - \bar{u}_i) (u_j^{(k)} - \bar{u}_j) + \frac{1}{2} \sum_{\ell \in \Omega} (x_i^{(\ell)} - x_i^{(k)}) f_j^{(k\ell)} \right)$$

where

- k and ℓ are atoms in the domain,
- Ω is the volume of the domain,
- $m^{(k)}$ is the mass of atom k ,
- $u_i^{(k)}$ is the i^{th} component of the velocity of atom k ,
- \bar{u}_j is the j^{th} component of the average velocity of atoms in the volume,
- $x_i^{(k)}$ is the i^{th} component of the position of atom k , and
- $f_i^{(k\ell)}$ is the i^{th} component of the force between atom k and ℓ .

At zero kelvin, all velocities are zero so we have

$$\tau_{ij} = \frac{1}{2\Omega} \sum_{k, \ell \in \Omega} (x_i^{(\ell)} - x_i^{(k)}) f_j^{(k\ell)}$$

This can be thought of as follows. The τ_{11} component of stress is the force in the 1 direction divided by the area of a plane perpendicular to that direction. Consider two adjacent volumes separated by such a plane. The 11-component of stress on that interface is the sum of all pairwise forces between atoms on the two sides....

Stress modeling (Cauchy)

In general, stress is not uniformly distributed over the cross-section of a material body, and consequently the stress at a point in a given region is different from the average stress over the entire area. Therefore, it is necessary to define the stress not over a given area but at a specific point in the body (Figure 1.1). According to Cauchy, the *stress at any point* in an object, assumed to behave as a continuum, is completely defined by the nine components σ_{ij} of a second-order tensor of type (0,2) known as the Cauchy stress tensor, $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The Cauchy stress tensor obeys the tensor transformation law under a change in the system of coordinates. A graphical representation of this transformation law is the Mohr's circle of stress distribution.

The Cauchy stress tensor is used for stress analysis of material bodies experiencing small deformations where the differences in stress distribution in most cases can be neglected. For large deformations, also called finite deformations, other measures of stress, such as the first and second Piola-Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor, are required.

According to the principle of conservation of linear momentum, if a continuous body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations (Cauchy's equations of motion for zero acceleration). At the same time, according to the principle of conservation of angular momentum, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, thus having only six independent stress components instead of the original nine.

There are certain invariants associated with the stress tensor, whose values do not depend upon the coordinate system chosen or the area element upon which the stress tensor operates. These are the three eigenvalues of the stress tensor, which are called the principal stresses. Solids, liquids, and gases have stress fields. Static fluids support normal stress but will flow under shear stress. Moving viscous fluids can support shear stress (dynamic pressure). Solids can support both shear and normal stress, with ductile materials failing under shear and brittle materials failing under normal stress. All materials have temperature dependent variations in stress-related properties, and non-Newtonian materials have rate-dependent variations.

Stress analysis

Stress analysis means the determination of the internal distribution of stresses in a structure. It is needed in engineering for the study and design of structures such as tunnels, dams, mechanical parts, and structural frames, under prescribed or expected loads. To determine the distribution of stress in a structure, the engineer needs to solve a boundary-value problem by specifying the boundary conditions. These are displacements and forces on the boundary of the structure.

Constitutive equations, such as Hooke's Law for linear elastic materials, describe the stress-strain relationship in these calculations.

When a structure is expected to deform elastically (and resume its original shape), a boundary-value problem based on the theory of elasticity is applied, with infinitesimal strains, under design loads.

When the applied loads permanently deform the structure, the theory of plasticity is used.

The stress analysis can be simplified when the physical dimensions and the distribution of loads allow the structure to be treated as one-dimensional or two-dimensional. For a two-dimensional analysis a plane stress or a plane strain condition can be assumed. Alternatively, experimental determination of stresses can be carried out.

Approximate computer-based solutions for boundary-value problems can be obtained through numerical methods such as the Finite Element Method, the Finite Difference Method, and the Boundary Element Method. Analytical or closed-form solutions can be obtained for simple geometries, constitutive relations, and boundary conditions.

Theoretical background

Continuum mechanics deals with deformable bodies, as opposed to rigid bodies. The stresses considered in continuum mechanics are only those produced by deformation of the body, *sc.* only relative changes in stress are considered, not the absolute values. A body is considered stress-free if the only forces present are those inter-atomic forces (ionic, metallic, and van der Waals forces) required to hold the body together and to keep its shape in the absence of all external influences, including gravitational attraction.

Stresses generated during manufacture of the body to a specific configuration are also excluded.

Following the classical dynamics of Newton and Euler, the motion of a material body is produced by the action of externally applied forces which are assumed to be of two kinds: surface forces and body forces.

Surface forces, or contact forces, can act either on the bounding surface of the body, as a result of mechanical contact with other bodies, or on imaginary internal surfaces that bound portions of the body, as a result of the mechanical interaction between the parts of the body to either side of the surface (Euler-Cauchy's stress principle). When a body is acted upon by external contact forces, internal contact forces are then transmitted from point to point inside the body to balance their action, according to Newton's second law of motion of conservation of linear momentum and angular momentum (for continuous bodies these laws are called the Euler's equations of motion). The internal contact forces are related to the body's deformation through constitutive equations.

The concept of stress can then be thought as a measure of the intensity of the internal contact forces acting between particles of the body across imaginary internal surfaces. In other words, stress is a measure of the average quantity of force exerted per unit area of the surface on which these internal forces act. The intensity of contact forces is related, specifically in an inverse proportion, to the area of contact. For example, if a force applied to a small area is compared to a distributed load of the same resultant magnitude applied to a larger area, one finds that the effects or intensities of these two forces are locally different because the stresses are not the same.

Body forces are forces originating from sources outside of the body that act on the volume (or mass) of the body. Saying that body forces are due to outside sources implies that the *internal forces* are manifested through the contact forces alone. These forces arise from the presence of the body in force fields, (e.g., a gravitational field). As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed. Thus, body forces are assumed to be continuous over the entire volume of the body.

The density of internal forces at every point in a deformable body are not necessarily equal, *i.e.* there is a distribution of stresses throughout the body. This variation of internal forces throughout the body is governed by Newton's second law of motion of conservation of linear momentum and angular momentum, which normally are applied to a mass particle but are extended in continuum mechanics to a body of continuously distributed mass. For continuous bodies these laws are called Euler's equations of motion. If a body is represented as an assemblage of discrete particles, each governed by Newton's laws of motion, then Euler's equations can be derived from Newton's laws. Euler's equations can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

Euler–Cauchy stress principle

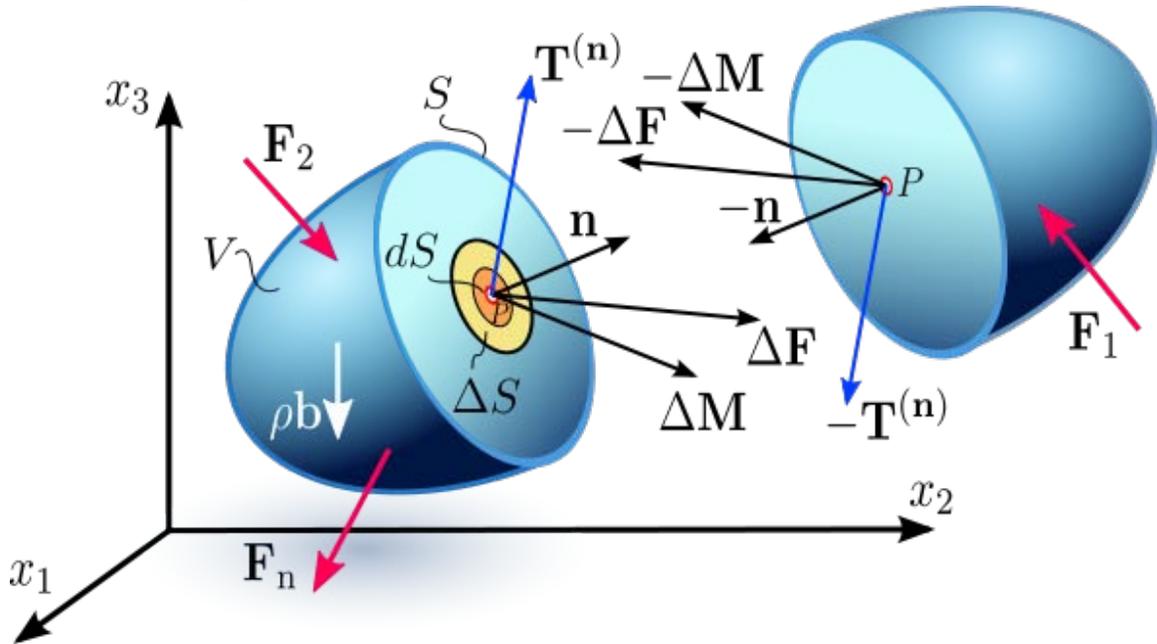


Figure 2.1a Internal distribution of contact forces and couple stresses on a differential dS of the internal surface S in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

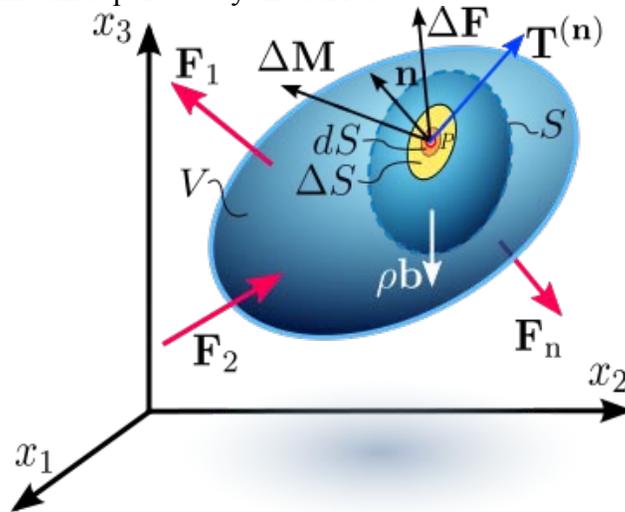


Figure 2.1b Internal distribution of contact forces and couple stresses on a differential dS of the internal surface S in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

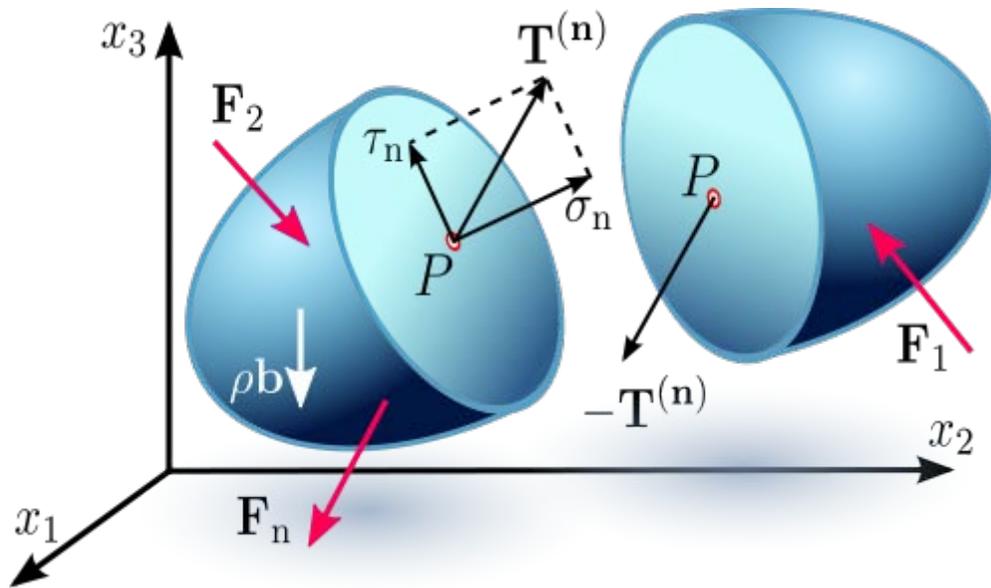


Figure 2.1c Stress vector on an internal surface S with normal vector \mathbf{n} . Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to \mathbf{n} , and can be resolved into two components: one component normal to the plane, called *normal stress* σ_n , and another component parallel to this plane, called the *shearing stress* τ .

The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equivalent (equipollent) to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field $\mathbf{T}^{(\mathbf{n})}$, called the stress vector, defined on the surface S and assumed to depend continuously on the surface's unit vector \mathbf{n} .

To explain this principle, we consider an imaginary surface S passing through an internal material point P dividing the continuous body into two segments, as seen in Figure 2.1a or 2.1b (some authors use the cutting plane diagram and others use the diagram with the arbitrary volume inside the continuum enclosed by the surface S). The body is subjected to external surface forces \mathbf{F} and body forces \mathbf{b} . The internal contact forces being transmitted from one segment to the other through the dividing plane, due to the action of one portion of the continuum onto the other, generate a force distribution on a small area ΔS , with a normal unit vector \mathbf{n} , on the dividing plane S . The force distribution is equipollent to a contact force $\Delta \mathbf{F}$ and a couple stress $\Delta \mathbf{M}$, as shown in Figure 2.1a and 2.1b. Cauchy's stress principle asserts that as ΔS becomes very small and tends to zero the ratio $\Delta \mathbf{F}/\Delta S$ becomes $d\mathbf{F}/dS$ and the couple stress vector $\Delta \mathbf{M}$ vanishes. In specific fields of continuum mechanics the couple stress is assumed not to vanish; however, as stated previously, in classical branches of continuum mechanics we deal with non-polar materials which do not consider couple stresses and body moments. The resultant vector $d\mathbf{F}/dS$ is defined as the *stress vector* or *traction vector* given by $\mathbf{T}^{(\mathbf{n})} = T_i^{(\mathbf{n})} \mathbf{e}_i$ at the point P associated with a plane with a normal vector \mathbf{n} :

$$T_i^{(\mathbf{n})} = \lim_{\Delta S \rightarrow 0} \frac{\Delta F_i}{\Delta S} = \frac{dF_i}{dS}.$$

This equation means that the stress vector depends on its location in the body and the orientation of the plane on which it is acting.

Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to \mathbf{n} , and can be resolved into two components:

- one normal to the plane, called *normal stress*

$$\sigma_n = \lim_{\Delta S \rightarrow 0} \frac{\Delta F_n}{\Delta S} = \frac{dF_n}{dS},$$

where dF_n is the normal component of the force $d\mathbf{F}$ to the differential area dS

- and the other parallel to this plane, called the *shear stress*

$$\tau = \lim_{\Delta S \rightarrow 0} \frac{\Delta F_s}{\Delta S} = \frac{dF_s}{dS},$$

where dF_s is the tangential component of the force $d\mathbf{F}$ to the differential surface area dS . The shear stress can be further decomposed into two mutually perpendicular vectors.

Cauchy's postulate

According to the *Cauchy Postulate*, the stress vector $\mathbf{T}^{(\mathbf{n})}$ remains unchanged for all surfaces passing through the point P and having the same normal vector \mathbf{n} at P , *i.e.* having a common tangent at P . This means that the stress vector is a function of the normal vector \mathbf{n} only, and it is not influenced by the curvature of the internal surfaces.

Cauchy's fundamental lemma

A consequence of Cauchy's postulate is *Cauchy's Fundamental Lemma*, also called the *Cauchy reciprocal theorem*, which states that the stress vectors acting on opposite sides of the same surface are equal in magnitude and opposite in direction. Cauchy's fundamental lemma is equivalent to Newton's third law of motion of action and reaction, and it is expressed as

$$-\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(-\mathbf{n})}.$$

Cauchy's stress theorem – stress tensor

The state of stress at a point in the body is then defined by all the stress vectors $\mathbf{T}^{(\mathbf{n})}$ associated with all planes (infinite in number) that pass through that point. However,

according to *Cauchy's fundamental theorem*, also called *Cauchy's stress theorem*, merely by knowing the stress vectors on three mutually perpendicular planes, the stress vector on any other plane passing through that point can be found through coordinate transformation equations.

Cauchy's stress theorem states that there exists a second-order tensor field $\boldsymbol{\sigma}(\mathbf{x}, t)$, called the *Cauchy stress tensor*, independent of \mathbf{n} , such that \mathbf{T} is a linear function of \mathbf{n} :

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \mathbf{n} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i.$$

This equation implies that the stress vector $\mathbf{T}^{(\mathbf{n})}$ at any point P in a continuum associated with a plane with normal vector \mathbf{n} can be expressed as a function of the stress vectors on the planes perpendicular to the coordinate axes, *i.e.* in terms of the components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$.

To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal area dA oriented in an arbitrary direction specified by a normal vector \mathbf{n} (Figure 2.2). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane \mathbf{n} . The stress vector on this plane is denoted by $\mathbf{T}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$, and are by definition the components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$. This tetrahedron is sometimes called the *Cauchy tetrahedron*. From equilibrium of forces, *i.e.* Euler's first law of motion (Newton's second law of motion), we have

$$\mathbf{T}^{(\mathbf{n})} dA - \mathbf{T}^{(\mathbf{e}_1)} dA_1 - \mathbf{T}^{(\mathbf{e}_2)} dA_2 - \mathbf{T}^{(\mathbf{e}_3)} dA_3 = \rho \left(\frac{h}{3} dA \right) \mathbf{a},$$

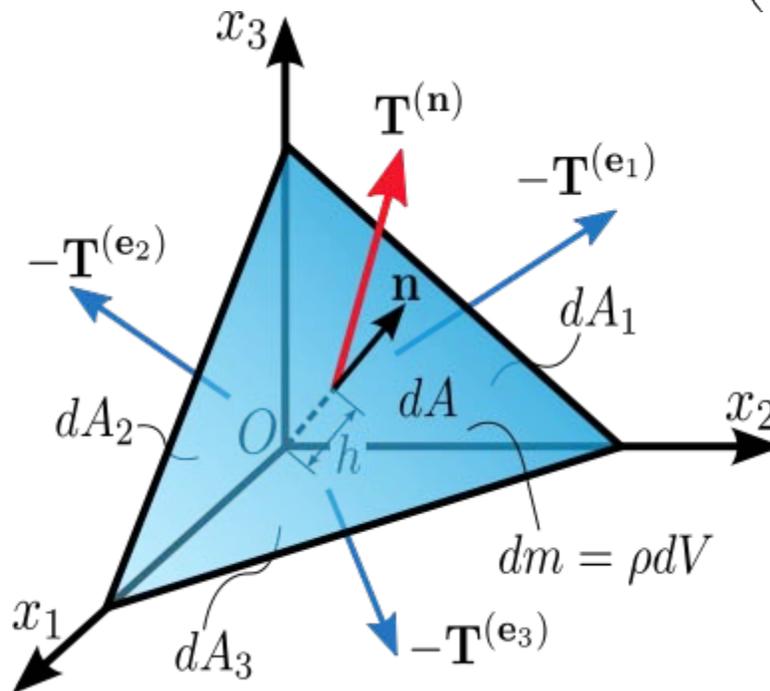


Figure 2.2. Stress vector acting on a plane with normal vector \mathbf{n} .

A note on the sign convention: The tetrahedron is formed by slicing a parallelepiped along an arbitrary plane \mathbf{n} . So, the force acting on the plane \mathbf{n} is the reaction exerted by the other half of the parallelepiped and has an opposite sign.

where the right-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ρ is the density, \mathbf{a} is the acceleration, and h is the height of the tetrahedron, considering the plane \mathbf{n} as the base. The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face (using the dot product):

$$\begin{aligned} dA_1 &= (\mathbf{n} \cdot \mathbf{e}_1) dA = n_1 dA, \\ dA_2 &= (\mathbf{n} \cdot \mathbf{e}_2) dA = n_2 dA, \\ dA_3 &= (\mathbf{n} \cdot \mathbf{e}_3) dA = n_3 dA, \end{aligned}$$

and then substituting into the equation to cancel out dA :

$$\mathbf{T}^{(\mathbf{n})} - \mathbf{T}^{(\mathbf{e}_1)} n_1 - \mathbf{T}^{(\mathbf{e}_2)} n_2 - \mathbf{T}^{(\mathbf{e}_3)} n_3 = \rho \left(\frac{h}{3} \right) \mathbf{a}.$$

To consider the limiting case as the tetrahedron shrinks to a point, h must go to 0 (intuitively, the plane \mathbf{n} is translated along \mathbf{n} toward O). As a result, the right-hand-side of the equation approaches 0, so

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(\mathbf{e}_1)} n_1 + \mathbf{T}^{(\mathbf{e}_2)} n_2 + \mathbf{T}^{(\mathbf{e}_3)} n_3.$$

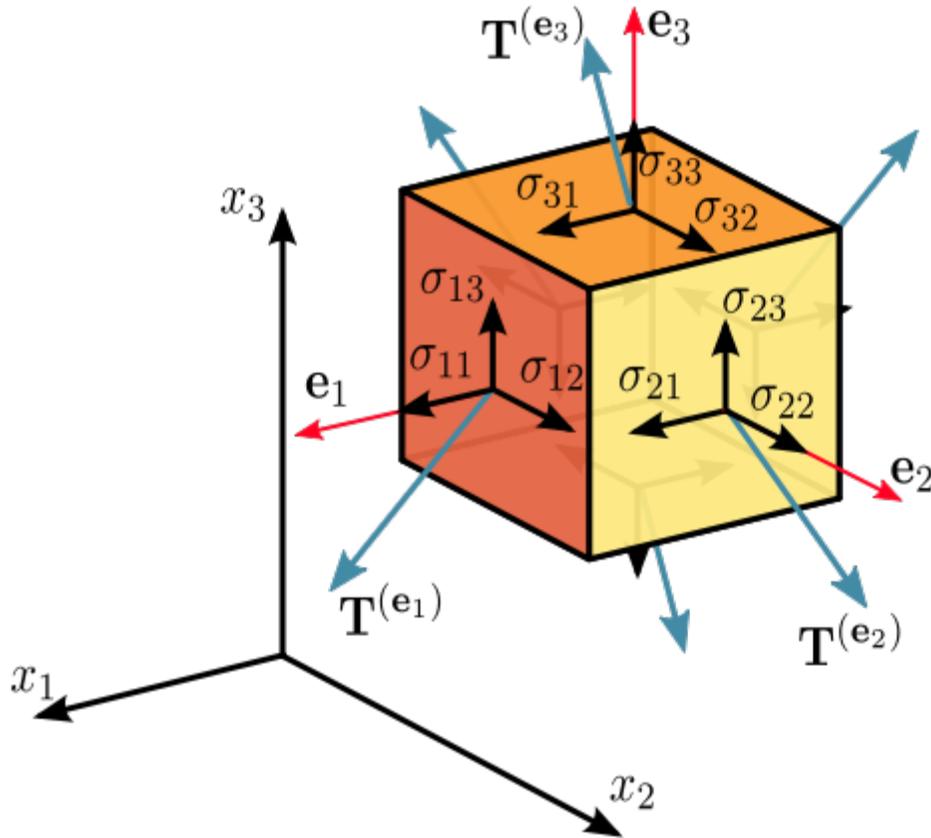


Figure 2.3 Components of stress in three dimensions

Assuming a material element (Figure 2.3) with planes perpendicular to the coordinate axes of a Cartesian coordinate system, the stress vectors associated with each of the element planes, *i.e.* $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$ can be decomposed into a normal component and two shear components, *i.e.* components in the direction of the three coordinate axes. For the particular case of a surface with normal unit vector oriented in the direction of the x_1 -axis, the normal stress is denoted by σ_{11} , and the two shear stresses are denoted as σ_{12} and σ_{13} :

$$\begin{aligned}\mathbf{T}^{(\mathbf{e}_1)} &= T_1^{(\mathbf{e}_1)} \mathbf{e}_1 + T_2^{(\mathbf{e}_1)} \mathbf{e}_2 + T_3^{(\mathbf{e}_1)} \mathbf{e}_3 = \sigma_{11} \mathbf{e}_1 + \sigma_{12} \mathbf{e}_2 + \sigma_{13} \mathbf{e}_3, \\ \mathbf{T}^{(\mathbf{e}_2)} &= T_1^{(\mathbf{e}_2)} \mathbf{e}_1 + T_2^{(\mathbf{e}_2)} \mathbf{e}_2 + T_3^{(\mathbf{e}_2)} \mathbf{e}_3 = \sigma_{21} \mathbf{e}_1 + \sigma_{22} \mathbf{e}_2 + \sigma_{23} \mathbf{e}_3, \\ \mathbf{T}^{(\mathbf{e}_3)} &= T_1^{(\mathbf{e}_3)} \mathbf{e}_1 + T_2^{(\mathbf{e}_3)} \mathbf{e}_2 + T_3^{(\mathbf{e}_3)} \mathbf{e}_3 = \sigma_{31} \mathbf{e}_1 + \sigma_{32} \mathbf{e}_2 + \sigma_{33} \mathbf{e}_3,\end{aligned}$$

In index notation this is

$$\mathbf{T}^{(\mathbf{e}_i)} = T_j^{(\mathbf{e}_i)} \mathbf{e}_j = \sigma_{ij} \mathbf{e}_j.$$

The nine components σ_{ij} of the stress vectors are the components of a second-order Cartesian tensor called the *Cauchy stress tensor*, which completely defines the state of stress at a point and is given by

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \mathbf{T}^{(\mathbf{e}_1)} \\ \mathbf{T}^{(\mathbf{e}_2)} \\ \mathbf{T}^{(\mathbf{e}_3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix},$$

where σ_{11} , σ_{22} , and σ_{33} are normal stresses, and σ_{12} , σ_{13} , σ_{21} , σ_{23} , σ_{31} , and σ_{32} are shear stresses. The first index i indicates that the stress acts on a plane normal to the x_i -axis, and the second index j denotes the direction in which the stress acts. A stress component is positive if it acts in the positive direction of the coordinate axes, and if the plane where it acts has an outward normal vector pointing in the positive coordinate direction.

Thus, using the components of the stress tensor

$$\begin{aligned} \mathbf{T}^{(\mathbf{n})} &= \mathbf{T}^{(\mathbf{e}_1)}n_1 + \mathbf{T}^{(\mathbf{e}_2)}n_2 + \mathbf{T}^{(\mathbf{e}_3)}n_3 \\ &= \sum_{i=1}^3 \mathbf{T}^{(\mathbf{e}_i)}n_i \\ &= (\sigma_{ij}\mathbf{e}_j)n_i \\ &= \sigma_{ij}n_i\mathbf{e}_j \end{aligned}$$

or, equivalently,

$$T_j^{(\mathbf{n})} = \sigma_{ij}n_i.$$

Alternatively, in matrix form we have

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

The Voigt notation representation of the Cauchy stress tensor takes advantage of the symmetry of the stress tensor to express the stress as a six-dimensional vector of the form:

$$\boldsymbol{\sigma} = [\sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6]^T \equiv [\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{31} \quad \sigma_{12}]^T.$$

The Voigt notation is used extensively in representing stress-strain relations in solid mechanics and for computational efficiency in numerical structural mechanics software.

Transformation rule of the stress tensor

It can be shown that the stress tensor is a contravariant second order tensor, which is a statement of how it transforms under a change of the coordinate system. From an x_i -system to an x'_i -system, the components σ_{ij} in the initial system are transformed into the components σ'_{ij} in the new system according to the tensor transformation rule (Figure 2.4):

$$\sigma'_{ij} = a_{im} a_{jn} \sigma_{mn} \quad \text{OR} \quad \boldsymbol{\sigma}' = \mathbf{A} \boldsymbol{\sigma} \mathbf{A}^T,$$

where \mathbf{A} is a rotation matrix with components a_{ij} . In matrix form this is

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

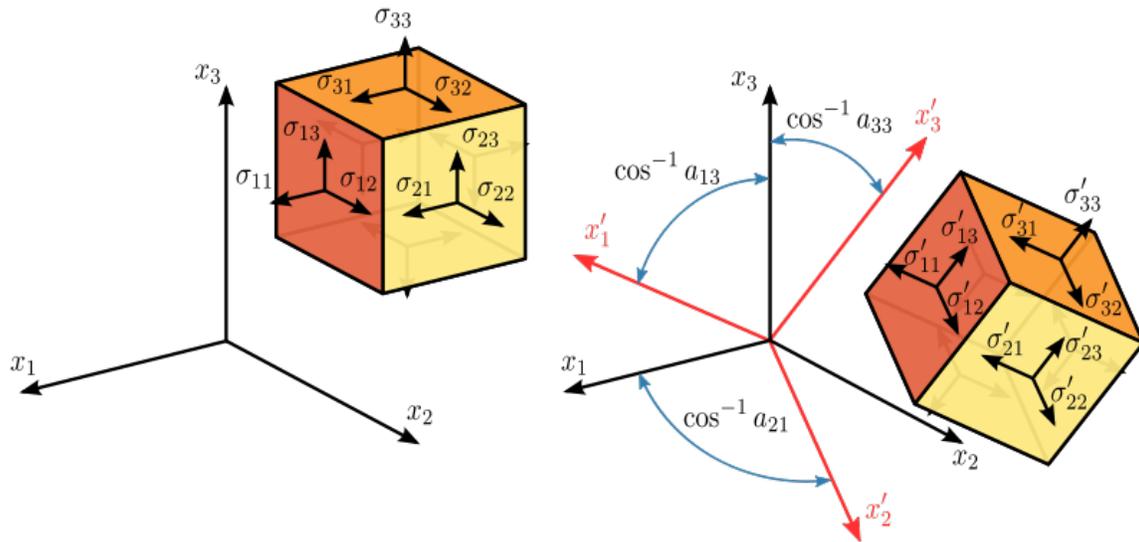


Figure 2.4 Transformation of the stress tensor

Expanding the matrix operation, and simplifying some terms by taking advantage of the symmetry of the stress tensor, gives

$$\begin{aligned} \sigma'_{11} &= a_{11}^2 \sigma_{11} + a_{12}^2 \sigma_{22} + a_{13}^2 \sigma_{33} + 2a_{11}a_{12}\sigma_{12} + 2a_{11}a_{13}\sigma_{13} + 2a_{12}a_{13}\sigma_{23}, \\ \sigma'_{22} &= a_{21}^2 \sigma_{11} + a_{22}^2 \sigma_{22} + a_{23}^2 \sigma_{33} + 2a_{21}a_{22}\sigma_{12} + 2a_{21}a_{23}\sigma_{13} + 2a_{22}a_{23}\sigma_{23}, \\ \sigma'_{33} &= a_{31}^2 \sigma_{11} + a_{32}^2 \sigma_{22} + a_{33}^2 \sigma_{33} + 2a_{31}a_{32}\sigma_{12} + 2a_{31}a_{33}\sigma_{13} + 2a_{32}a_{33}\sigma_{23}, \\ \sigma'_{12} &= a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + a_{13}a_{23}\sigma_{33} \\ &\quad + (a_{11}a_{22} + a_{12}a_{21})\sigma_{12} + (a_{12}a_{23} + a_{13}a_{22})\sigma_{23} + (a_{11}a_{23} + a_{13}a_{21})\sigma_{13}, \end{aligned}$$

$$\begin{aligned}\sigma'_{23} &= a_{21}a_{31}\sigma_{11} + a_{22}a_{32}\sigma_{22} + a_{23}a_{33}\sigma_{33} \\ &\quad + (a_{21}a_{32} + a_{22}a_{31})\sigma_{12} + (a_{22}a_{33} + a_{23}a_{32})\sigma_{23} + (a_{21}a_{33} + a_{23}a_{31})\sigma_{13}, \\ \sigma'_{13} &= a_{11}a_{31}\sigma_{11} + a_{12}a_{32}\sigma_{22} + a_{13}a_{33}\sigma_{33} \\ &\quad + (a_{11}a_{32} + a_{12}a_{31})\sigma_{12} + (a_{12}a_{33} + a_{13}a_{32})\sigma_{23} + (a_{11}a_{33} + a_{13}a_{31})\sigma_{13}.\end{aligned}$$

The Mohr circle for stress is a graphical representation of this transformation of stresses.

Normal and shear stresses

The magnitude of the normal stress component σ_n of any stress vector $\mathbf{T}^{(\mathbf{n})}$ acting on an arbitrary plane with normal vector \mathbf{n} at a given point, in terms of the components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$, is the dot product of the stress vector and the normal vector:

$$\begin{aligned}\sigma_n &= \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= T_i^{(\mathbf{n})} n_i \\ &= \sigma_{ij} n_i n_j.\end{aligned}$$

The magnitude of the shear stress component τ_n , acting in the plane spanned by the two vectors $\mathbf{T}^{(\mathbf{n})}$ and \mathbf{n} , can then be found using the Pythagorean theorem:

$$\begin{aligned}\tau_n &= \sqrt{(T^{(\mathbf{n})})^2 - \sigma_n^2} \\ &= \sqrt{T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} - \sigma_n^2},\end{aligned}$$

where

$$(T^{(\mathbf{n})})^2 = T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} = (\sigma_{ij} n_j) (\sigma_{ik} n_k) = \sigma_{ij} \sigma_{ik} n_j n_k.$$

Equilibrium equations and symmetry of the stress tensor

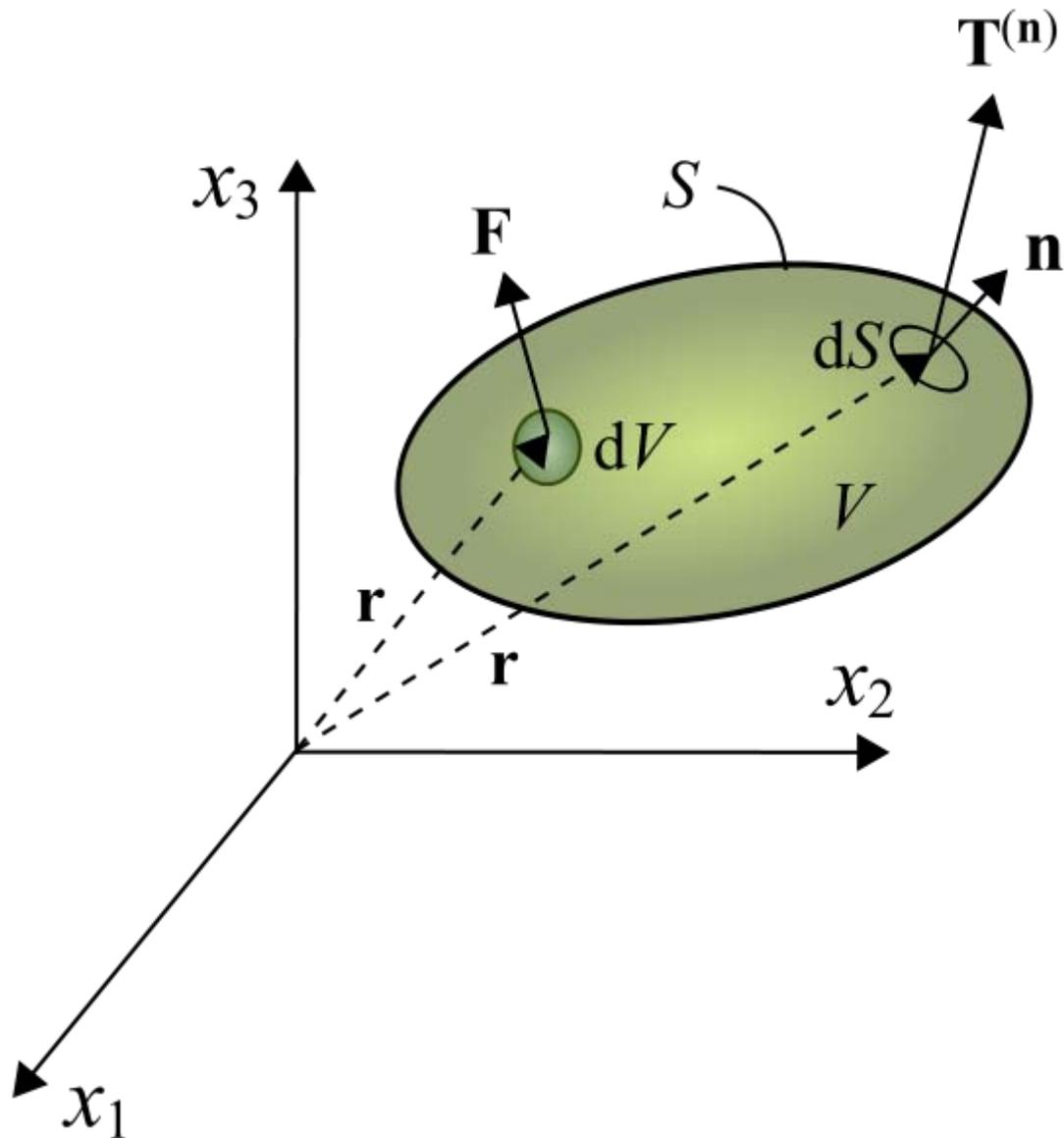


Figure 4. Continuum body in equilibrium

When a body is in equilibrium the components of the stress tensor in every point of the body satisfy the equilibrium equations,

$$\sigma_{ji,j} + F_i = 0$$

For example, for a hydrostatic fluid in equilibrium conditions, the stress tensor takes on the form:

$$\sigma_{ij} = -p\delta_{ij},$$

where p is the hydrostatic pressure, and δ_{ij} is the Kronecker delta.

At the same time, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}$$

However, in the presence of couple-stresses, i.e. moments per unit volume, the stress tensor is non-symmetric. This also is the case when the Knudsen number is close to one, $K_n \rightarrow 1$, or the continuum is a non-Newtonian fluid, which can lead to rotationally non-invariant fluids, such as polymers.

Principal stresses and stress invariants

At every point in a stressed body there are at least three planes, called *principal planes*, with normal vectors \mathbf{n} , called *principal directions*, where the corresponding stress vector is perpendicular to the plane, i.e., parallel or in the same direction as the normal vector \mathbf{n} , and where there are no normal shear stresses τ_n . The three stresses normal to these principal planes are called *principal stresses*.

The components σ_{ij} of the stress tensor depend on the orientation of the coordinate system at the point under consideration. However, the stress tensor itself is a physical quantity and as such, it is independent of the coordinate system chosen to represent it. There are certain invariants associated with every tensor which are also independent of the coordinate system. For example, a vector is a simple tensor of rank one. In three dimensions, it has three components. The value of these components will depend on the coordinate system chosen to represent the vector, but the length of the vector is a physical quantity (a scalar) and is independent of the coordinate system chosen to represent the vector. Similarly, every second rank tensor (such as the stress and the strain tensors) has three independent invariant quantities associated with it. One set of such invariants are the principal stresses of the stress tensor, which are just the eigenvalues of the stress tensor. Their direction vectors are the principal directions or eigenvectors.

A stress vector parallel to the normal vector \mathbf{n} is given by:

$$\mathbf{T}^{(\mathbf{n})} = \lambda \mathbf{n} = \sigma_n \mathbf{n}$$

where λ is a constant of proportionality, and in this particular case corresponds to the magnitudes σ_n of the normal stress vectors or principal stresses.

Knowing that $T_i^{(n)} = \sigma_{ij} n_j$ and $n_i = \delta_{ij} n_j$, we have

$$\begin{aligned}
T_i^{(n)} &= \lambda n_i \\
\sigma_{ij} n_j &= \lambda n_i \\
\sigma_{ij} n_j - \lambda n_i &= 0 \\
(\sigma_{ij} - \lambda \delta_{ij}) n_j &= 0
\end{aligned}$$

This is a homogeneous system, i.e. equal to zero, of three linear equations where n_j are the unknowns. To obtain a nontrivial (non-zero) solution for n_j , the determinant matrix of the coefficients must be equal to zero, i.e. the system is singular. Thus,

$$|\sigma_{ij} - \lambda \delta_{ij}| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0$$

Expanding the determinant leads to the *characteristic equation*

$$|\sigma_{ij} - \lambda \delta_{ij}| = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0$$

where

$$\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_{kk} \\
I_2 &= \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \\
&= \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{11} \sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \\
&= \frac{1}{2} (\sigma_{ii} \sigma_{jj} - \sigma_{ij} \sigma_{ji}) \\
I_3 &= \det(\sigma_{ij}) \\
&= \sigma_{11} \sigma_{22} \sigma_{33} + 2 \sigma_{12} \sigma_{23} \sigma_{31} - \sigma_{12}^2 \sigma_{33} - \sigma_{23}^2 \sigma_{11} - \sigma_{13}^2 \sigma_{22}
\end{aligned}$$

The characteristic equation has three real roots λ , i.e. not imaginary due to the symmetry of the stress tensor. The three roots $\lambda_1 = \sigma_1$, $\lambda_2 = \sigma_2$, and $\lambda_3 = \sigma_3$ are the eigenvalues or principal stresses, and they are the roots of the Cayley–Hamilton theorem. The principal stresses are unique for a given stress tensor. Therefore, from the characteristic equation it is seen that the coefficients I_1 , I_2 and I_3 , called the first, second, and third *stress invariants*, respectively, have always the same value regardless of the orientation of the coordinate system chosen.

For each eigenvalue, there is a non-trivial solution for n_j in the equation $(\sigma_{ij} - \lambda\delta_{ij}) n_j = 0$. These solutions are the principal directions or eigenvectors defining the plane where the principal stresses act. The principal stresses and principal directions characterize the stress at a point and are independent of the orientation of the coordinate system.

If we choose a coordinate system with axes oriented to the principal directions, then the normal stresses will be the principal stresses and the stress tensor is represented by a diagonal matrix:

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The principal stresses may be combined to form the stress invariants, I_1 , I_2 , and I_3 . The first and third invariant are the trace and determinant respectively, of the stress tensor. Thus,

$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}$$

Because of its simplicity, working and thinking in the principal coordinate system is often very useful when considering the state of the elastic medium at a particular point.

Principal stresses are often expressed in the following equation for evaluating stresses in the x and y directions or axial and bending stresses on a part. The principal normal stresses can then be used to calculate the Von Mises stress and ultimately the safety factor and margin of safety.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Using just the part of the equation under the square root is equal to the maximum and minimum shear stress for plus and minus. This is shown as:

$$\tau_{max}, \tau_{min} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum and minimum shear stresses

The maximum shear stress or maximum principal shear stress is equal to one-half the difference between the largest and smallest principal stresses, and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses, i.e. the plane of the maximum shear stress is oriented 45° from the principal stress planes. The maximum shear stress is expressed as

$$\tau_{\max} = \frac{1}{2} |\sigma_{\max} - \sigma_{\min}|$$

Assuming $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then

$$\tau_{\max} = \frac{1}{2} |\sigma_1 - \sigma_3|$$

The normal stress component acting on the plane for the maximum shear stress is non-zero and it is equal to

$$\sigma_n = \frac{1}{2} (\sigma_1 + \sigma_3)$$

Stress deviator tensor

The stress tensor σ_{ij} can be expressed as the sum of two other stress tensors:

1. a *mean hydrostatic stress tensor* or *volumetric stress tensor* or *mean normal stress tensor*, $p\delta_{ij}$, which tends to change the volume of the stressed body; and
2. a deviatoric component called the *stress deviator tensor*, s_{ij} , which tends to distort it.

So:

$$\sigma_{ij} = s_{ij} + p\delta_{ij},$$

where P is the mean stress given by

$$p = \frac{\sigma_{kk}}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \frac{1}{3} I_1.$$

Note that convention in solid mechanics differs slightly from what is listed above. In solid mechanics, pressure is generally defined as negative one-third the trace of the stress tensor.

The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the stress tensor:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}.$$

Invariants of the stress deviator tensor

As it is a second order tensor, the stress deviator tensor also has a set of invariants, which can be obtained using the same procedure used to calculate the invariants of the stress tensor. It can be shown that the principal directions of the stress deviator tensor s_{ij} are the same as the principal directions of the stress tensor σ_{ij} . Thus, the characteristic equation is

$$|s_{ij} - \lambda\delta_{ij}| = \lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0,$$

where J_1 , J_2 and J_3 are the first, second, and third *deviatoric stress invariants*, respectively. Their values are the same (invariant) regardless of the orientation of the coordinate system chosen. These deviatoric stress invariants can be expressed as a function of the components of s_{ij} or its principal values s_1 , s_2 , and s_3 , or alternatively, as a function of σ_{ij} or its principal values σ_1 , σ_2 , and σ_3 . Thus,

$$J_1 = s_{kk} = 0,$$

$$J_2 = \frac{1}{2}s_{ij}s_{ji}$$

$$= -s_1s_2 - s_2s_3 - s_3s_1$$

$$= \frac{1}{6} [(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2$$

$$= \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= \frac{1}{3}I_1^2 - I_2,$$

$$J_3 = \det(s_{ij})$$

$$= \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

$$= s_1s_2s_3$$

$$= \frac{2}{27}I_1^3 - \frac{1}{3}I_1I_2 + I_3.$$

Because $s_{kk} = 0$, the stress deviator tensor is in a state of pure shear.

A quantity called the equivalent stress or von Mises stress is commonly used in solid mechanics. The equivalent stress is defined as

$$\sigma_e = \sqrt{3 J_2} = \sqrt{\frac{1}{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}.$$

Octahedral stresses

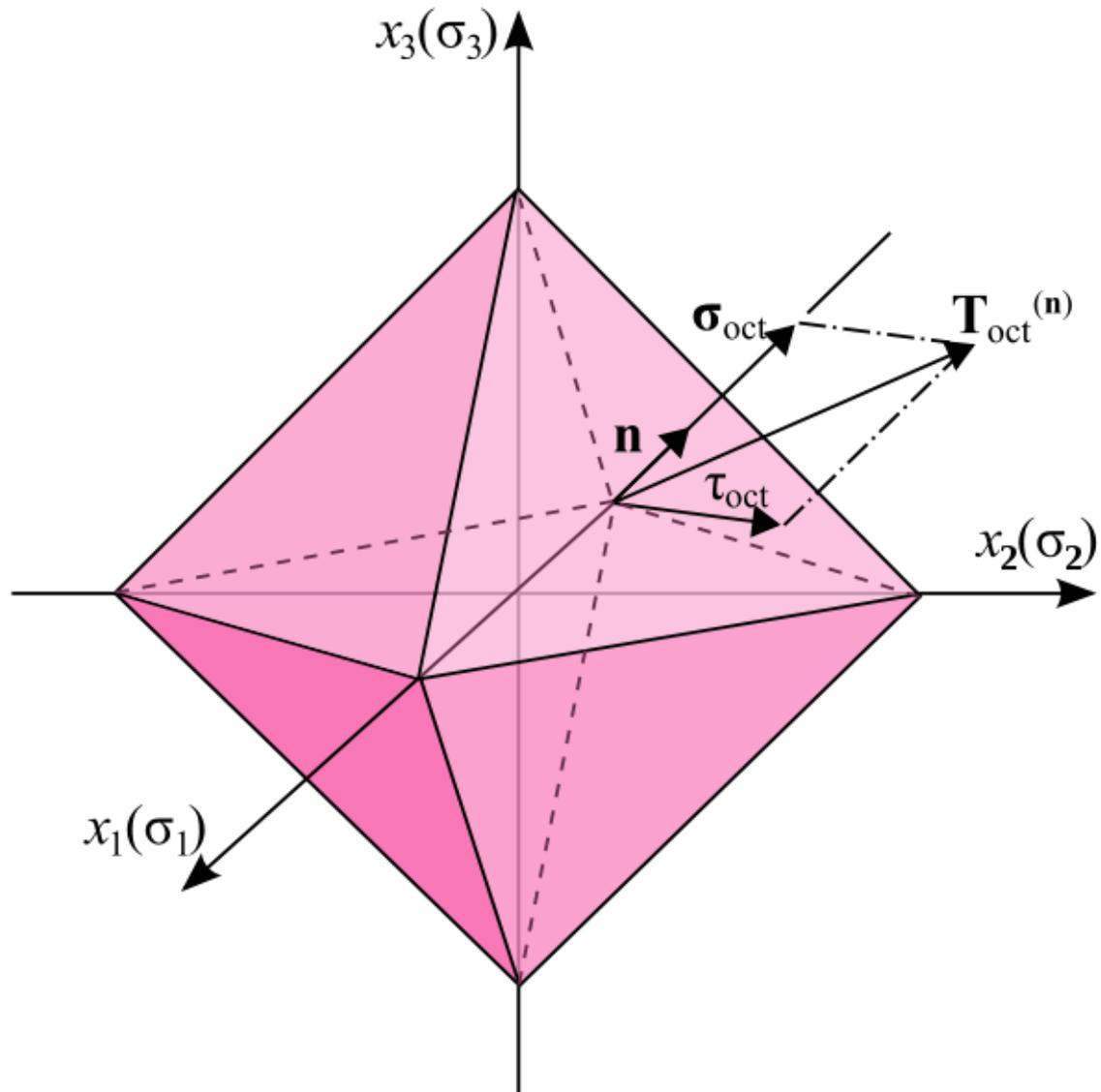


Figure 6. Octahedral stress planes

Considering the principal directions as the coordinate axes, a plane whose normal vector makes equal angles with each of the principal axes (i.e. having direction cosines equal to

$\{1/\sqrt{3}\}$ is called an *octahedral plane*. There are a total of eight octahedral planes (Figure 6). The normal and shear components of the stress tensor on these planes are called *octahedral normal stress* σ_{oct} and *octahedral shear stress* τ_{oct} , respectively.

Knowing that the stress tensor of point O (Figure 6) in the principal axes is

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

the stress vector on an octahedral plane is then given by:

$$\begin{aligned} \mathbf{T}_{\text{oct}}^{(n)} &= \sigma_{ij} n_i \mathbf{e}_j \\ &= \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \\ &= \frac{1}{\sqrt{3}} (\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 + \sigma_3 \mathbf{e}_3) \end{aligned}$$

The normal component of the stress vector at point O associated with the octahedral plane is

$$\begin{aligned} \sigma_{\text{oct}} &= T_i^{(n)} n_i \\ &= \sigma_{ij} n_i n_j \\ &= \sigma_1 n_1 n_1 + \sigma_2 n_2 n_2 + \sigma_3 n_3 n_3 \\ &= \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_1 \end{aligned}$$

which is the mean normal stress or hydrostatic stress. This value is the same in all eight octahedral planes. The shear stress on the octahedral plane is then

$$\begin{aligned} \tau_{\text{oct}} &= \sqrt{T_i^{(n)} T_i^{(n)} - \sigma_{\text{oct}}^2} \\ &= \left[\frac{1}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{1}{9} (\sigma_1 + \sigma_2 + \sigma_3)^2 \right]^{1/2} \\ &= \frac{1}{3} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2} = \frac{1}{3} \sqrt{2I_1^2 - 6I_2} = \sqrt{\frac{2}{3} J_2} \end{aligned}$$

Alternative measures of stress

The Cauchy stress tensor is not the only measure of stress that is used in practice. Other measures of stress include the first and second Piola–Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor.

Piola–Kirchhoff stress tensor

In the case of finite deformations, the *Piola–Kirchhoff stress tensors* are used to express the stress relative to the reference configuration. This is in contrast to the Cauchy stress tensor which expresses the stress relative to the present configuration. For infinitesimal deformations or rotations, the Cauchy and Piola–Kirchhoff tensors are identical. These tensors take their names from Gabrio Piola and Gustav Kirchhoff.

Whereas the Cauchy stress tensor, $\boldsymbol{\sigma}$ relates stresses in the current configuration, the deformation gradient and strain tensors are described by relating the motion to the reference configuration; thus not all tensors describing the state of the material are in either the reference or current configuration. Having the stress, strain and deformation all described either in the reference or current configuration would make it easier to define constitutive models (for example, the Cauchy Stress tensor is variant to a pure rotation, while the deformation strain tensor is invariant; thus creating problems in defining a constitutive model that relates a varying tensor, in terms of an invariant one during pure rotation; as by definition constitutive models have to be invariant to pure rotations). The 1st Piola–Kirchhoff stress tensor, \mathbf{P} is one possible solution to this problem. It defines a family of tensors, which describe the configuration of the body in either the current or the reference state.

The 1st Piola–Kirchhoff stress tensor, \mathbf{P} relates forces in the *present* configuration with areas in the *reference* ("material") configuration.

$$\mathbf{P} = J \boldsymbol{\sigma} \cdot \mathbf{F}^{-T}$$

where \mathbf{F} is the deformation gradient and $J = \det \mathbf{F}$ is the Jacobian determinant.

In terms of components with respect to an orthonormal basis, the first Piola–Kirchhoff stress is given by

$$P_{iL} = J \sigma_{ik} F_{Lk}^{-1} = J \sigma_{ik} \frac{\partial X_L}{\partial x_k}$$

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. In general, it is not symmetric. The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress state (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the deformation gradient.

2nd Piola–Kirchhoff stress tensor

Whereas the 1st Piola–Kirchhoff stress relates forces in the current configuration to areas in the reference configuration, the 2nd Piola–Kirchhoff stress tensor \mathbf{S} relates forces in the reference configuration to areas in the reference configuration. The force in the reference configuration is obtained via a mapping that preserves the relative relationship between the force direction and the area normal in the current configuration.

$$\mathbf{S} = J \mathbf{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \mathbf{F}^{-T} .$$

In index notation with respect to an orthonormal basis,

$$S_{IL} = J F_{Ik}^{-1} F_{Lm}^{-1} \sigma_{km} = J \frac{\partial X_I}{\partial x_k} \frac{\partial X_L}{\partial x_m} \sigma_{km}$$

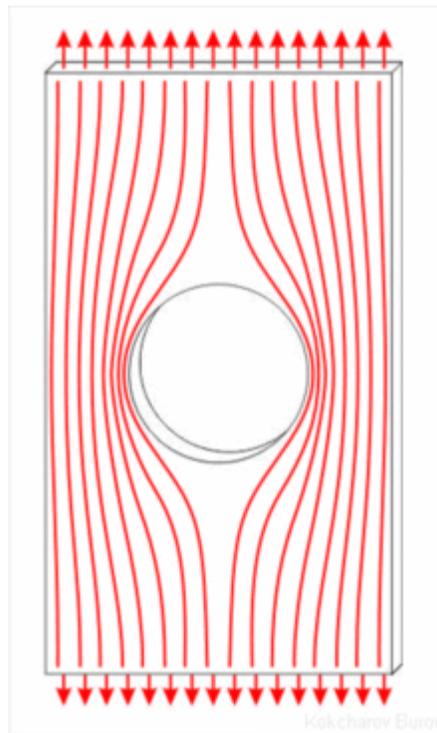
This tensor is symmetric.

If the material rotates without a change in stress state (rigid rotation), the components of the 2nd Piola–Kirchhoff stress tensor will remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

Chapter 4

Stress Concentration



Internal force lines are denser near the hole

A **stress concentration** (often called **stress raisers** or **stress risers**) is a location in an object where stress is concentrated. An object is strongest when force is evenly distributed over its area, so a reduction in area, e.g. caused by a crack, results in a localized increase in stress. A material can fail, via a propagating crack, when a concentrated stress exceeds the material's theoretical cohesive strength. The real fracture strength of a material is always lower than the theoretical value because most materials

contain small cracks that concentrate stress. Fatigue cracks always start at stress raisers, so removing such defects increases the fatigue strength.

Causes



The sharp corner at the brick has acted as a stress concentrator within the concrete causing it to crack

Geometric discontinuities cause an object to experience a local increase in the intensity of a stress field. The examples of shapes that cause these concentrations are: cracks, sharp corners, holes and, changes in the cross-sectional area of the object. High local stresses can cause the object to fail more quickly than if it wasn't there. Engineers must design the geometry to minimize stress concentrations.

Prevention

A counter-intuitive method of reducing one of the worst types of stress concentrations, a crack, is to drill a large hole at the end of the crack. The drilled hole, with its relatively large diameter, causes a smaller stress concentration than the sharp end of a crack. This is however, a temporary solution that must be corrected at the first opportune time.

It is important to systematically check for possible stress concentrations caused by cracks—there is a critical crack length of $2a$ for which, when this value is exceeded, the crack proceeds to definite catastrophic failure. This ultimate failure is definite since the crack will propagate on its own once the length is greater than $2a$. (There is no additional energy required to increase the crack length so the crack will continue to enlarge until the material fails.) The origins of the value $2a$ can be understood through Griffith's theory of brittle fracture.

Examples



This orthosis is implanted to support the femur after a fracture, but the concentration of stress at its bend increases the possibility that it may break under force.

The term "stress raiser" is used in orthopedics; a focus point of stress on an implanted orthosis is very likely to be its point of failure.

Classic cases of metal failures due to stress concentrations include metal fatigue at the corners of the windows of the De Havilland Comet aircraft and brittle fractures at the corners of hatches in Liberty ships in cold and stressful conditions in winter storms in the Atlantic Ocean.

Concentration factor for cracks

The maximum stress felt near a crack occurs in the area of lowest radius of curvature. In an elliptical crack of length $2a$ and width $2b$, under an applied external stress σ , the stress at the ends of the major axes is given by:

$$\sigma_{max} = \sigma \left(1 + 2\frac{a}{b} \right) = \sigma \left(1 + 2\sqrt{\frac{a}{\rho}} \right)$$

where ρ is the radius of curvature of the crack tip. A **stress concentration factor** is the ratio of the highest stress (σ_{max}) to a reference stress (σ) of the gross cross-section. As the radius of curvature approaches zero, the maximum stress approaches infinity. Note that the stress concentration factor is a function of the geometry of a crack, and not of its size. These factors can be found in typical engineering reference materials to predict the stresses that could otherwise not be analyzed using strength of materials approaches. This is not to be confused with 'Stress Intensity Factor'.

Concentration factor calculation

There are experimental methods for measuring stress concentration factor including photoelastic stress analysis, brittle coatings or strain gauges. While all these approaches have been successful, all also have experimental, environmental, accuracy and/or measurement disadvantages.

During the design phase, there are multiple approaches to estimating stress concentration factors. Several catalogs of stress concentration factors have been published. Perhaps most famous is *Stress Concentration Design Factors* by Peterson, first published in 1953. Finite element methods are commonly used in design today. Theoretical approaches, using elasticity or strength of material considerations, can lead to equations similar to the one shown above.

There may be small differences between the catalog, FEM and theoretical values calculated. Each method has advantages and disadvantages. Many catalog curves were derived from experimental data. FEM calculates the peak stresses directly and nominal stresses may be easily found by integrating stresses in the surrounding material. The result is that engineering judgment may have to be used when selecting which data

applies to making a design decision. Many theoretical stress concentration factors have been derived for infinite or semi-infinite geometries which may not be analyzable and are not testable in a stress lab, but tackling a problem using two or more of these approaches will allow an engineer to achieve an accurate conclusion.

Chapter 5

Yield Surface

A **yield surface** is a five-dimensional surface in the six-dimensional space of stresses. The yield surface is usually convex and the state of stress of *inside* the yield surface is elastic. When the stress state lies on the surface the material is said to have reached its yield point and the material is said to have become plastic. Further deformation of the material causes the stress state to remain on the yield surface, even though the surface itself may change shape and size as the plastic deformation evolves. This is because stress states that lie outside the yield surface are non-permissible in rate-independent plasticity, though not in some models of viscoplasticity.

The yield surface is usually expressed in terms of (and visualized in) a three-dimensional principal stress space $(\sigma_1, \sigma_2, \sigma_3)$, a two- or three-dimensional space spanned by stress invariants (I_1, J_2, J_3) or a version of the three-dimensional Haigh–Westergaard stress space. Thus we may write the equation of the yield surface (that is, the yield function) in the forms:

- $f(\sigma_1, \sigma_2, \sigma_3) = 0$ where σ_i are the principal stresses.
- $f(I_1, J_2, J_3) = 0$ where I_1 is the first principal invariant of the Cauchy stress and J_2, J_3 are the second and third principal invariants of the deviatoric part of the Cauchy stress.
- $f(p, q, r) = 0$ where p, q are scaled versions of I_1 and J_2 and r is a function of J_2, J_3 .
- $f(\xi, \rho, \theta) = 0$ where ξ, ρ are scaled versions of I_1 and J_2 , and θ is the **Lode angle**.

Invariants used to describe yield surfaces

The first principal invariant of the Cauchy stress (I_1), and the second and third principal invariants of the deviatoric part of the Cauchy stress (J_2, J_3) are defined as

$$I_1 = \text{Tr}(\boldsymbol{\sigma}) = \sigma_1 + \sigma_2 + \sigma_3$$

$$J_2 = \frac{1}{2} \mathbf{s} : \mathbf{s} = \frac{1}{6} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$J_3 = \det(\mathbf{s}) = \frac{1}{3} (\mathbf{s} \cdot \mathbf{s}) : \mathbf{s} = s_1 s_2 s_3$$

where $\boldsymbol{\sigma}$ is the Cauchy stress and $\sigma_1, \sigma_2, \sigma_3$ are its principal values, \mathbf{s} is the deviatoric part of the Cauchy stress and s_1, s_2, s_3 are its principal values.

The quantities p, q, r are usually used to describe yield surfaces for cohesive frictional materials such as rocks, soils, and ceramics. These quantities are defined as

$$p = \frac{1}{3} I_1 \quad ; \quad q = \sqrt{3} J_2 = \sigma_{\text{eq}} \quad ; \quad r = 3 \left(\frac{J_3}{2} \right)^{1/3}$$

where σ_{eq} is the **equivalent stress**.

The quantities ξ, ρ, θ describe a cylindrical coordinate system (the **Haigh–Westergaard** coordinates) and are defined as

$$\xi = \frac{1}{\sqrt{3}} I_1 = \sqrt{3} p \quad ; \quad \rho = \sqrt{2} J_2 = \sqrt{\frac{2}{3}} q \quad ; \quad \cos(3\theta) = \left(\frac{r}{q} \right)^3 = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$$

The $\xi - \rho$ plane is also called the **Rendulic plane**. The angle θ is called the **Lode angle** and the relation between θ and J_2, J_3 was first given by Nayak and Zienkiewicz in 1972

The principal stresses and the Haigh–Westergaard coordinates are related by

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} \xi \\ \xi \\ \xi \end{bmatrix} + \sqrt{\frac{2}{3}} \rho \begin{bmatrix} \cos \theta \\ \cos \left(\theta - \frac{2\pi}{3} \right) \\ \cos \left(\theta + \frac{2\pi}{3} \right) \end{bmatrix}$$

Examples of yield surfaces

There are several different yield surfaces known in engineering, and those most popular are listed below.

Tresca yield surface

The Tresca or *maximum shear stress* yield criterion is taken to be the work of Henri Tresca. It is also referred as the Tresca–Guest (TG) criterion. The functional form of this yield criterion is

$$f(\sigma_1, \sigma_2, \sigma_3) = 0 .$$

In terms of the principal stresses the Tresca criterion is expressed as

$$\max(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) = \sigma_0$$

Figure 1 shows the Tresca–Guest yield surface in the three-dimensional space of principal stresses. It is a prism of six sides and having infinite length. This means that the material remains elastic when all three principal stresses are roughly equivalent (a hydrostatic pressure), no matter how much it is compressed or stretched. However, when one of principal stresses becomes smaller (or larger) than the others the material is subject to shearing. In such situations, if the shear stress reaches the yield limit then the material enters the plastic domain. Figure 2 shows the Tresca–Guest yield surface in two-dimensional stress space, it is a cross section of the prism along the σ_1, σ_2 plane.

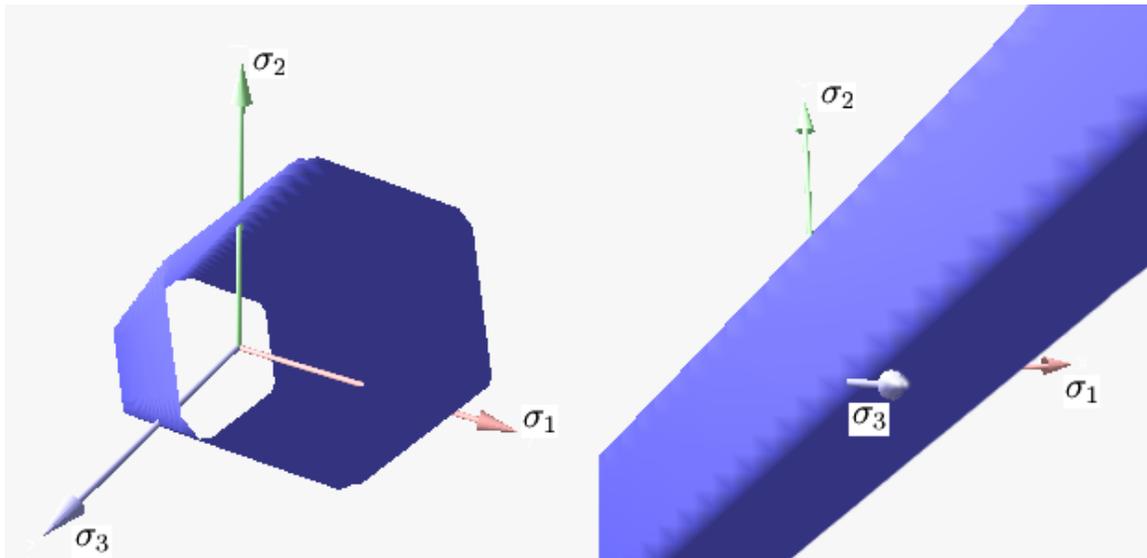


Figure 1: View of Tresca–Guest yield surface in 3D space of principal stresses

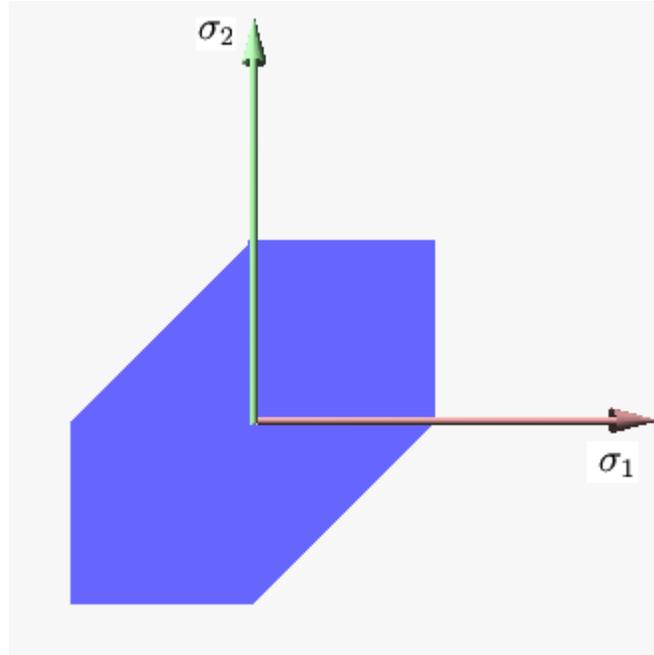


Figure 2: Tresca–Guest yield surface in 2D space (σ_1, σ_2)

Huber–von Mises yield surface

The von Mises yield criterion (also known as Prandtl–Reuss yield criterion) has the functional form

$$f(J_2) = 0 .$$

This yield criterion is often credited to Maximilian Huber and Richard von Mises. It is also referred to as the Huber–Mises–Hencky (HMH) criterion.

The von Mises yield criterion is expressed in the principal stresses as

$$\sqrt{3J_2} = \sigma_y \quad \text{or,} \quad (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_y^2$$

where σ_y is the yield stress in uniaxial tension.

Figure 3 shows the von Mises yield surface in the three-dimensional space of principal stresses. It is a circular cylinder of infinite length with its axis inclined at equal angles to the three principal stresses. Figure 4 shows the von Mises yield surface in two-dimensional space compared with Tresca–Guest criterion. A cross section of the von Mises cylinder on the plane of σ_1, σ_2 produces the elliptical shape of the yield surface.

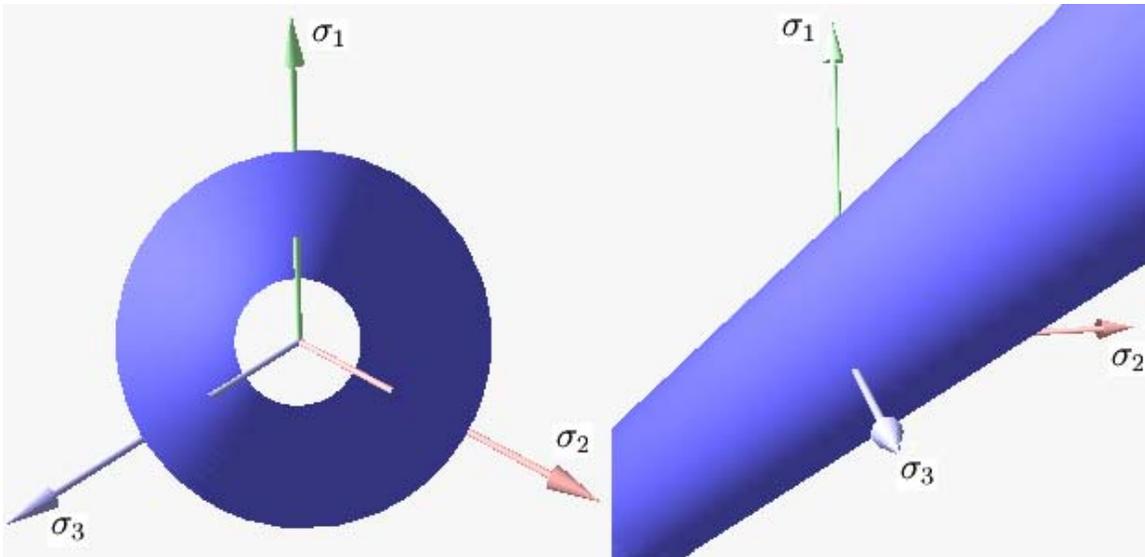


Figure 3: View of Huber–Mises–Hencky yield surface in 3D space of principal stresses

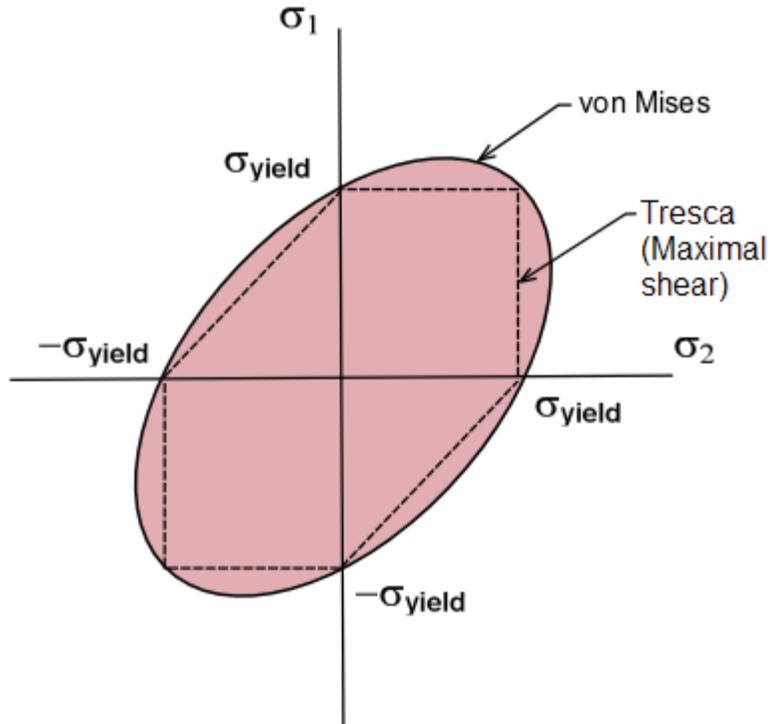


Figure 4: Comparison of Tresca–Guest and Huber–Mises–Hencky criteria in 2D space (σ_1, σ_2)

Mohr–Coulomb yield surface

The Mohr–Coulomb yield (failure) criterion is a two-parameter yield criterion which has the functional form

$$f(\sigma_1, \sigma_2, \sigma_3) = 0$$

This model is often used to model concrete, soil or granular materials.

The Mohr–Coulomb yield criterion may be expressed as:

$$\pm \frac{\sigma_1 - \sigma_2}{2} = c - K \left(\frac{\sigma_1 + \sigma_2}{2} \right); \quad \pm \frac{\sigma_2 - \sigma_3}{2} = c - K \left(\frac{\sigma_2 + \sigma_3}{2} \right); \quad \pm \frac{\sigma_3 - \sigma_1}{2} = c - K \left(\frac{\sigma_3 + \sigma_1}{2} \right)$$

where

$$m = \frac{\sigma_c}{\sigma_t}, \quad K = \frac{m - 1}{m + 1}; \quad c = \left(\frac{1}{m + 1} \right) \sigma_c = \left(\frac{m}{m + 1} \right) \sigma_t$$

and the parameters σ_c and σ_t are the yield (failure) stresses of the material in uniaxial compression and tension, respectively. If $K = 0$ then the Mohr–Coulomb criterion reduces to the Tresca–Guest criterion.

Figure 5 shows Mohr–Coulomb yield surface in the three-dimensional space of principal stresses. It is a conical prism and K determines the inclination angle of conical surface. Figure 6 shows Mohr–Coulomb yield surface in two-dimensional stress space. It is a cross section of this conical prism on the plane of σ_1, σ_2 .

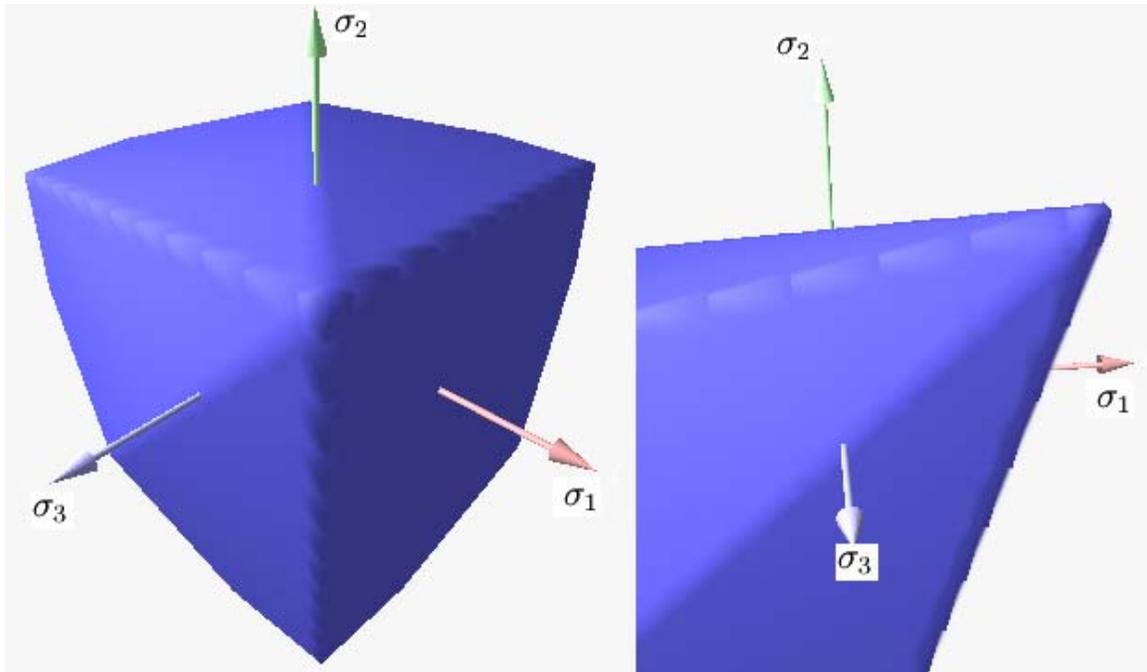


Figure 5: View of Mohr–Coulomb yield surface in 3D space of principal stresses

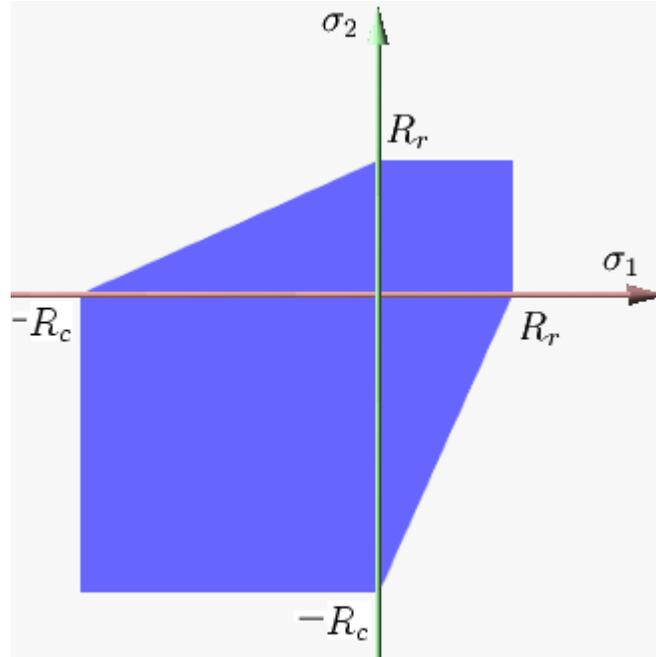


Figure 6: Mohr–Coulomb yield surface in 2D space (σ_1, σ_2)

The following formula was used to plot the surface in Fig. 5 :

$$\max \left(\frac{|\sigma_1 - \sigma_2|}{2} - c + K \frac{\sigma_1 + \sigma_2}{2}, \frac{|\sigma_2 - \sigma_3|}{2} - c + K \frac{\sigma_2 + \sigma_3}{2}, \frac{|\sigma_3 - \sigma_1|}{2} - c + K \frac{\sigma_3 + \sigma_1}{2} \right) = 0$$

Drucker–Prager yield surface

The Drucker–Prager yield criterion has the function form

$$f(I_1, J_2) = 0 .$$

This criterion is most often used for concrete where both normal and shear stresses can determine failure. The Drucker–Prager yield criterion may be expressed as

$$\alpha (\sigma_1 + \sigma_2 + \sigma_3) + \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{6}} = K$$

where

$$m = \frac{\sigma_c}{\sigma_t} ; \quad K = \frac{2\sigma_c}{\sqrt{3}(m+1)} ; \quad \alpha = \frac{m-1}{\sqrt{3}(m+1)}$$

and σ_c, σ_t are the uniaxial yield stresses in compression and tension respectively.

Figure 7 shows Drucker–Prager yield surface in the three-dimensional space of principal stresses. It is a regular cone. Figure 8 shows Drucker–Prager yield surface in two-dimensional space. The ellipsoidal-shaped elastic domain is a cross section of the cone on the plane of σ_1, σ_2 ; here it is shown enclosing the elastic domain for the Mohr–Coulomb yield criterion, although the converse scenario is also possible.

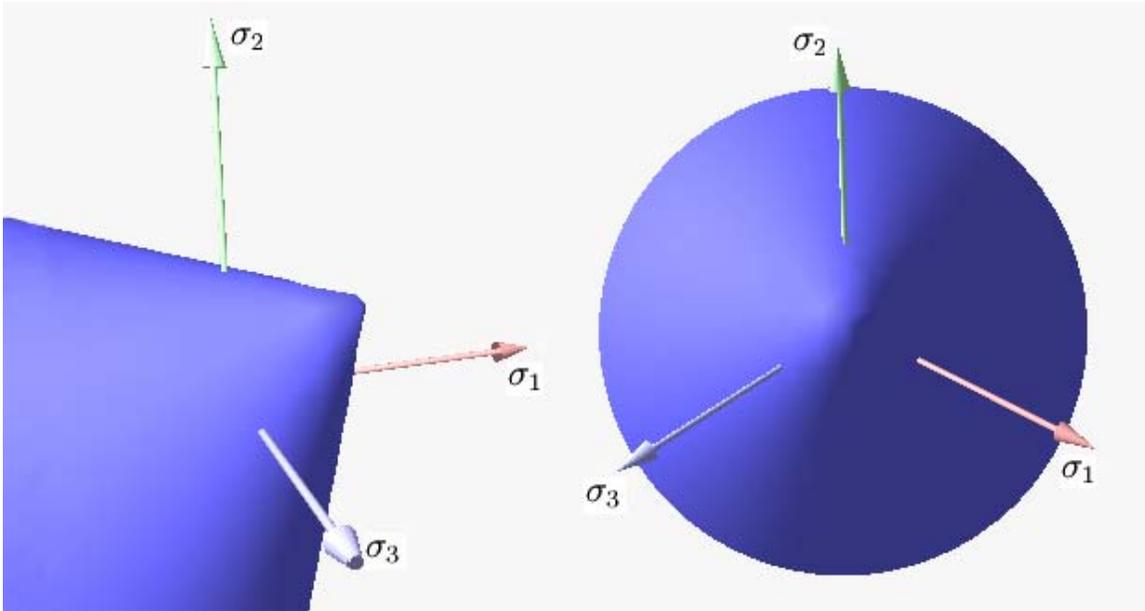


Figure 7: View of Drucker–Prager yield surface in 3D space of principal stresses

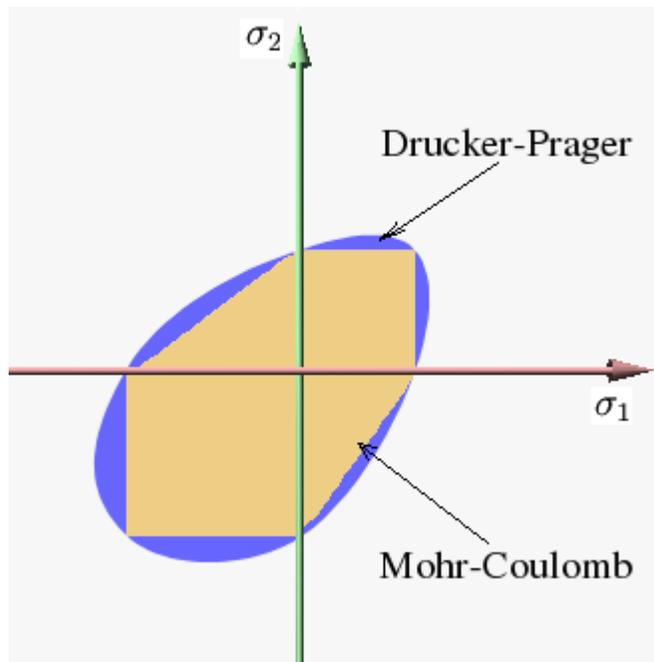


Figure 8: Drucker–Prager and Mohr–Coulomb yield surface in 2D space (σ_1, σ_2)

Bresler–Pister yield surface

The Bresler–Pister yield criterion is an extension of the Drucker–Prager yield criterion that uses three parameters.

The Bresler–Pister yield surface has the functional form

$$f(I_1, J_2) = 0 .$$

In terms of the principal stresses, this yield criterion may be expressed as

$$f := \frac{1}{\sqrt{6}} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} - c_0 - c_1 (\sigma_1 + \sigma_2 + \sigma_3) - c_2 (\sigma_1 + \sigma_2 + \sigma_3)^2$$

where c_0, c_1, c_2 are material constants. The additional parameter c_2 gives the yield surface a ellipsoidal cross section when viewed from a direction perpendicular to its axis. If σ_c is the yield stress in uniaxial compression, σ_t is the yield stress in uniaxial tension, and σ_b is the yield stress in biaxial compression, the parameters can be expressed as

$$c_1 = \left(\frac{\sigma_t - \sigma_c}{\sqrt{3}(\sigma_t + \sigma_c)} \right) \left(\frac{4\sigma_b^2 - \sigma_b(\sigma_c + \sigma_t) + \sigma_c\sigma_t}{4\sigma_b^2 + 2\sigma_b(\sigma_t - \sigma_c) - \sigma_c\sigma_t} \right)$$

$$c_2 = \left(\frac{1}{\sqrt{3}(\sigma_t + \sigma_c)} \right) \left(\frac{\sigma_b(3\sigma_t - \sigma_c) - 2\sigma_c\sigma_t}{4\sigma_b^2 + 2\sigma_b(\sigma_t - \sigma_c) - \sigma_c\sigma_t} \right)$$

$$c_0 = \frac{\sigma_c}{\sqrt{3}} + c_1\sigma_c - c_2\sigma_c^2$$

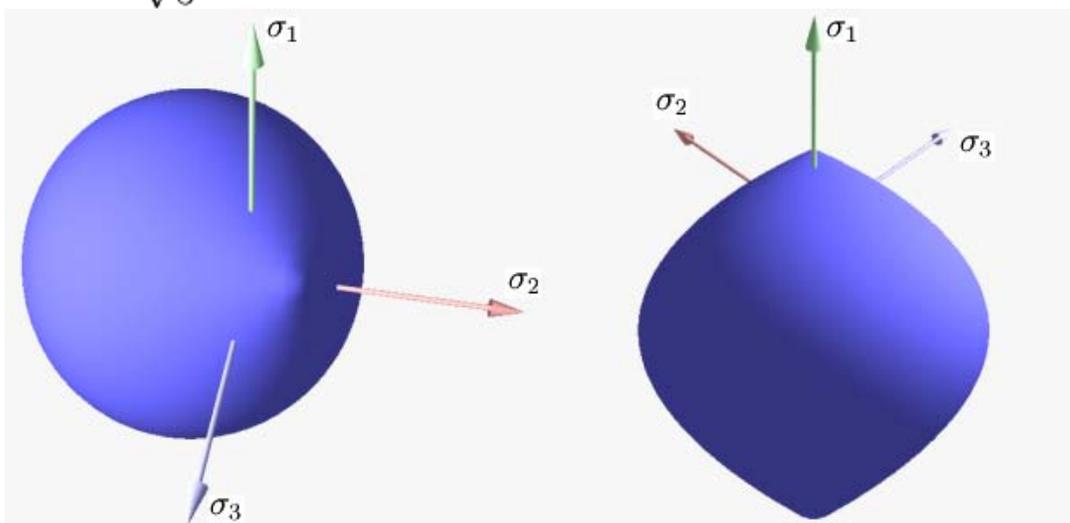


Figure 9: View of Bresler–Pister yield surface in 3D space of principal stresses

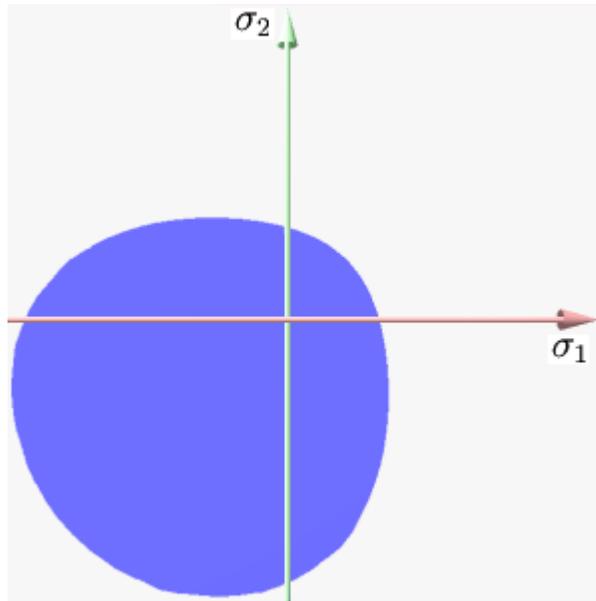


Figure 10: Bresler–Pister yield surface in 2D space (σ_1, σ_2)

Willam–Warnke yield surface

The Willam–Warnke yield criterion is a three-parameter smoothed version of the Mohr–Coulomb yield criterion that has similarities in form to the Drucker–Prager and Bresler–Pister yield criteria.

The yield criterion has the functional form

$$f(I_1, J_2, J_3) = 0 .$$

However, it is more commonly expressed in Haigh–Westergaard coordinates as

$$f(\xi, \rho, \theta) = 0 .$$

The cross-section of the surface when viewed along its axis is a smoothed triangle (unlike Mohr–Coulomb). The Willam–Warnke yield surface is convex and has unique and well defined first and second derivatives on every point of its surface. Therefore the Willam–Warnke model is computationally robust and has been used for a variety of cohesive-frictional materials.

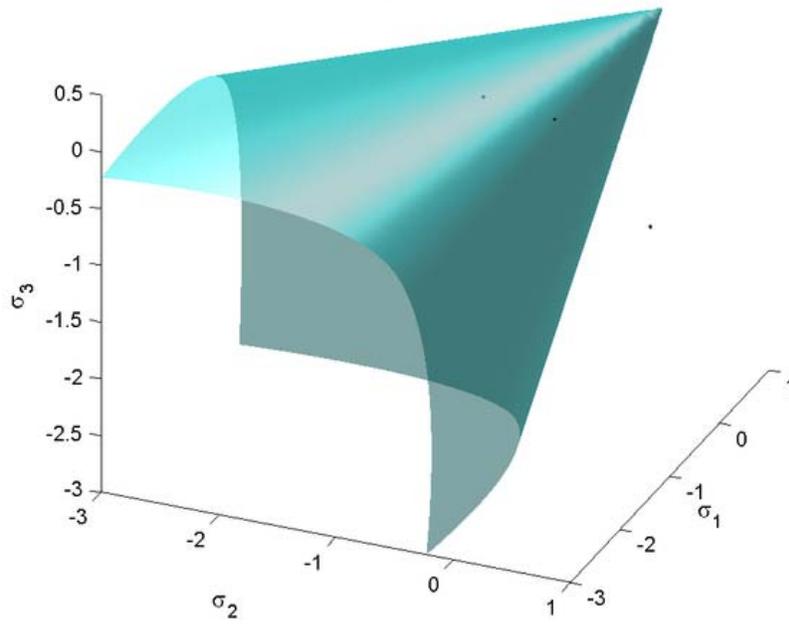


Figure 11: View of Willam–Warnke yield surface in 3D space of principal stresses

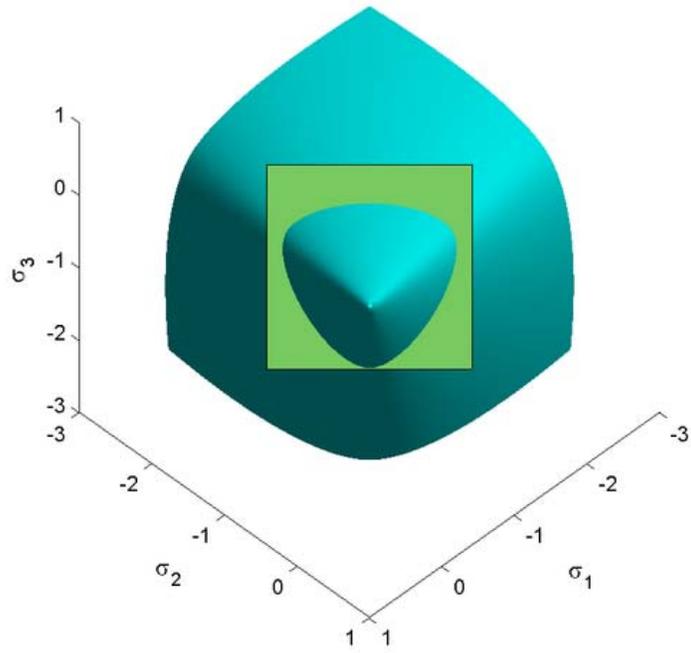


Figure 12: Willam–Warnke yield surface in the π -plane

Chapter 6

Infinitesimal Strain Theory

In continuum mechanics, the **infinitesimal strain theory**, sometimes called **small deformation theory**, **small displacement theory**, or **small displacement-gradient theory**, deals with *infinitesimal deformations* of a continuum body. For an infinitesimal deformation the displacements \mathbf{u} and the displacement gradients $\nabla \mathbf{u}$ are small compared to unity, i.e., $\|\mathbf{u}\| \ll 1$ and $\|\nabla \mathbf{u}\| \ll 1$, allowing for the *geometric linearization* of the Lagrangian finite strain tensor \mathbf{E} , and the Eulerian finite strain tensor \mathbf{e} , i.e. the non-linear or second-order terms of the finite strain tensor can be neglected. The linearized Lagrangian and Eulerian strain tensors are approximately the same and can be approximated by the **infinitesimal strain tensor** or **Cauchy's strain tensor**, $\boldsymbol{\varepsilon}$. Thus,

$$\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon} = \frac{1}{2} ((\nabla \mathbf{u})^T + \nabla \mathbf{u})$$

or

$$E_{KL} \approx e_{rs} \approx \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

The infinitesimal strain theory is used in the analysis of deformations of materials exhibiting elastic behaviour, such as materials found in mechanical and civil engineering applications, e.g. concrete and steel.

Infinitesimal strain tensor

For *infinitesimal deformations* of a continuum body, in which the displacements and the displacement gradients are small compared to unity, i.e., $\|\mathbf{u}\| \ll 1$ and $\|\nabla \mathbf{u}\| \ll 1$, it is possible to perform a *geometric linearization* of the Lagrangian finite strain tensor \mathbf{E} , and the Eulerian finite strain tensor \mathbf{e} . In such a linearization, the non-linear or second-order terms of the finite strain tensor are neglected. Thus we have

$$\mathbf{E} = \frac{1}{2} (\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T + \nabla_{\mathbf{x}}\mathbf{u}(\nabla_{\mathbf{x}}\mathbf{u})^T) \approx \frac{1}{2} (\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T)$$

or

$$E_{KL} = \frac{1}{2} \left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K} \frac{\partial U_M}{\partial X_L} \right) \approx \frac{1}{2} \left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} \right)$$

and

$$\mathbf{e} = \frac{1}{2} (\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T + \nabla_{\mathbf{x}}\mathbf{u}(\nabla_{\mathbf{x}}\mathbf{u})^T) \approx \frac{1}{2} (\nabla_{\mathbf{x}}\mathbf{u} + (\nabla_{\mathbf{x}}\mathbf{u})^T)$$

or

$$e_{rs} = \frac{1}{2} \left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r} - \frac{\partial u_k}{\partial x_r} \frac{\partial u_k}{\partial x_s} \right) \approx \frac{1}{2} \left(\frac{\partial u_r}{\partial x_s} + \frac{\partial u_s}{\partial x_r} \right)$$

This linearization implies that the Lagrangian description and the Eulerian description are approximately the same as there is little difference in the material and spatial coordinates of a given material point in the continuum. Therefore, the material displacement gradient components and the spatial displacement gradient components are approximately equal. Thus we have

$$\mathbf{E} \approx \mathbf{e} \approx \boldsymbol{\varepsilon} = \frac{1}{2} ((\nabla\mathbf{u})^T + \nabla\mathbf{u}) \quad \text{or} \quad E_{KL} \approx e_{rs} \approx \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i})$$

where ε_{ij} are the components of the *infinitesimal strain tensor* $\boldsymbol{\varepsilon}$, also called *Cauchy's strain tensor*, *linear strain tensor*, or *small strain tensor*.

$$\varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) & \frac{1}{2} \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \\ \frac{1}{2} \left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \right) & \frac{\partial u_2}{\partial x_2} & \frac{1}{2} \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \\ \frac{1}{2} \left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) & \frac{1}{2} \left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

or using different notation:

$$\begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) & \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) \\ \frac{1}{2} \left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} \right) & \frac{\partial u_y}{\partial y} & \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) \\ \frac{1}{2} \left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z} \right) & \frac{1}{2} \left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} \right) & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

Furthermore, since the deformation gradient can be expressed as $\mathbf{F} = \nabla \mathbf{u} + \mathbf{I}$ where \mathbf{I} is the second-order identity tensor, we have

$$\boldsymbol{\varepsilon} = \frac{1}{2} (\mathbf{F}^T + \mathbf{F}) - \mathbf{I}$$

Also, from the general expression for the Lagrangian and Eulerian finite strain tensors we have

$$\mathbf{E}_{(m)} = \frac{1}{2m} (\mathbf{U}^{2m} - \mathbf{I}) = \frac{1}{2m} [(\mathbf{F}^T \mathbf{F})^m - \mathbf{I}] \approx \frac{1}{2m} [\{\nabla \mathbf{u} + (\nabla \mathbf{u})^T + \mathbf{I}\}^m - \mathbf{I}] \approx \boldsymbol{\varepsilon}$$

$$\mathbf{e}_{(m)} = \frac{1}{2m} (\mathbf{V}^{2m} - \mathbf{I}) = \frac{1}{2m} [(\mathbf{F} \mathbf{F}^T)^m - \mathbf{I}] \approx \boldsymbol{\varepsilon}$$

Geometric derivation of the infinitesimal strain tensor

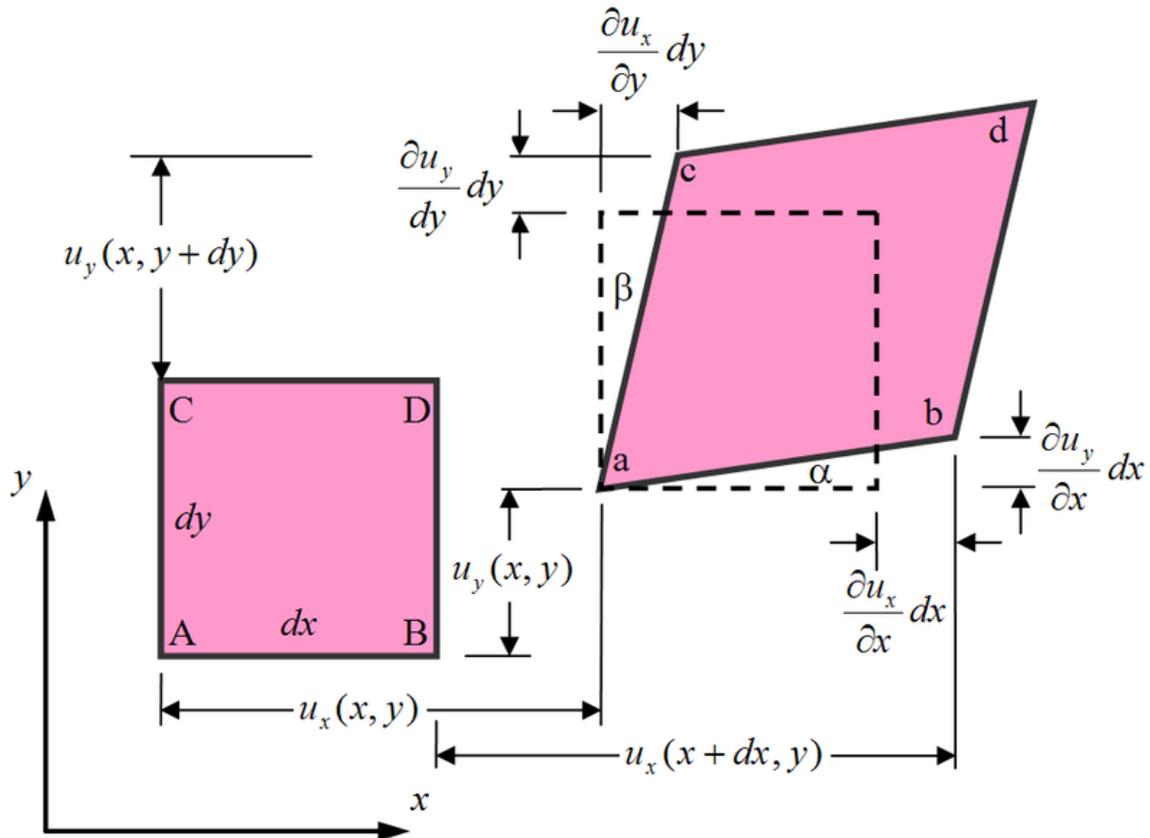


Figure 1. Two-dimensional geometric deformation of an infinitesimal material element.

Considering a two-dimensional deformation of an infinitesimal rectangular material element with dimensions dx by dy (Figure 1), which after deformation, takes the form of a rhombus. From the geometry of Figure 1 we have

$$\begin{aligned}\overline{ab} &= \sqrt{\left(dx + \frac{\partial u_x}{\partial x} dx\right)^2 + \left(\frac{\partial u_y}{\partial x} dx\right)^2} \\ &= \sqrt{1 + 2\frac{\partial u_x}{\partial x} + \left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2} dx\end{aligned}$$

For very small displacement gradients, i.e., $\|\nabla \mathbf{u}\| \ll 1$, we have

$$\overline{ab} \approx dx + \frac{\partial u_x}{\partial x} dx$$

The normal strain in the x -direction of the rectangular element is defined by

$$\varepsilon_x = \frac{\overline{ab} - \overline{AB}}{\overline{AB}}$$

and knowing that $\overline{AB} = dx$, we have

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

Similarly, the normal strain in the y -direction, and z -direction, becomes

$$\varepsilon_y = \frac{\partial u_y}{\partial y} \quad , \quad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

The engineering shear strain, or the change in angle between two originally orthogonal material lines, in this case line AC and AB , is defined as

$$\gamma_{xy} = \alpha + \beta$$

From the geometry of Figure 1 we have

$$\tan \alpha = \frac{\frac{\partial u_y}{\partial x} dx}{dx + \frac{\partial u_x}{\partial x} dx} = \frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}} \quad , \quad \tan \beta = \frac{\frac{\partial u_x}{\partial y} dy}{dy + \frac{\partial u_y}{\partial y} dy} = \frac{\frac{\partial u_x}{\partial y}}{1 + \frac{\partial u_y}{\partial y}}$$

For small rotations, i.e. α and β are $\ll 1$ we have

$$\tan \alpha \approx \alpha \quad , \quad \tan \beta \approx \beta$$

and, again, for small displacement gradients, we have

$$\alpha = \frac{\partial u_y}{\partial x} \quad , \quad \beta = \frac{\partial u_x}{\partial y}$$

thus

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

By interchanging x and y and u_x and u_y , it can be shown that $\gamma_{xy} = \gamma_{yx}$

Similarly, for the y - z and x - z planes, we have

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \quad , \quad \gamma_{zx} = \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

It can be seen that the tensorial shear strain components of the infinitesimal strain tensor can then be expressed using the engineering strain definition, γ , as

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \epsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \epsilon_{zz} \end{bmatrix}$$

Physical interpretation of the infinitesimal strain tensor

From finite strain theory we have

$$d\mathbf{x}^2 - d\mathbf{X}^2 = d\mathbf{X} \cdot 2\mathbf{E} \cdot d\mathbf{X} \quad \text{or} \quad (dx)^2 - (dX)^2 = 2E_{KL} dX_K dX_L$$

For infinitesimal strains then we have

$$d\mathbf{x}^2 - d\mathbf{X}^2 = d\mathbf{X} \cdot 2\boldsymbol{\epsilon} \cdot d\mathbf{X} \quad \text{or} \quad (dx)^2 - (dX)^2 = 2\epsilon_{KL} dX_K dX_L$$

Dividing by $(dX)^2$ we have

$$\frac{dx - dX}{dX} \frac{dx + dX}{dX} = 2\epsilon_{ij} \frac{dX_i}{dX} \frac{dX_j}{dX}$$

For small deformations we assume that $dx \approx dX$, thus the second term of the left hand

side becomes: $\frac{dx + dX}{dX} \approx 2$.

Then we have

$$\frac{dx - dX}{dX} = \varepsilon_{ij} N_i N_j = \mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N}$$

where $N_i = \frac{dX_i}{dX}$, is the unit vector in the direction of $d\mathbf{X}$, and the left-hand-side expression is the normal strain $e^{(\mathbf{N})}$ in the direction of \mathbf{N} . For the particular case of \mathbf{N} in the X_1 direction, i.e. $\mathbf{N} = \mathbf{I}_1$, we have

$$e_{(\mathbf{I}_1)} = \mathbf{I}_1 \cdot \boldsymbol{\varepsilon} \cdot \mathbf{I}_1 = \varepsilon_{11}$$

Similarly, for $\mathbf{N} = \mathbf{I}_2$ and $\mathbf{N} = \mathbf{I}_3$ we can find the normal strains ε_{22} and ε_{33} , respectively. Therefore, the diagonal elements of the infinitesimal strain tensor are the normal strains in the coordinate directions.

Strain transformation rules

If we choose an orthonormal coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ we can write the tensor in terms of components with respect to those base vectors as

$$\boldsymbol{\varepsilon} = \sum_{i=1}^3 \sum_{j=1}^3 \varepsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

In matrix form,

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix}$$

We can easily choose to use another orthonormal coordinate system $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$ instead. In that case the components of the tensor are different, say

$$\boldsymbol{\varepsilon} = \sum_{i=1}^3 \sum_{j=1}^3 \hat{\varepsilon}_{ij} \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_j \quad \Rightarrow \quad \underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13} \\ \hat{\varepsilon}_{12} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23} \\ \hat{\varepsilon}_{13} & \hat{\varepsilon}_{23} & \hat{\varepsilon}_{33} \end{bmatrix}$$

The components of the strain in the two coordinate systems are related by

$$\hat{\varepsilon}_{ij} = l_{ip} l_{jq} \varepsilon_{pq}$$

where the Einstein summation convention for repeated indices has been used and $l_{ij} = \hat{\mathbf{e}}_i \cdot \mathbf{e}_j$. In matrix form

$$\underline{\hat{\boldsymbol{\varepsilon}}} = \underline{\mathbf{L}} \underline{\boldsymbol{\varepsilon}} \underline{\mathbf{L}}^T$$

or

$$\begin{bmatrix} \hat{\varepsilon}_{11} & \hat{\varepsilon}_{12} & \hat{\varepsilon}_{13} \\ \hat{\varepsilon}_{21} & \hat{\varepsilon}_{22} & \hat{\varepsilon}_{23} \\ \hat{\varepsilon}_{31} & \hat{\varepsilon}_{32} & \hat{\varepsilon}_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix}^T$$

Strain invariants

Certain operations on the strain tensor give the same result without regard to which orthonormal coordinate system is used to represent the components of strain. The results of these operations are called **strain invariants**. The most commonly used strain invariants are

$$\begin{aligned} I_1 &= \text{tr}(\boldsymbol{\varepsilon}) \\ I_2 &= \frac{1}{2} \{ \text{tr}(\boldsymbol{\varepsilon}^2) - [\text{tr}(\boldsymbol{\varepsilon})]^2 \} \\ I_3 &= \det(\boldsymbol{\varepsilon}) \end{aligned}$$

In terms of components

$$\begin{aligned} I_1 &= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \\ I_2 &= \varepsilon_{12}^2 + \varepsilon_{23}^2 + \varepsilon_{31}^2 - \varepsilon_{11}\varepsilon_{22} - \varepsilon_{22}\varepsilon_{33} - \varepsilon_{33}\varepsilon_{11} \\ I_3 &= \varepsilon_{11}(\varepsilon_{22}\varepsilon_{33} - \varepsilon_{23}^2) - \varepsilon_{12}(\varepsilon_{12}\varepsilon_{33} - \varepsilon_{23}\varepsilon_{31}) + \varepsilon_{13}(\varepsilon_{12}\varepsilon_{23} - \varepsilon_{22}\varepsilon_{31}) \end{aligned}$$

Principal strains

It can be shown that it is possible to find a coordinate system ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) in which the components of the strain tensor are

$$\underline{\boldsymbol{\varepsilon}} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix} \implies \boldsymbol{\varepsilon} = \varepsilon_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \varepsilon_2 \mathbf{n}_2 \otimes \mathbf{n}_2 + \varepsilon_3 \mathbf{n}_3 \otimes \mathbf{n}_3$$

The components of the strain tensor in the ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) coordinate system are called the **principal strains** and the directions \mathbf{n}_i are called the directions of principal strain. Since there are no shear strain components in this coordinate system, the principal strains represent the maximum and minimum stretches of an elemental volume.

If we are given the components of the strain tensor in an arbitrary orthonormal coordinate system, we can find the principal strains using an eigenvalue decomposition determined by solving the system of equations

$$(\underline{\underline{\epsilon}} - \epsilon_i \underline{\underline{\mathbf{I}}}) \mathbf{n}_i = \underline{\underline{\mathbf{0}}}$$

This system of equations is equivalent to finding the vector \mathbf{n}_i along which the strain tensor becomes a pure stretch with no shear component.

Volumetric strain

The *dilatation* (the relative variation of the volume) is the trace of the tensor:

$$\delta = \frac{\Delta V}{V_0} = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

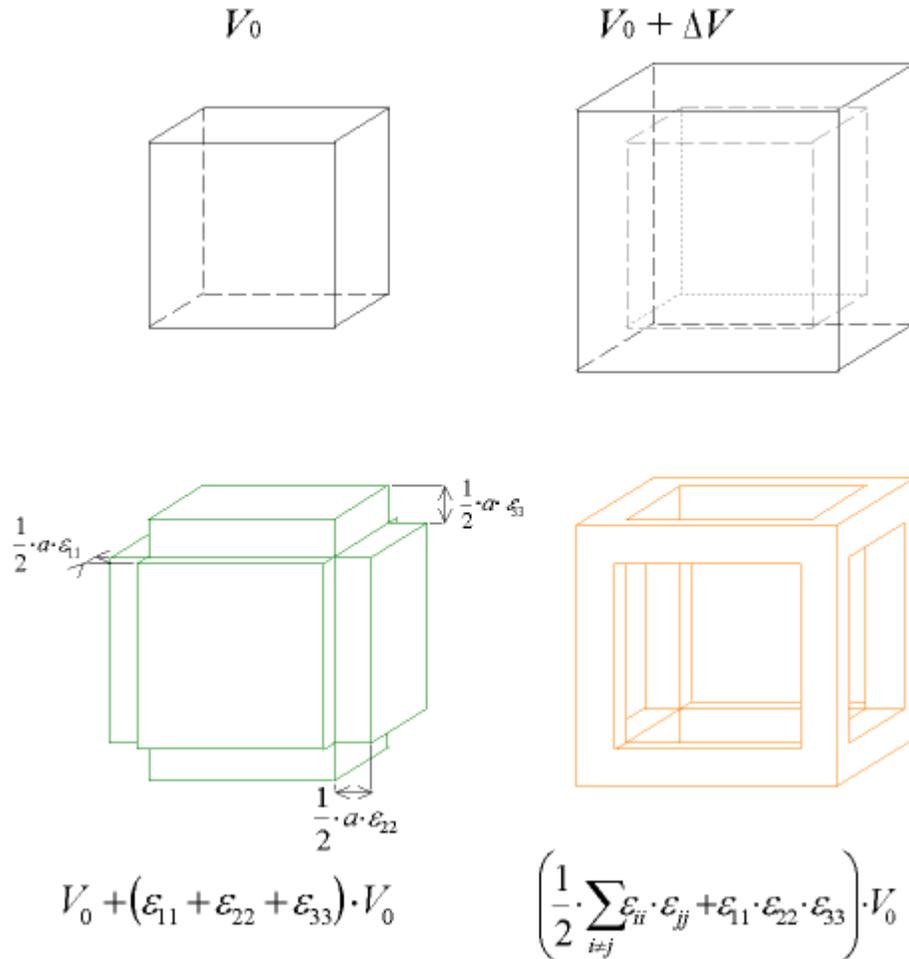
Actually, if we consider a cube with an edge length a , it is a quasi-cube after the deformation (the variations of the angles do not change the volume) with the dimensions $a \cdot (1 + \epsilon_{11}) \times a \cdot (1 + \epsilon_{22}) \times a \cdot (1 + \epsilon_{33})$ and $V_0 = a^3$, thus

$$\frac{\Delta V}{V_0} = \frac{(1 + \epsilon_{11} + \epsilon_{22} + \epsilon_{33} + \epsilon_{11} \cdot \epsilon_{22} + \epsilon_{11} \cdot \epsilon_{33} + \epsilon_{22} \cdot \epsilon_{33} + \epsilon_{11} \cdot \epsilon_{22} \cdot \epsilon_{33}) \cdot a^3 - a^3}{a^3}$$

as we consider small deformations,

$$1 \gg \epsilon_{ii} \gg \epsilon_{ii} \cdot \epsilon_{jj} \gg \epsilon_{11} \cdot \epsilon_{22} \cdot \epsilon_{33}$$

therefore the formula.



Real variation of volume (top) and the approximated one (bottom): the green drawing shows the estimated volume and the orange drawing the neglected volume

In case of pure shear, we can see that there is no change of the volume.

Strain deviator tensor

The infinitesimal strain tensor ε_{ij} , similarly to the stress tensor, can be expressed as the sum of two other tensors:

1. a **mean strain tensor** or **volumetric strain tensor** or **spherical strain tensor**, $\varepsilon_M \delta_{ij}$, related to dilation or volume change; and
2. a deviatoric component called the **strain deviator tensor**, ε'_{ij} , related to distortion.

$$\varepsilon_{ij} = \varepsilon'_{ij} + \varepsilon_M \delta_{ij}$$

where ε_M is the mean strain given by

$$\varepsilon_M = \frac{\varepsilon_{kk}}{3} = \frac{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}{3} = \frac{1}{3} I_1^e$$

The deviatoric strain tensor can be obtained by subtracting the mean strain tensor from the infinitesimal strain tensor:

$$\varepsilon'_{ij} = \varepsilon_{ij} - \frac{\varepsilon_{kk}}{3} \delta_{ij}$$

$$\begin{bmatrix} \varepsilon'_{11} & \varepsilon'_{12} & \varepsilon'_{13} \\ \varepsilon'_{21} & \varepsilon'_{22} & \varepsilon'_{23} \\ \varepsilon'_{31} & \varepsilon'_{32} & \varepsilon'_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} - \begin{bmatrix} \varepsilon_M & 0 & 0 \\ 0 & \varepsilon_M & 0 \\ 0 & 0 & \varepsilon_M \end{bmatrix}$$

$$= \begin{bmatrix} \varepsilon_{11} - \varepsilon_M & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} - \varepsilon_M & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} - \varepsilon_M \end{bmatrix}$$

Octahedral strains

Let ($\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$) be the directions of the three principal strains. An **octahedral plane** is whose normal makes equal angles with the three principal directions. The engineering shear strain on an octahedral plane is called the **octahedral shear strain** and is given by

$$\gamma_{\text{oct}} = \frac{2}{3} \sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the principal strains.

The normal strain on an octahedral plane is given by

$$\varepsilon_{\text{oct}} = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

Equivalent strain

A scalar quantity called the **equivalent strain**, or the von Mises equivalent strain, is often used to describe the state of strain in solids. Several definitions of equivalent strain can be found in the literature. A definition that is commonly used in the literature on plasticity is

$$\varepsilon_{\text{eq}} = \sqrt{\frac{2}{3} \boldsymbol{\varepsilon}^{\text{dev}} : \boldsymbol{\varepsilon}^{\text{dev}}} = \sqrt{\frac{2}{3} \varepsilon_{ij}^{\text{dev}} \varepsilon_{ij}^{\text{dev}}}; \quad \boldsymbol{\varepsilon}^{\text{dev}} = \boldsymbol{\varepsilon} - \frac{1}{3} \text{tr}(\boldsymbol{\varepsilon}) \mathbf{1}$$

This quantity is work conjugate to the equivalent stress defined as

$$\sigma_{\text{eq}} = \sqrt{\frac{3}{2} \boldsymbol{\sigma}^{\text{dev}} : \boldsymbol{\sigma}^{\text{dev}}}$$

Compatibility equations

For prescribed strain components ϵ_{ij} the strain tensor equation $u_{i,j} + u_{j,i} = 2\epsilon_{ij}$ represents a system of six differential equations for the determination of three displacement components u_i , giving an over-determined system. Thus, a solution does not generally exist for an arbitrary choice of strain components. Therefore, some restrictions, named *compatibility equations*, are imposed upon the strain components. With the addition of the three compatibility equations the number of independent equations is reduced to three, matching the number of unknown displacement components. These constraints on the strain tensor were discovered by Saint-Venant, and are called the "Saint Venant compatibility equations".

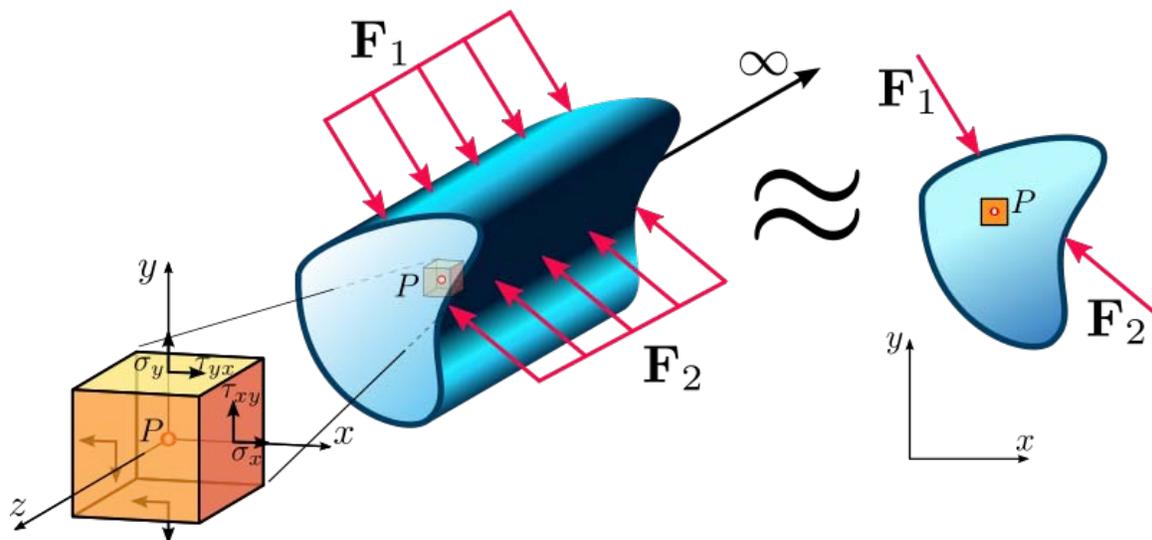
The compatibility functions serve to assure a single-valued continuous displacement function u_i . If the elastic medium is visualized as a set of infinitesimal cubes in the unstrained state, after the medium is strained, an arbitrary strain tensor may not yield a situation in which the distorted cubes still fit together without overlapping.

In index notation, the compatibility equations are expressed as

$$\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0$$

Special cases

Plane strain



Plane strain state in a continuum.

In real engineering components, stress (and strain) are 3-D tensors but in prismatic structures such as a long metal billet, the length of the structure is much greater than the other two dimensions. The strains associated with length, i.e. the normal strain ϵ_{33} and the shear strains ϵ_{13} and ϵ_{23} (if the length is the 3-direction) are constrained by nearby

material and are small compared to the *cross-sectional strains*. The strain tensor can then be approximated by:

$$\underline{\underline{\epsilon}} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

in which the double underline indicates a second order tensor. This strain state is called *plane strain*. The corresponding stress tensor is:

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

in which the non-zero σ_{33} is needed to maintain the constraint $\epsilon_{33} = 0$. This stress term can be temporarily removed from the analysis to leave only the in-plane terms, effectively reducing the 3-D problem to a much simpler 2-D problem.

Antiplane strain

Antiplane strain is another special state of strain that can occur in a body, for instance in a region close to a screw dislocation. The strain tensor for antiplane strain is given by

$$\underline{\underline{\epsilon}} = \begin{bmatrix} 0 & 0 & \epsilon_{13} \\ 0 & 0 & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 0 \end{bmatrix}$$

Infinitesimal rotation tensor

The infinitesimal strain tensor is defined as

$$\epsilon = \frac{1}{2}[\nabla \mathbf{u} + (\nabla \mathbf{u})^T]$$

Therefore the displacement gradient can be expressed as

$$\nabla \mathbf{u} = \epsilon + \omega$$

where

$$\omega := \frac{1}{2}[\nabla \mathbf{u} - (\nabla \mathbf{u})^T]$$

The quantity $\underline{\omega}$ is the **infinitesimal rotation tensor**. This tensor is skew symmetric. For infinitesimal deformations the scalar components of $\underline{\omega}$ satisfy the condition $|\omega_{ij}| \ll 1$. Note that the displacement gradient is small only if **both** the strain tensor and the rotation tensor are infinitesimal.

The axial vector

A skew symmetric second-order tensor has three independent scalar components. These three components are used to define an **axial vector**, \mathbf{w} , as follows

$$\omega_{ij} = -e_{ijk} w_k ; \quad w_i = \frac{1}{2} e_{ijk} \omega_{jk}$$

where e_{ijk} is the permutation symbol. In matrix form

$$\underline{\underline{\omega}} = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} ; \quad \underline{\underline{\mathbf{w}}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

The axial vector is also called the **infinitesimal rotation vector**. The rotation vector is related to the displacement gradient by the relation

$$\mathbf{w} = \frac{1}{2} \nabla \times \mathbf{u}$$

In index notation

$$w_i = \frac{1}{2} e_{ijk} u_{k,j}$$

If $\|\underline{\omega}\| \ll 1$ and $\underline{\epsilon} = \mathbf{0}$ then the material undergoes an approximate rigid body rotation of magnitude $|\mathbf{w}|$ around the vector \mathbf{w} .

Relation between the strain tensor and the rotation vector

Given a continuous, single-valued displacement field \mathbf{u} and the corresponding infinitesimal strain tensor $\underline{\epsilon}$, we have

$$\nabla \times \underline{\epsilon} = e_{ijk} \epsilon_{lj,i} \mathbf{e}_k \otimes \mathbf{e}_l = \frac{1}{2} e_{ijk} [u_{l,ji} + u_{j,li}] \mathbf{e}_k \otimes \mathbf{e}_l$$

Since a change in the order of differentiation does not change the result, $u_{l,ji} = u_{l,ij}$. Therefore

$$e_{ijk} u_{l,ji} = (e_{12k} + e_{21k}) u_{l,12} + (e_{13k} + e_{31k}) u_{l,13} + (e_{23k} + e_{32k}) u_{l,32} = 0$$

Also

$$\frac{1}{2} e_{ijk} u_{j,li} = \left(\frac{1}{2} e_{ijk} u_{j,i} \right)_{,l} = \left(\frac{1}{2} e_{kij} u_{j,i} \right)_{,l} = w_{k,l}$$

Hence

$$\nabla \times \boldsymbol{\varepsilon} = w_{k,l} \mathbf{e}_k \otimes \mathbf{e}_l = \nabla \mathbf{w}$$

Relation between rotation tensor and rotation vector

From an important identity regarding the curl of a tensor we know that for a continuous, single-valued displacement field \mathbf{u} ,

$$\nabla \times (\nabla \mathbf{u}) = \mathbf{0}.$$

Since $\nabla \mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$ we have $\nabla \times \boldsymbol{\omega} = -\nabla \times \boldsymbol{\varepsilon} = -\nabla \mathbf{w}$.

Strain tensor in cylindrical coordinates

In cylindrical polar coordinates (r, θ, z) , the displacement vector can be written as

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_z \mathbf{e}_z$$

The components of the strain tensor in a cylindrical coordinate system are given by

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\ \varepsilon_{\theta\theta} &= \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right) \\ \varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\ \varepsilon_{r\theta} &= \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \\ \varepsilon_{\theta z} &= \frac{1}{2} \left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \\ \varepsilon_{zr} &= \frac{1}{2} \left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \end{aligned}$$

Strain tensor in spherical coordinates

In spherical coordinates (r, θ, ϕ) , the displacement vector can be written as

$$\mathbf{u} = u_r \mathbf{e}_r + u_\theta \mathbf{e}_\theta + u_\phi \mathbf{e}_\phi$$

The components of the strain tensor in a spherical coordinate system are given by

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r} \left(\frac{\partial u_\theta}{\partial \theta} + u_r \right)$$

$$\varepsilon_{\phi\phi} = \frac{1}{r \sin \theta} \left(\frac{\partial u_\phi}{\partial \phi} + u_r \sin \theta + u_\theta \cos \theta \right)$$

$$\varepsilon_{r\theta} = \frac{1}{2} \left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

$$\varepsilon_{\theta\phi} = \frac{1}{2r} \left(\frac{1}{\sin \theta} \frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - u_\phi \cot \theta \right)$$

$$\varepsilon_{\phi r} = \frac{1}{2} \left(\frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right)$$

Chapter 7

Ultimate Tensile Strength

Ultimate tensile strength (UTS), often shortened to **tensile strength (TS)** or **ultimate strength**, is the maximum stress that a material can withstand while being stretched or pulled before *necking*, which is when the specimen's cross-section starts to significantly contract. Tensile strength is the opposite of compressive strength and the values can be quite different.

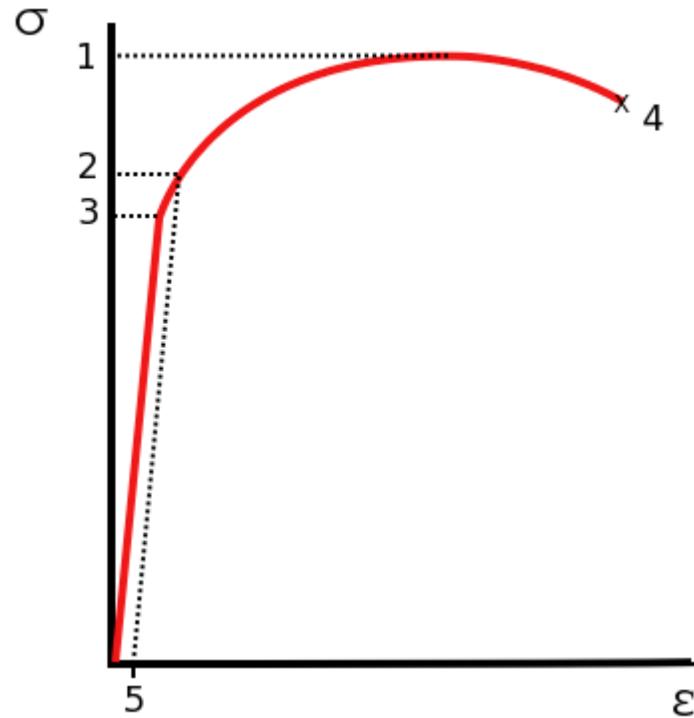
The UTS is usually found by performing a tensile test and recording the stress versus strain; the highest point of the stress-strain curve is the UTS. It is an intensive property; therefore its value does not depend on the size of the test specimen. However, it is dependent on other factors, such as the preparation of the specimen, the presence or otherwise of surface defects, and the temperature of the test environment and material.

Tensile strengths are rarely used in the design of ductile members, but they are important in brittle members. They are tabulated for common materials such as alloys, composite materials, ceramics, plastics, and wood.

Tensile strength is defined as a stress, which is measured as force per unit area. In the SI system, the unit is pascal (Pa) or, equivalently, newtons per square metre (N/m²). The customary unit is pounds-force per square inch (lbf/in² or psi), or kilo-pounds per square inch (ksi), which is equal to 1000 psi; kilo-pounds per square inch are commonly used for convenience when measuring tensile strengths.

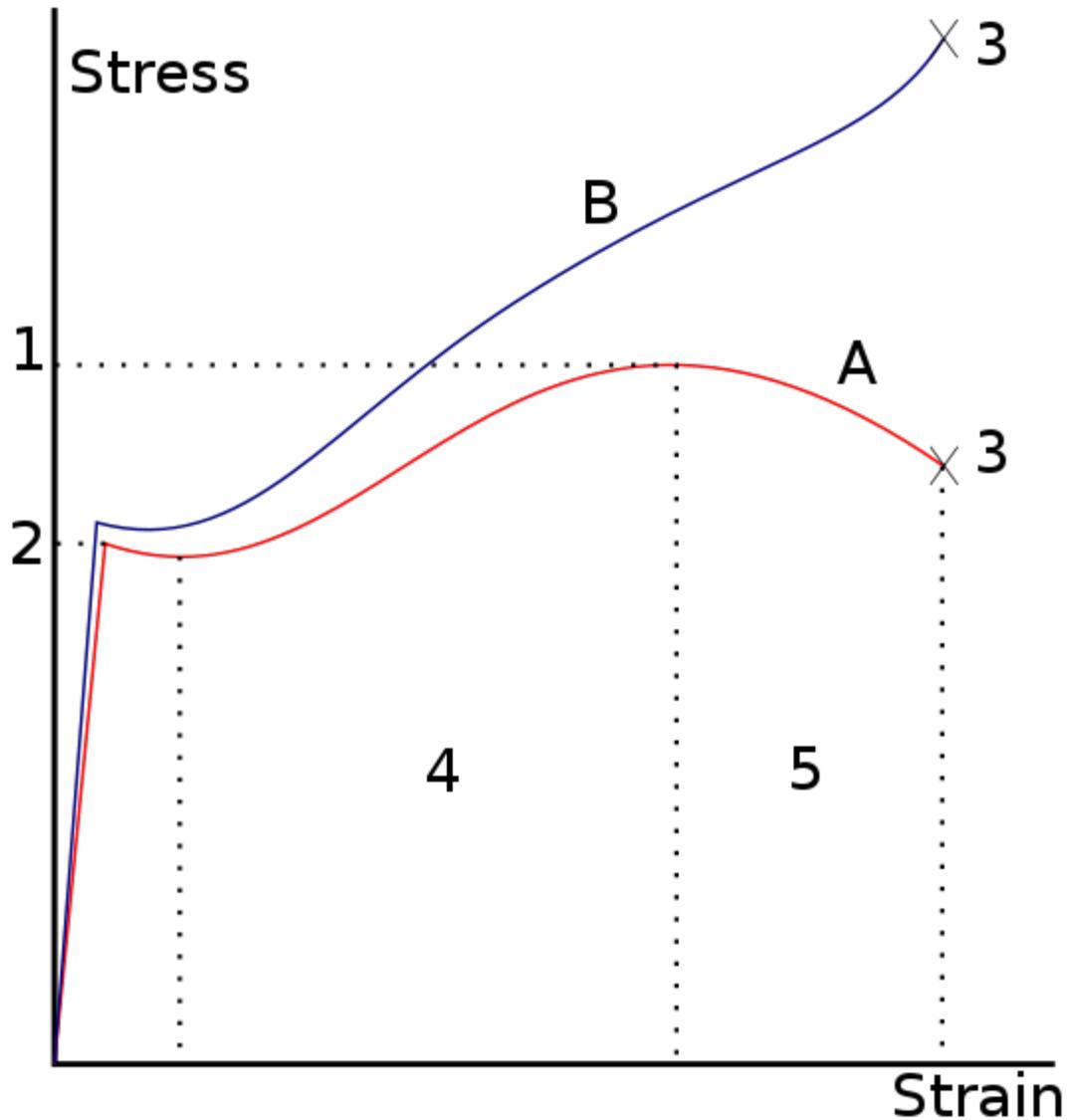
Concept

Ductile materials



Stress vs. Strain curve typical of aluminum

1. Ultimate strength
2. Yield strength
3. Proportional limit stress
4. Fracture
5. Offset strain (typically 0.2%)



Stress vs. strain curve typical of structural steel

- 1. Ultimate strength
- 2. Yield strength
- 3. Fracture
- 4. Strain hardening region
- 5. Necking region
- A: Engineering stress
- B: True stress

Many materials display linear elastic behavior, defined by a linear stress-strain relationship, as shown in the figure up to point 2, in which deformations are completely recoverable upon removal of the load; that is, a specimen loaded elastically in tension will elongate, but will return to its original shape and size when unloaded. Beyond this

linear region, for ductile materials, such as steel, deformations are plastic. A plastically deformed specimen will not return to its original size and shape when unloaded. Note that there will be elastic recovery of a *portion* of the deformation. For many applications, plastic deformation is unacceptable, and is used as the design limitation.

After the yield point, ductile metals will undergo a period of strain hardening, in which the stress increases again with increasing strain, and they begin to neck, as the cross-sectional area of the specimen decreases due to plastic flow. In a sufficiently ductile material, when necking becomes substantial, it causes a reversal of the engineering stress-strain curve (curve A); this is because the *engineering stress* is calculated assuming the original cross-sectional area before necking. The reversal point is the maximum stress on the engineering stress-strain curve, and the engineering stress coordinate of this point is the tensile ultimate strength, given by point 1.

The UTS is not used in the design of ductile static members because design practices dictate the use of the yield stress. It is, however, used to for quality control, because of the ease of testing. It is also used to roughly determine material types for unknown samples.

Brittle materials

Brittle materials, such as concrete and carbon fiber, are characterized by failure at small strains. They often fail while still behaving in a linear elastic manner, and thus do not have a defined yield point. Because strains are low, there is negligible difference between the engineering stress and the true stress. Testing of several identical specimens will result in different failure stresses, this is due to the Weibull Modulus of the brittle material.

The UTS is a common engineering parameter when design brittle members, because there is no yield point.

Liquids and gases

Tensile strength can be defined for liquids and gases as well as solids. For example, when drawing water through a straw, the column of water is pulled upwards from the top by suction. The same holds for the column of air between the top of the water column and the top of the straw.

In liquids, the force is mostly transmitted down the column by cohesion. Cohesion translates in mechanical terms as inexpandability (which is in mechanical terms the same thing as incompressibility). The compression modulus measures the tensile strength of liquids and gases. Failure can also occur in liquids when cohesion breaks down locally, this is the phenomenon of cavitation. In gases, there is no cohesion and the material can expand.

Testing



Round bar tensile specimen after testing

Typically, the testing involves taking a small sample with a fixed cross-section area, and then pulling it with a controlled, gradually increasing force until the sample changes shape or breaks.

When testing metals, indentation hardness correlates linearly with tensile strength. This important relation permits economically important nondestructive testing of bulk metal deliveries with lightweight, even portable equipment, such as hand-held Rockwell hardness testers.

Typical tensile strengths

Typical tensile strengths of some materials

Material	Yield strength (MPa)	Ultimate strength (MPa)	Density (g/cm ³)
first carbon nanotube ropes	?	3,600	1.3
Structural steel ASTM A36 steel	250	400	7.8
carbon steel 1090		841	7.58
Steel, API 5L X65	448	531	7.8
Steel, high strength alloy ASTM A514	690	760	7.8
Steel, prestressing strands	1,650	1,860	7.8
Steel (AISI 1060 0.6% carbon) Piano wire		2,200-2,482	7.8
High density polyethylene (HDPE)	26-33	37	0.95
Polypropylene	12-43	19.7-80	0.91
Stainless steel AISI 302 - Cold-rolled	520	860	8.19
Cast iron 4.5% C, ASTM A-48	130	200	
4130 steel quench & temper at 1200°F	703	814	7.75
"Liquidmetal" alloy	1723	550-1600	6.1
Titanium alloy (6% Al, 4% V)	830	900	4.51
Beryllium 99.9% Be	345	448	1.84
Aluminium alloy 2014-T6	414	483	2.8
Aluminium alloy 6063-T6		248	2.63
Copper 99.9% Cu	70	220	8.92
Cupronickel 10% Ni, 1.6% Fe, 1% Mn, balance Cu	130	350	8.94
Brass	200+	550	5.3
Tungsten		1,510	19.25
Glass		33	2.53
E-Glass	N/A	3,450	2.57
S-Glass	N/A	4,710	2.48
Basalt fiber	N/A	4,840	2.7
Marble	N/A	15	
Concrete	N/A	3(tension) 30(compression)	2.7
Carbon fiber	N/A	5,650	1.75

Carbon fiber (Toray T1000G)		6,370	1.80
Human hair		380	
Bamboo		350-500	0.4
Spider silk		1,000	1.3
Darwin's bark spider silk	1,652		
Silkworm silk	500		1.3
Aramid (Kevlar or Twaron)	3,620	2,757	1.44
UHMWPE	23	46	0.97
UHMWPE fibers (Dyneema or Spectra)		2,300-3,500	0.97
Vectran		2,850-3,340	
Polybenzoxazole (Zylon)		5,800	1.56
Pine wood (parallel to grain)		40	
Bone (limb)	104-121	130	1.6
Nylon, type 6/6	45	75	1.15
Epoxy adhesive	-	12 - 30	-
Rubber	-	15	
Boron	N/A	3,100	2.46
Silicon, monocrystalline (m-Si)	N/A	7,000	2.33
Silicon carbide (SiC)	N/A	3,440	
Ultra-pure silica glass fiber-optic strands		4100	
Sapphire (Al ₂ O ₃)	N/A	1,900	3.9-4.1
diamond	N/A	2,800	3.5
Graphene	N/A	130,000	1.0
Colossal carbon tube	N/A	7,000	0.116
Carbon nanotube	N/A	11,000-63,000	0.037-1.34
Carbon nanotube composites	N/A	1,200	N/A

^a Many of the values depend on manufacturing process and purity/composition.

^b Multiwalled carbon nanotubes have the highest tensile strength of any material yet measured, with labs producing them at a tensile strength of 63 GPa, still well below their theoretical limit of 300 GPa. The first nanotube ropes (20mm in length) whose tensile strength was published (in 2000) had a strength of 3.6 GPa. The density depends on the manufacturing method, and the lowest value is 0.037 or 0.55 (solid).

^c The strength of spider silk is highly variable. It depends on many factors including kind of silk (Every spider can produce several for sundry purposes.), species, age of silk, temperature, humidity, swiftness at which stress is applied during testing, length stress is applied, and way the silk is gathered (forced silking or natural spinning). The value shown in the table, 1000 MPa, is roughly representative of the results from a few studies involving several different species of spider however specific results varied greatly.

^d Human hair strength varies by ethnicity and chemical treatments.

Typical properties for annealed elements

Element	Young's modulus (GPa)	Offset or yield strength (MPa)	Ultimate strength (MPa)
silicon	107		5000–9000
tungsten	411	550	0550–0620
iron	211	080–100	0350
titanium	120	100–225	0240–0370
copper	130	033	0210
tantalum	186	180	0200
tin	047	009–014	0015–0200
zinc (wrought)	105		0110–0200
nickel	170	014–035	0140–0195
silver	083		0170
gold	079		0100
aluminium	070	015–020	0040-0050
lead	016		0012