Heat Conduction Handbook

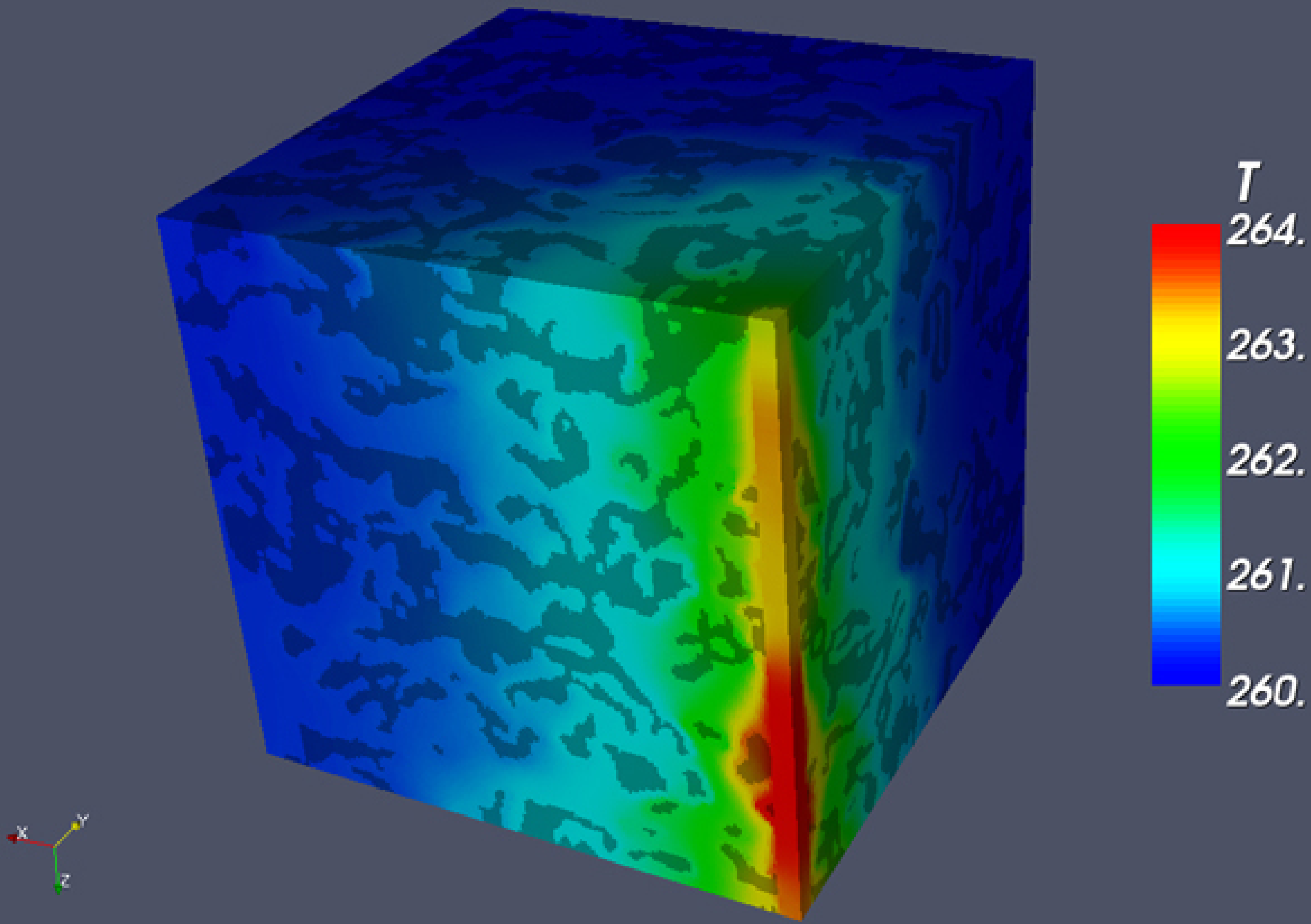

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Table of Contents

Chapter 1

Conduction (Heat)

In heat transfer, **conduction** (or **heat conduction**) is the transfer of thermal energy between regions of matter due to a temperature gradient. Heat spontaneously flows from a region of higher temperature to a region of lower temperature, and reduces temperature differences over time, approaching thermal equilibrium. The previous statement can be argued to apply to heat transfer in general, but to distinguish conduction specifically, it should be stated that the heat flows through the region of matter itself, as opposed to requiring electromagnetic waves as does radiation or to requiring bulk motion of the matter as does convection. Conduction takes place in all forms of matter, viz. solids, liquids, gases and plasmas, but does not require any bulk motion of matter. In solids, it is due to the combination of vibrations of the molecules in a lattice or phonons with the energy transported by free electrons. In gases and liquids, conduction is due to the collisions and diffusion of the molecules during their random motion.

In the engineering sciences, heat transfer includes the processes of thermal radiation, convection, and sometimes mass transfer and often more than one of these processes occurs in a given situation.

Overview

On a microscopic scale, conduction occurs as rapidly moving or vibrating atoms and molecules interact with neighboring particles, transferring some of their kinetic energy. Heat is transferred by conduction when adjacent atoms vibrate against one another, or as electrons move from one atom to another. Conduction is the most significant means of heat transfer within a solid or between solid objects in thermal contact. Conduction is greater in solids because the network of relatively fixed spacial relationships between atoms helps to transfer energy between them by vibration.

As density decreases so does conduction. Therefore, fluids (and especially gases) are less conductive. This is due to the large distance between atoms in a gas: fewer collisions between atoms means less conduction. Conductivity of gases increases with temperature. Conductivity increases with increasing pressure from vacuum up to a critical point that the density of the gas is such that molecules of the gas may be expected to collide with each other before they transfer heat from one surface to another. After this point conductivity increases only slightly with increasing pressure and density.

Thermal contact conductance is the study of heat conduction between solid bodies in contact. A temperature drop is often observed at the interface between the two surfaces in contact. This phenomenon is said to be a result of a thermal contact resistance existing between the contacting surfaces. Interfacial thermal resistance is a measure of an interface's resistance to thermal flow. This thermal resistance differs from contact resistance, as it exists even at atomically perfect interfaces. Understanding the thermal resistance at the interface between two materials is of primary significance in the study of its thermal properties. Interfaces often contribute significantly to the observed properties of the materials.

The inter-molecular transfer of energy could be primarily by elastic impact as in fluids or by free electron diffusion as in metals or phonon vibration as in insulators. In insulators the heat flux is carried almost entirely by phonon vibrations.

Metals (e.g. copper, platinum, gold,etc.) are usually the best conductors of thermal energy. This is due to the way that metals are chemically bonded: metallic bonds (as opposed to covalent or ionic bonds) have free-moving electrons which are able to transfer thermal energy rapidly through the metal. The "electron fluid" of a conductive metallic solid conducts nearly all of the heat flux through the solid. Phonon flux is still present, but carries less than 1% of the energy. Electrons also conduct electric current through conductive solids, and the thermal and electrical conductivities of most metals have about the same ratio. A good electrical conductor, such as copper, usually also conducts heat well. The Peltier-Seebeck effect exhibits the propensity of electrons to conduct heat through an electrically conductive solid. Thermoelectricity is caused by the relationship between electrons, heat fluxes and electrical currents. Heat conduction within a solid is directly analogous to diffusion of particles within a fluid, in the situation where there are no fluid currents.

To quantify the ease with which a particular medium conducts, engineers employ the thermal conductivity, also known as the conductivity constant or conduction coefficient, *k*. In thermal conductivity *k* is defined as "the quantity of heat, Q, transmitted in time (t) through a thickness (L), in a direction normal to a surface of area (A), due to a temperature difference (ΔT) [...]." Thermal conductivity is a material *property* that is primarily dependent on the medium's phase, temperature, density, and molecular bonding. Thermal effusivity is a quantity derived from conductivity which is a measure of its ability to exchange thermal energy with its surroundings.

Steady-state conduction

Steady state conduction is the form of conduction that happens when the temperature difference(s) driving the conduction are constant, so that (after an equilibration time), the spatial distribution of temperatures (temperature field) in the conducting object does not change any further. Thus, all partial derivatives of temperature *with respect to space* may either be zero or have values, but all derivatives of temperature at any point *with respect to time* are uniformly zero. In steady state conduction, the amount of heat entering any region of an object is equal to amount of heat coming out (if this were not so, the temperature would be rising or falling, as thermal energy was tapped or trapped in a region).

For example, a bar may be cold at one end and hot at the other, but after a state of steady state conduction is reached, the spacial gradient of temperatures along the bar does not change any further, as time proceeds. Instead, the temperature at any given section of the rod remains constant, and this temperature varies linearly in space, along the direction of heat transfer.

In steady state conduction, all the laws of direct current electrical conduction can be applied to "heat currents". In such cases, it is possible to take "thermal resistances" as the analog to electrical resistances. In such cases, temperature plays the role of voltage, and heat transferred per unit time (heat power) is the analog of electrical current.

Transient conduction

In non-steady-state situations, in which a new temperature change at a boundary, or a new source of heat has been suddenly introduced, a system will change in time to reach a new equilibrium. Such systems eventually establish steady-state thermal gradients, if the introduction and removal of heat is continuous. During the time-period of establishment of such new conditions, the mode of thermal energy flow is termed *transient conduction*.

Transient conduction occurs in any situation where there is a change in temperature from external or internal sources or sinks of heat, causing a new source of heat-input (or output) within an object, or into or out of the object (to the external surroundings). New sources of heat "turning on" within an object can cause the phenomenon (for example, an engine starting in a car). New external conditions also cause this process: the copper bar in the above example would experience transient conduction as soon as one end was subjected to a different temperature from the other. Over time, the field of temperatures inside the bar would reach a new steady-state, in which a constant temperature gradient along the bar will finally be set up, and this gradient would then stay constant, in time. Typically, such a new steady state gradient is approached exponentially with time, after a new temperature-or-heat source or sink, has been introduced. To this extent, the "transient conduction" phase is then over, even though heat flow may be still continue at high power.

For example, in an automobile, the transient thermal conduction phase for the entire machine would be over, and the steady state phase would appear, as soon as the engine had reached steady-state operating temperature. In this state, temperature would vary greatly from cylinders to other parts of the car, but at no point would the temperature be increasing or decreasing.

An example of transient conduction which does not end with steady-state conduction, but rather no conduction, occurs when a hot copper ball is dropped into oil at a low temperature. Here the temperature field within the object begins to change as a function of time, as the heat is removed from the metal, and the interest lies in analyzing this spatial change of temperature within the object over time, until all gradients disappear entirely (the ball has reached the same temperature as the oil). Mathematically, this condition is also approached exponentially; in theory it takes infinite time, but in practice it is over, for all intents and purposes, in a much shorter period. At the end of this process with no heat sink but the internal parts of the ball (which are finite), there is no steady state heat conduction to be reached. Such a state never occurs in this situation, but rather the end of the process is when there is no heat conduction at all.

Analysis of non steady-state conduction systems is more complex than steady state systems, and (except for simple shapes) calls for the application of approximation theories, and/or numerical analysis by computer. One popular graphical method involves the use of Heisler Charts.

Relativistic conduction

The theory of relativistic heat conduction is a model that is compatible with the theory of special relativity. For most of the last century, it was recognized that Fourier equation is in contradiction with the theory of relativity because it admits an infinite speed of propagation of heat signals. For example, according to Fourier equation, a pulse of heat at the origin would be felt at infinity instantaneously. The speed of information propagation is faster than the speed of light in vacuum, which is physically inadmissible within the framework of relativity. Alterations to the Fourier model provided for a relativistic model of heat conduction, avoiding this problem.

Quantum conduction

Second sound is a quantum mechanical phenomenon in which heat transfer occurs by wave-like motion, rather than by the more usual mechanism of diffusion. Heat takes the place of pressure in normal sound waves. This leads to a very high thermal conductivity. It is known as "second sound" because the wave motion of heat is similar to the propagation of sound in air.

Fourier's law

The law of Heat Conduction, also known as Fourier's law, states that the time rate of heat transfer through a material is proportional to the negative gradient in the temperature and to the area, at right angles to that gradient, through which the heat is flowing. We can state this law in two equivalent forms: the integral form, in which we look at the amount of energy flowing into or out of a body as a whole, and the differential form, in which we look at the flow rates or fluxes of energy locally.

Differential form

The differential form of Fourier's Law of thermal conduction shows that the local heat flux density, $\overrightarrow{q}$, is equal to the product of thermal conductivity, *k*, and the negative local temperature gradient, $-\nabla T$. The heat flux density is the amount of energy that flows through a unit area per unit time.

$$\overrightarrow{q} = -k\nabla T$$

where (including the SI units)

$\overrightarrow{q}$ is the local heat flux, $W \cdot m^{-2}$
k is the material's conductivity, $W \cdot m^{-1} \cdot K^{-1}$,
∇T is the temperature gradient, $K \cdot m^{-1}$.

The thermal conductivity, *k*, is often treated as a constant, though this is not always true. While the thermal conductivity of a material generally varies with temperature, the variation can be small over a significant range of temperatures for some common materials. In anisotropic materials, the thermal conductivity typically varies with orientation; in this case *k* is represented by a second-order tensor. In nonuniform materials, *k* varies with spatial location.

For many simple applications, Fourier's law is used in its one-dimensional form. In the x-direction,

$$q_x = -k\frac{dT}{dx}$$

Integral form

By integrating the differential form over the material's total surface *S*, we arrive at the integral form of Fourier's law:

$$\frac{\partial Q}{\partial t} = -k \oint_S \overrightarrow{\nabla} T \cdot \overrightarrow{dA}$$

where (including the SI units)

$\frac{\partial Q}{\partial t}$ is the amount of heat transferred per unit time (in W) and
$\overrightarrow{dA}$ is an oriented surface area element (in m^2)

The above differential equation, when integrated for a homogeneous material of 1-D geometry between two endpoints at constant temperature, gives the heat flow rate as:

$$\frac{\Delta Q}{\Delta t} = -kA\frac{\Delta T}{\Delta x}$$

where

A is the cross-sectional surface area,
ΔT is the temperature difference between the ends,
Δx is the distance between the ends.

This law forms the basis for the derivation of the heat equation. Ohm's law is the electrical analogue of Fourier's law.

Conductance

Writing

$$U = \frac{k}{\Delta x},$$

where U is the conductance, in W/(m^2 K).

Fourier's law can also be stated as:

$$\frac{\Delta Q}{\Delta t} = UA\,(-\Delta T).$$

The reciprocal of conductance is resistance, R, given by:

$$R = \frac{1}{U} = \frac{\Delta x}{k} = \frac{A\,(-\Delta T)}{\frac{\Delta Q}{\Delta t}},$$

and it is resistance which is additive when several conducting layers lie between the hot and cool regions, because A and Q are the same for all layers. In a multilayer partition, the total conductance is related to the conductance of its layers by:

$$\frac{1}{U} = \frac{1}{U_1} + \frac{1}{U_2} + \frac{1}{U_3} + \cdots$$

So, when dealing with a multilayer partition, the following formula is usually used:

$$\frac{\Delta Q}{\Delta t} = \frac{A\,(-\Delta T)}{\frac{\Delta x_1}{k_1} + \frac{\Delta x_2}{k_2} + \frac{\Delta x_3}{k_3} + \cdots}.$$

When heat is being conducted from one fluid to another through a barrier, it is sometimes important to consider the conductance of the thin film of fluid which remains stationary next to the barrier. This thin film of fluid is difficult to quantify, its characteristics depending upon complex conditions of turbulence and viscosity, but when dealing with thin high-conductance barriers it can sometimes be quite significant.

Intensive-property representation

The previous conductance equations, written in terms of extensive properties, can be reformulated in terms of intensive properties.

Ideally, the formulae for conductance should produce a quantity with dimensions independent of distance, like Ohm's Law for electrical resistance: $R = V/I$, and conductance: $G = I/V$.

From the electrical formula: $R = \rho x/A$, where ρ is resistivity, x = length, and A is cross-sectional area, we have $G = kA/x$, where G is conductance, k is conductivity, x = length, and A = cross-sectional area.

For Heat,

$$U = \frac{kA}{\Delta x},$$

where *U* is the conductance.

Fourier's law can also be stated as:

$$\dot{Q} = U\,\Delta T$$

analogous to Ohm's law: $I = V/R$ or $I = VG.$

The reciprocal of conductance is resistance, R, given by:

$$R = \frac{\Delta T}{\dot{Q}},$$

analogous to Ohm's law: $R = V/I.$

The rules for combining resistances and conductances (in series and in parallel) are the same for both heat flow and electric current.

Cylinders

Conduction through cylinders can be calculated when variables such as the internal radius r_1, the external radius r_2, and the length denoted as ℓ.

The temperature difference between the inner and outer wall can be expressed as $T_2 - T_1$.

The area of the heat flow: $A_r = 2\pi r\ell$

When Fourier's equation is applied:

$$Q = -kA_r\frac{\mathrm{d}T}{\mathrm{d}r} = -2k\pi r\ell\frac{\mathrm{d}T}{\mathrm{d}r}$$

Rearranged:

$$Q\int_{r_1}^{r_2}\frac{1}{r}\mathrm{d}r = -2k\pi\ell\int_{T_1}^{T_2}\mathrm{d}T$$

Therefore the rate of heat transfer is

$$Q = 2k\pi\ell\frac{T_1 - T_2}{\ln(r_2/r_1)}$$

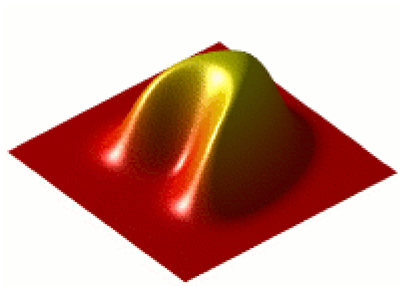
The thermal resistance is

$$R_c = \frac{\Delta T}{Q} = \frac{\ln(r_2/r_1)}{2\pi k\ell}$$

And $Q = 2\pi k\ell r_m\frac{T_1 - T_2}{r_2 - r_1}$, where $r_m = \frac{r_2 - r_1}{\ln(r_2/r_1)}$ and it is important to note that this is the log-mean radius.

Chapter 2

Heat Equation



The heat equation predicts that if a hot body is placed in a box of cold water, the temperature of the body will decrease, and eventually (after infinite time, and subject to no external heat sources) the temperature in the box will equalize.

The **heat equation** is an important partial differential equation which describes the distribution of heat (or variation in temperature) in a given region over time. For a function *u*(*x*,*y*,*z*,*t*) of three spatial variables (*x*,*y*,*z*) and the time variable *t*, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

also written

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

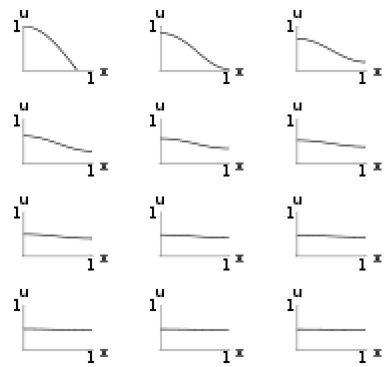
or sometimes

$$\frac{\partial u}{\partial t} - \alpha \Delta u = 0$$

where α is a positive constant and Δor ∇^2denotes the Laplace operator. In the physical problem of temperature variation, *u*(*x*,*y*,*z*,*t*) is the temperature and α is the thermal diffusivity. For the mathematical treatment it is sufficient to consider the case α = 1.

The heat equation is of fundamental importance in diverse scientific fields. In mathematics, it is the prototypical parabolic partial differential equation. In probability theory, the heat equation is connected with the study of Brownian motion via the Fokker–Planck equation. In financial mathematics it is used to solve the Black–Scholes partial differential equation. The diffusion equation, a more general version of the heat equation, arises in connection with the study of chemical diffusion and other related processes.

General description



Graphical representation of the solution to a 1D heat equation PDE.

Suppose one has a function *u* which describes the temperature at a given location (x, y, z). This function will change over time as heat spreads throughout space. The heat equation is used to determine the change in the function *u* over time. The image to the right is animated and describes the way heat changes in time along a metal bar. One of the interesting properties of the heat equation is the maximum principle which says that the maximum value of *u* is either earlier in time than the region of concern or on the edge of the region of concern. This is essentially saying that temperature comes either from some source or from earlier in time because heat permeates but is not created from nothingness. This is a property of parabolic partial differential equations and is not difficult to prove mathematically.

Another interesting property is that even if *u* has a discontinuity at an initial time $t = t_0$, the temperature becomes smooth as soon as $t > t_0$. For example, if a bar of metal has temperature 0 and another has temperature 100 and they are stuck together end to end, then very quickly the temperature at the point of connection is 50 and the graph of the temperature is smoothly running from 0 to 100.

The heat equation is used in probability and describes random walks. It is also applied in financial mathematics for this reason.

It is also important in Riemannian geometry and thus topology: it was adapted by Richard Hamilton when he defined the Ricci flow that was later used by Grigori Perelman to solve the topological Poincaré conjecture.

The physical problem and the equation

Derivation in one dimension

The heat equation is derived from Fourier's law and conservation of energy (Cannon 1984). By Fourier's law, the flow rate of heat energy through a surface is proportional to the negative temperature gradient across the surface,

$$\mathbf{q} = -k\nabla u$$

where k is the thermal conductivity and u is the temperature. In one dimension, the gradient is an ordinary spatial derivative, and so Fourier's law is

$$\mathbf{q} = -ku_x$$

where u_x is du/dx. In the absence of work done, a change in internal energy per unit volume in the material, ΔQ, is proportional to the change in temperature, Δu. That is,

$$\Delta Q = c_p \rho \Delta u$$

where c_p is the specific heat capacity and ρ is the mass density of the material. (In this section only, Δ is the ordinary difference operator, not the Laplacian.) Choosing zero energy at absolute zero temperature, this can be rewritten as

$$Q = c_p \rho u.$$

The increase in internal energy in a small spatial region of the material

$$x - \Delta x \le \xi \le x + \Delta x$$

over the time period

$$t - \Delta t \le \tau \le t + \Delta t$$

is given by

$$c_p \rho \int_{x-\Delta x}^{x+\Delta x} [u(\xi, t+\Delta t) - u(\xi, t-\Delta t)]\, d\xi = c_p \rho \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial u}{\partial \tau}\, d\xi d\tau$$

where the fundamental theorem of calculus was used. Additionally, with no work done and absent any heat sources or sinks, the change in internal energy in the interval [x-Δx, x+Δx] is accounted for entirely by the flux of heat across the boundaries. By Fourier's law, this is

$$k \int_{t-\Delta t}^{t+\Delta t} \left[\frac{\partial u}{\partial x}(x+\Delta x, \tau) - \frac{\partial u}{\partial x}(x-\Delta x, \tau) \right] d\tau = k \int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} \frac{\partial^2 u}{\partial \xi^2} \, d\xi \, d\tau$$

again by the fundamental theorem of calculus. By conservation of energy,

$$\int_{t-\Delta t}^{t+\Delta t} \int_{x-\Delta x}^{x+\Delta x} [c_p \rho u_\tau - k u_{\xi\xi}] \, d\xi \, d\tau = 0.$$

This is true for any rectangle $[t-\Delta t, t+\Delta t] \times [x-\Delta x, x+\Delta x]$. Consequently, the integrand must vanish identically:

$$c_p \rho u_t - k u_{xx} = 0.$$

Which can be rewritten as:

$$u_t = \frac{k}{c_p \rho} u_{xx},$$

or:

$$\frac{\partial u}{\partial t} = \frac{k}{c_p \rho} \left(\frac{\partial^2 u}{\partial x^2} \right)$$

which is the heat equation. The coefficient $k/(c_p\rho)$ is called thermal diffusivity and is often denoted α.

Three-dimensional problem

In the special case of heat propagation in an isotropic and homogeneous medium in a 3-dimensional space, this equation is

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \alpha (u_{xx} + u_{yy} + u_{zz})$$

where:

- $u = u(x, y, z, t)$ is temperature as a function of space and time;
- $\frac{\partial u}{\partial t}$ is the rate of change of temperature at a point over time;

- u_{xx}, u_{yy}, and u_{zz} are the second spatial derivatives (*thermal conductions*) of temperature in the x, y, and z directions, respectively;

- $\alpha = k / c_p\rho$ is the thermal diffusivity, a material-specific quantity depending on the *thermal conductivity*, k, the *mass density*, ρ, and the *specific heat capacity*, c_p.

The heat equation is a consequence of Fourier's law of cooling.

If the medium is not the whole space, in order to solve the heat equation uniquely we also need to specify boundary conditions for u. To determine uniqueness of solutions in the whole space it is necessary to assume an exponential bound on the growth of solutions, this assumption is consistent with observed experiments.

Solutions of the heat equation are characterized by a gradual smoothing of the initial temperature distribution by the flow of heat from warmer to colder areas of an object. Generally, many different states and starting conditions will tend toward the same stable equilibrium. As a consequence, to reverse the solution and conclude something about earlier times or initial conditions from the present heat distribution is very inaccurate except over the shortest of time periods.

The heat equation is the prototypical example of a parabolic partial differential equation.

Using the Laplace operator, the heat equation can be simplified, and generalized to similar equations over spaces of arbitrary number of dimensions, as

$$u_t = \alpha \nabla^2 u = \alpha \Delta u,$$

where the Laplace operator, Δ or ∇^2, the divergence of the gradient, is taken in the spatial variables.

The heat equation governs heat diffusion, as well as other diffusive processes, such as particle diffusion or the propagation of action potential in nerve cells. Although they are not diffusive in nature, some quantum mechanics problems are also governed by a mathematical analog of the heat equation. It also can be used to model some phenomena arising in finance, like the Black-Scholes or the Ornstein-Uhlenbeck processes. The equation, and various non-linear analogues, has also been used in image analysis.

The heat equation is, technically, in violation of special relativity, because its solutions involve instantaneous propagation of a disturbance. The part of the disturbance outside the forward light cone can usually be safely neglected, but if it is necessary to develop a reasonable speed for the transmission of heat, a hyperbolic problem should be considered instead – like a partial differential equation involving a second-order time derivative.

Internal heat generation

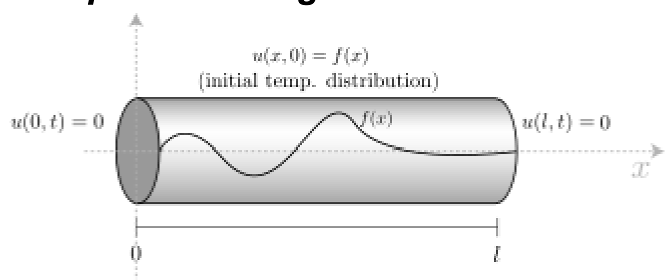
The function u above represents temperature of a body. Alternatively, it is sometimes convenient to change units and represent u as the heat density of a medium. Since heat density is proportional to temperature in a homogeneous medium, the heat equation is still obeyed in the new units.

Suppose that a body obeys the heat equation and, in addition, generates its own heat per unit volume (e.g., in watts/L) at a rate given by a known function q varying in space and time. Then the heat per unit volume u satisfies an equation

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + q.$$

For example, a tungsten light bulb filament generates heat, so it would have a positive nonzero value for q when turned on. While the light is turned off, the value of q for the tungsten filament would be zero.

Solving the heat equation using Fourier series



Idealized physical setting for heat conduction in a rod with homogeneous boundary conditions.

The following solution technique for the heat equation was proposed by Joseph Fourier in his treatise *Théorie analytique de la chaleur*, published in 1822. Let us consider the heat equation for one space variable. This could be used to model heat conduction in a rod. The equation is

$$(1)\ u_t = \alpha u_{xx}$$

where $u = u(x, t)$ is a function of two variables x and t. Here

- x is the space variable, so $x \in [0,L]$, where L is the length of the rod.
- t is the time variable, so $t \geq 0$.

We assume the initial condition

$$(2)\ u(x,0) = f(x) \quad \forall\, x \in [0, L]$$

where the function *f* is given and the boundary conditions

$$(3)\ u(0,t) = 0 = u(L,t) \quad \forall\, t > 0.$$

Let us attempt to find a solution of (1) which is not identically zero satisfying the boundary conditions (3) but with the following property: *u* is a product in which the dependence of *u* on *x*, *t* is separated, that is:

$$(4)\ u(x,t) = X(x)T(t).$$

This solution technique is called separation of variables. Substituting *u* back into equation (1),

$$\frac{T'(t)}{\alpha T(t)} = \frac{X''(x)}{X(x)}.$$

Since the right hand side depends only on *x* and the left hand side only on *t*, both sides are equal to some constant value $-\lambda$. Thus:

$$(5)\ T'(t) = -\lambda\alpha T(t)$$

and

$$(6)\ X''(x) = -\lambda X(x).$$

We will now show that solutions for (6) for values of $\lambda \leq 0$ cannot occur:

1. Suppose that $\lambda < 0$. Then there exist real numbers *B*, *C* such that

   $$X(x) = Be^{\sqrt{-\lambda}\,x} + Ce^{-\sqrt{-\lambda}\,x}.$$

   From (3) we get

   $$X(0) = 0 = X(L).$$

   and therefore $B = 0 = C$ which implies *u* is identically 0.

2. Suppose that $\lambda = 0$. Then there exist real numbers *B*, *C* such that

   $$X(x) = Bx + C.$$

   From equation (3) we conclude in the same manner as in 1 that *u* is identically 0.

3. Therefore, it must be the case that $\lambda > 0$. Then there exist real numbers A, B, C such that

$$T(t) = Ae^{-\lambda\alpha t}$$

and

$$X(x) = B\sin(\sqrt{\lambda}\,x) + C\cos(\sqrt{\lambda}\,x).$$

From (3) we get $C = 0$ and that for some positive integer n,

$$\sqrt{\lambda} = n\frac{\pi}{L}.$$

This solves the heat equation in the special case that the dependence of u has the special form (4).

In general, the sum of solutions to (1) which satisfy the boundary conditions (3) also satisfies (1) and (3). We can show that the solution to (1), (2) and (3) is given by

$$u(x,t) = \sum_{n=1}^{+\infty} D_n \left(\sin\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2\alpha t}{L^2}}$$

where

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx.$$

Generalizing the solution technique

The solution technique used above can be greatly extended to many other types of equations. The idea is that the operator u_{xx} with the zero boundary conditions can be represented in terms of its eigenvectors. This leads naturally to one of the basic ideas of the spectral theory of linear self-adjoint operators.

Consider the linear operator $\Delta u = u_{xx}$. The infinite sequence of functions

$$e_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$

for $n \geq 1$ are eigenvectors of Δ. Indeed

$$\Delta e_n = -\frac{n^2\pi^2}{L^2}e_n.$$

Moreover, any eigenvector f of Δ with the boundary conditions $f(0)=f(L)=0$ is of the form e_n for some $n \geq 1$. The functions e_n for $n \geq 1$ form an orthonormal sequence with respect to a certain inner product on the space of real-valued functions on $[0, L]$. This means

$$\langle e_n, e_m \rangle = \int_0^L e_n(x) e_m(x) dx = \begin{cases} 0 & n \neq m \\ 1 & m = n \end{cases}.$$

Finally, the sequence $\{e_n\}_{n \in \mathbf{N}}$ spans a dense linear subspace of $L^2(0, L)$. This shows that in effect we have diagonalized the operator Δ.

Heat conduction in non-homogeneous anisotropic media

In general, the study of heat conduction is based on several principles. Heat flow is a form of energy flow, and as such it is meaningful to speak of the time rate of flow of heat into a region of space.

- The time rate of heat flow into a region V is given by a time-dependent quantity $q_t(V)$. We assume q has a density, so that

  $$q_t(V) = \int_V Q(x,t)\, dx$$

- Heat flow is a time-dependent vector function $\mathbf{H}(x)$ characterized as follows: the time rate of heat flowing through an infinitesimal surface element with area $d\,S$ and with unit normal vector $\mathbf{n}$ is

  $$\mathbf{H}(x) \cdot \mathbf{n}(x)\, dS$$

Thus the rate of heat flow into V is also given by the surface integral

$$q_t(V) = - \int_{\partial V} \mathbf{H}(x) \cdot \mathbf{n}(x)\, dS$$

where $\mathbf{n}$(x) is the outward pointing normal vector at x.

- The Fourier law states that heat energy flow has the following linear dependence on the temperature gradient

  $$\mathbf{H}(x) = -\mathbf{A}(x) \cdot \nabla u(x)$$
  where $\mathbf{A}(x)$ is a 3×3 real matrix that is symmetric and positive definite.

By Green's theorem, the previous surface integral for heat flow into V can be transformed into the volume integral

$$q_t(V) = -\int_{\partial V} \mathbf{H}(x) \cdot \mathbf{n}(x)\, dS$$
$$= \int_{\partial V} \mathbf{A}(x) \cdot \nabla u(x) \cdot \mathbf{n}(x)\, dS$$
$$= \int_V \sum_{i,j} \partial_{x_i} \big(a_{ij}(x) \partial_{x_j} u(x,t)\big)\, dx$$

- The time rate of temperature change at *x* is proportional to the heat flowing into an infinitesimal volume element, where the constant of proportionality is dependent on a constant κ

$$\partial_t u(x,t) = \kappa(x) Q(x,t)$$

Putting these equations together gives the general equation of heat flow:

$$\partial_t u(x,t) = \kappa(x) \sum_{i,j} \partial_{x_i} \big(a_{ij}(x) \partial_{x_j} u(x,t)\big)$$

Remarks.

- The coefficient κ(*x*) is the inverse of specific heat of the substance at *x* × density of the substance at *x*.

- In the case of an isotropic medium, the matrix **A** is a scalar matrix equal to thermal conductivity.

- In the anisotropic case where the coefficient matrix **A** is not scalar (i.e., if it depends on *x*), then an explicit formula for the solution of the heat equation can seldom be written down. Though, it is usually possible to consider the associated abstract Cauchy problem and show that it is a well-posed problem and/or to show some qualitative properties (like preservation of positive initial data, infinite speed of propagation, convergence toward an equilibrium, smoothing properties). This is usually done by one-parameter semigroups theory: for instance, if *A* is a symmetric matrix, then the elliptic operator defined by

  $$Au(x) := \sum_{i,j} \partial_{x_i} a_{ij}(x) \partial_{x_j} u(x)$$

  is self-adjoint and dissipative, thus by the spectral theorem it generates a one-parameter semigroup.

Fundamental solutions

A fundamental solution, also called a *heat kernel*, is a solution of the heat equation corresponding to the initial condition of an initial point source of heat at a known position. These can be used to find a general solution of the heat equation over certain domains; see, for instance, (Evans 1998) for an introductory treatment.

In one variable, the Green's function is a solution of the initial value problem

$$\begin{cases} u_t(x,t) - k u_{xx}(x,t) = 0 & -\infty < x < \infty, \quad 0 < t < \infty \\ u(x,t=0) = \delta(x) \end{cases}$$

where δ is the Dirac delta function. The solution to this problem is the fundamental solution

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right).$$

One can obtain the general solution of the one variable heat equation with initial condition $u(x,0) = g(x)$ for $-\infty<x<\infty$ and $0<t<\infty$ by applying a convolution:

$$u(x,t) = \int \Phi(x-y,t) g(y) dy.$$

In several spatial variables, the fundamental solution solves the analogous problem

$$\begin{cases} u_t(\mathbf{x},t) - k \sum_{i=1}^n u_{x_i x_i}(\mathbf{x},t) = 0 \\ u(\mathbf{x},t=0) = \delta(\mathbf{x}) \end{cases}$$

in $-\infty<x_i<\infty$, $i=1,\dots,n$, and $0<t<\infty$. The n-variable fundamental solution is the product of the fundamental solutions in each variable; i.e.,

$$\Phi(\mathbf{x},t) = \Phi(x_1,t)\Phi(x_2,t)\dots\Phi(x_n,t) = \frac{1}{(4\pi k t)^{n/2}} e^{-\mathbf{x}\cdot\mathbf{x}/4kt}.$$

The general solution of the heat equation on $\mathbf{R}^n$ is then obtained by a convolution, so that to solve the initial value problem with $u(\mathbf{x},t=0)=g(\mathbf{x})$, one has

$$u(\mathbf{x},t) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}-\mathbf{y},t) g(\mathbf{y}) d\mathbf{y}.$$

The general problem on a domain Ω in $\mathbf{R}^n$ is

$$\begin{cases} u_t(\mathbf{x},t) - k\sum_{i=1}^n u_{x_i x_i}(\mathbf{x},t) = 0 & \mathbf{x}\in\Omega \quad 0<t<\infty \\ u(\mathbf{x},t=0) = g(\mathbf{x}) & \mathbf{x}\in\Omega \end{cases}$$

with either Dirichlet or Neumann boundary data. A Green's function always exists, but unless the domain Ω can be readily decomposed into one-variable problems, it may not be possible to write it down explicitly. The method of images provides one additional technique for obtaining Green's functions for non-trivial domains.

Some Green's function solutions in 1D

A variety of elementary Green's function solutions in one-dimension are recorded here. In some of these, the spatial domain is the entire real line (-∞,∞). In others, it is the semi-infinite interval (0,∞) with either Neumann or Dirichlet boundary conditions. One further variation is that some of these solve the inhomogeneous equation

$$u_t = ku_{xx} + f.$$

where f is some given function of x and t.

Homogeneous heat equation

Initial value problem on (-∞,∞)

$$\begin{cases} u_t = ku_{xx} & -\infty < x < \infty,\ 0<t<\infty \\ u(x,0) = g(x) & IC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) g(y)\,dy$$

Comment. This solution is the convolution with respect to the variable x of the fundamental solution $\Phi(x,t) := \frac{1}{\sqrt{4kt}}\exp\left(-\frac{x^2}{4kt}\right)$ and the function $g(x)$. Therefore, according to the general properties of the convolution with respect to differentiation, $u = g*\Phi$ is a solution of the same heat equation, for $(\partial_t - k\,\partial_x^2)(\Phi*g) = [(\partial_t - k\,\partial_x^2)\Phi]*g) = 0$. Moreover, $\Phi(x,t) = \frac{1}{\sqrt{t}}\,\Phi(\frac{x}{\sqrt{t}})$ and $\int_{-\infty}^{+\infty}\Phi(x,t)\,dx = 1$, so that, by general facts about approximation to the identity, $\Phi(\cdot,t)*g \to g$ as $t\to 0$ in various senses, according to the specific g. For instance, if g is assumed bounded and continuous on $\mathbb{R}$, then $\Phi(\cdot,t)*g$ converges uniformly to g as $t\to 0$, meaning that $u(x,t)$ is continuous on $\mathbb{R}\times[0,\infty)$ with $u(x,0) = g(x)$.

Initial value problem on (0,∞) with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t = ku_{xx} & 0 \le x < \infty,\ 0<t<\infty \\ u(x,0) = g(x) & IC \\ u(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right) \right) g(y)\,dy$$

Comment. This solution is obtained from the preceding formula as applied to the data *g*(*x*) suitably extended to $\mathbb{R}$, so as to be an odd function, that is, letting $g(-x) := -g(x)$ for all *x*. Correspondingly, the solution of the initial value problem on $(-\infty,+\infty)$ is an odd function with respect to the variable *x* for all values of *t*, and in particular it satisfies the homogeneous Dirichlet boundary conditions $u(0,\, t) = 0.$

Initial value problem on $(0,\infty)$ with homogeneous Neumann boundary conditions

$$\begin{cases} u_t = ku_{xx} & 0 \le x < \infty,\ 0 < t < \infty \\ u(x,0) = g(x) & IC \\ u_x(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) + \exp\left(-\frac{(x+y)^2}{4kt}\right) \right) g(y)\,dy$$

Comment. This solution is obtained from the first solution formula as applied to the data *g*(*x*) suitably extended to $\mathbb{R}$, so as to be an even function, that is, letting $g(-x) := g(x)$ for all *x*. Correspondingly, the solution of the initial value problem on $(-\infty,+\infty)$ is an even function with respect to the variable *x* for all values of $t > 0$, and in particular, being smooth, it satisfies the homogeneous Neumann boundary conditions $u_x(0,\, t) = 0.$

Problem on $(0,\infty)$ with homogeneous initial conditions and non-homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t = ku_{xx} & 0 \le x < \infty,\ 0 < t < \infty \\ u(x,0) = 0 & IC \\ u(0,t) = h(t) & BC \end{cases}$$

$$u(x,t) = \int_0^t \frac{x}{\sqrt{4\pi k(t-s)^3}} \exp\left(-\frac{x^2}{4k(t-s)}\right) h(s)\,ds, \qquad \forall x > 0$$

Comment. This solution is the convolution with respect to the variable *t* of $\psi(x,t) := -2k\,\partial_x \Phi(x,t) = \frac{x}{\sqrt{4kt^3}} \exp\left(-\frac{x^2}{4kt}\right)$ and the function *h*(*t*). Since $\Phi(x,t)$ is the fundamental solution of $\partial_t - k\,\partial_x^2$, the function $\psi(x,t)$ is also a solution of the same heat equation, and so is $u := \psi * h$, thanks to general properties of the convolution with respect to differentiation. Moreover, $\psi(x,t) = \frac{1}{x^2}\,\psi(1, \frac{t}{x^2})$ and $\int_0^{+\infty} \psi(x,t)\,dt = 1$, so that, by general facts about approximation to the identity, $\psi(x,\cdot) * h \to h$ as $x \to 0$ in various senses, according to the specific *h*. For instance, if *h* is assumed continuous on $\mathbb{R}$ with support in $[0,\infty)$, then $\psi(x,\cdot) * h$ converges uniformly on compacta to *h* as $x \to 0$, meaning that *u*(*x*,*t*) is continuous on $[0,\infty) \times [0,\infty)$ with $u(0,t) = h(t)$.

Inhomogeneous heat equation

Problem on (-∞,∞) homogeneous initial conditions

$$\begin{cases} u_t = ku_{xx} + f(x,t) & -\infty < x < \infty,\ 0 < t < \infty \\ u(x,0) = 0 & IC \end{cases}$$

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) f(y,s)\,dy\,ds$$

Comment. This solution is the convolution in $\mathbb{R}^2$, that is with respect to both the variables *x* and *t*, of the fundamental solution $\Phi(x,t) := \frac{1}{\sqrt{4kt}} \exp\left(-\frac{x^2}{4kt}\right)$ and the function $f(x,t)$, both meant as defined on the whole $\mathbb{R}^2$ and identically 0 for all $t \le 0$. One verifies that $(\partial_t - k\,\partial_x^2)(\Phi * f) = f$, which is expressed in the language of distributions as $(\partial_t - k\,\partial_x^2)\Phi = \delta$, where the distribution δ is the Dirac's delta function, that is the evaluation at 0.

Problem on (0,∞) with homogeneous Dirichlet boundary conditions and initial conditions

$$\begin{cases} u_t = ku_{xx} + f(x,t) & 0 \le x < \infty,\ 0 < t < \infty \\ u(x,0) = 0 & IC \\ u(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left(\exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) - \exp\left(-\frac{(x+y)^2}{4k(t-s)}\right)\right) f(y,s)\,dy\,ds$$

Comment. This solution is obtained from the preceding formula as applied to the data $f(x,t)$ suitably extended to $\mathbb{R} \times [0,\infty)$, so as to be an odd function of the variable *x*, that is, letting $f(-x,t) := -f(x,t)$ for all *x* and *t*. Correspondingly, the solution of the inhomogeneous problem on $(-\infty,+\infty)$ is an odd function with respect to the variable *x* for all values of *t*, and in particular it satisfies the homogeneous Dirichlet boundary conditions $u(0,t) = 0$.

Problem on (0,∞) with homogeneous Neumann boundary conditions and initial conditions

$$\begin{cases} u_t = ku_{xx} + f(x,t) & 0 \le x < \infty,\ 0 < t < \infty \\ u(x,0) = 0 & IC \\ u_x(0,t) = 0 & BC \end{cases}$$

$$u(x,t) = \int_0^t \int_0^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} \left(\exp\left(-\frac{(x-y)^2}{4k(t-s)}\right) + \exp\left(-\frac{(x+y)^2}{4k(t-s)}\right)\right) f(y,s)\,dy\,ds$$

Comment. This solution is obtained from the first formula as applied to the data $f(x,t)$suitably extended to $\mathbb{R}\times[0,\infty)$, so as to be an even function of the variable *x*, that is, letting $f(-x,t):=f(x,t)$for all *x* and *t*. Correspondingly, the solution of the inhomogeneous problem on $(-\infty,+\infty)$is an even function with respect to the variable *x* for all values of *t*, and in particular, being a smooth function, it satisfies the homogeneous Neumann boundary conditions $u_x(0,t)=0$.

Examples

Since the heat equation is linear, solutions of other combinations of boundary conditions, inhomogeneous term, and initial conditions can be found by taking an appropriate linear combination of the above Green's function solutions.

For example, to solve

$$\begin{cases} u_t = ku_{xx} + f & -\infty < x < \infty,\, 0 < t < \infty \\ u(x,0) = g(x) & IC \end{cases}$$

let

$$u = w + v$$

where *u* and *v* solve the problems

$$\begin{cases} v_t = kv_{xx} + f,\, w_t = kw_{xx} & -\infty < x < \infty,\, 0 < t < \infty \\ v(x,0) = 0,\, w(x,0) = g(x) & IC \end{cases}$$

Similarly, to solve

$$\begin{cases} u_t = ku_{xx} + f & 0 \le x < \infty,\, 0 < t < \infty \\ u(x,0) = g(x) & IC \\ u(0,t) = h(t) & BC \end{cases}$$

let

$$u = w + v + r$$

where *w*, *v*, and *r* solve the problems

$$\begin{cases} v_t = kv_{xx} + f,\ w_t = kw_{xx},\ r_t = kr_{xx} & 0 \le x < \infty,\ 0 < t < \infty \\ v(x,0) = 0,\ w(x,0) = g(x),\ r(x,0) = 0 & IC \\ v(0,t) = 0,\ w(0,t) = 0,\ r(0,t) = h(t) & BC \end{cases}$$

Mean-value property for the heat equation

Solutions of the heat equations $(\partial_t - \Delta)u = 0$ satisfy a mean-value property analogous to the mean-value properties of harmonic functions (solutions of $\Delta u = 0$), though a bit more complicated. Precisely, if u solves $(\partial_t - \Delta)u = 0$ and $(x,t) + E_\lambda \subset \mathrm{dom}(u)$ then

$$u(x,t) = \frac{\lambda}{4}\int_{E_\lambda} u(x-y,t-s)\frac{|y|^2}{s^2}\,ds\,dy,$$

where E_λ is a "heat-ball", that is a super-level set of the fundamental solution of the heat equation:

$$E_\lambda := \{(y,s)\ :\ \Phi(y,s) > \lambda\},$$
$$\Phi(x,t) := (4t\pi)^{-\frac{n}{2}}\exp\Big(-\frac{|x|^2}{4t}\Big).$$

Notice that $\mathrm{diam}(E_\lambda) = o(1)$ as $\lambda \to \infty$, so the above formula holds for any (x,t) in the (open) set $\mathrm{dom}(u)$ for λ large enough. Conversely, any function u satisfying the above mean-value property on an open domain of $\mathbb{R}^n \times \mathbb{R}$ is a solution of the heat equation. This can be shown by an argument similar to the analogous one for harmonic functions.

Stationary Heat Equation

The stationary heat equation is not dependent on time. This happens in all those problems, where the time equilibrium constant is fast enough to approximate the more complex time dependent heat equation to the stationary case. This equation is much simpler and can help to understand better the physic of the materials without focusing on the dynamic of the heat transport process. It is widely used for simple engineering problems assuming there is equilibrium with time.

Stationary condition:

$$\frac{\partial u}{\partial t} = 0$$

Stationary heat equation with heat source (inhomogeneous case), which is also the Poisson's equation:

$$\alpha\nabla^2 u = q$$

Stationary heat equation without heat source (homogeneous case), which is also the Laplace's equation:

$$\nabla^2 u = 0$$

- where *u* is the temperature (°K), α (or *k*) is the thermal conductivity ($\frac{W}{m^\circ K}$), *q* the heat source density ($\frac{W}{m^3}$).

Applications

Particle diffusion

One can model particle diffusion by an equation involving either:

- the volumetric concentration of particles, denoted *c*, in the case of collective diffusion of a large number of particles, or
- the probability density function associated with the position of a single particle, denoted *P*.

In either case, one uses the heat equation

$$c_t = D\Delta c,$$

or

$$P_t = D\Delta P.$$

Both *c* and *P* are functions of position and time. *D* is the diffusion coefficient that controls the speed of the diffusive process, and is typically expressed in meters squared over second. If the diffusion coefficient *D* is not constant, but depends on the concentration *c* (or *P* in the second case), then one gets the nonlinear diffusion equation.

Brownian motion

The random trajectory of a single particle subject to the particle diffusion equation (or heat equation) is a Brownian motion. If a particle is placed at $\vec{R} = \vec{0}$ at time *t* = 0, then the probability density function associated with the position vector of the particle $\vec{R}$ will be the following:

$$P(\vec{R},t) = G(\vec{R},t) = \frac{1}{(2\pi)^{3/2}(\det(D)t)^{1/2}} \exp\left\{-\frac{\vec{R}^T D^{-1}\vec{R}}{2t}\right\}$$

which is a (multivariate) normal distribution evolving in time.

Schrödinger equation for a free particle

With a simple division, the Schrödinger equation for a single particle of mass *m* in the absence of any applied force field can be rewritten in the following way:

$\psi_t = \frac{i\hbar}{2m}\Delta\psi$, where *i* is the unit imaginary number, and $\hbar$ is Planck's constant divided by 2π, and ψ is the wavefunction of the particle.

This equation is formally similar to the particle diffusion equation, which one obtains through the following transformation:

$$c(\vec{R},t) \to \psi(\vec{R},t)$$
$$D \to \frac{i\hbar}{2m}.$$

Applying this transformation to the expressions of the Green functions determined in the case of particle diffusion yields the Green functions of the Schrödinger equation, which in turn can be used to obtain the wavefunction at any time through an integral on the wavefunction at *t=0*:

$$\psi(\vec{R},t) = \int \psi(\vec{R}^0, t=0) G(\vec{R}-\vec{R}^0, t) dR_x^0\, dR_y^0\, dR_z^0$$, with

$$G(\vec{R},t) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{-\frac{\vec{R}^2 m}{2i\hbar t}}.$$

Remark: this analogy between quantum mechanics and diffusion is a purely formal one. Physically, the evolution of the wavefunction satisfying Schrödinger's equation might have an origin other than diffusion.

Thermal diffusivity in polymers

A direct practical application of the heat equation, in conjunction with Fourier theory, in spherical coordinates, is the measurement of the thermal diffusivity in polymers (Unsworth and Duarte). The dual theoretical-experimental method demonstrated by these authors is applicable to rubber and various other materials of practical interest.

Further applications

The heat equation arises in the modeling of a number of phenomena and is often used in financial mathematics in the modeling of options. The famous Black–Scholes option pricing model's differential equation can be transformed into the heat equation allowing relatively easy solutions from a familiar body of mathematics. Many of the extensions to the simple option models do not have closed form solutions and thus must be solved numerically to obtain a modeled option price. The heat equation is also widely used in image analysis (Perona & Malik 1990) and in machine-learning as the driving theory behind scale-space or graph Laplacian methods. The heat equation can be efficiently solved numerically using the Crank–Nicolson method of (Crank & Nicolson 1947).

An abstract form of heat equation on manifolds provides a major approach to the Atiyah–Singer index theorem, and has led to much further work on heat equations in Riemannian geometry.

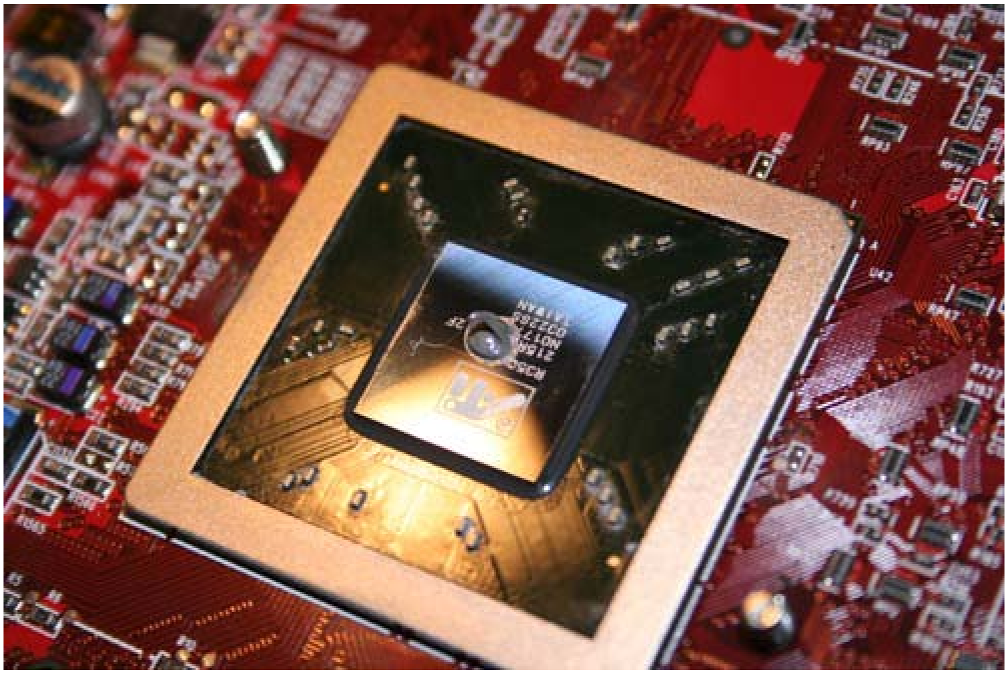
Chapter 3

Thermal Grease

From left to right: Arctic Cooling MX-2 and MX-3, Tuniq TX-3, Cool Laboratory Liquid Metal Pro(Liquid Metal based), Shin-Etsu MicroSi G751, Arctic Silver 5, Powdered Diamond. In background Arctic Silver grease remover

Silicone thermal compound

Metal (silver) thermal compound



Metal thermal grease applied to a chip

Surface imperfections

Thermal grease (also called **thermal gel**, **thermal compound**, **thermal paste**, **heat paste**, **heat sink paste**, **heat transfer compound**, or **heat sink compound**) is a fluid substance, originally with properties akin to grease, which increases the thermal conductivity of a thermal interface by filling microscopic air-gaps present due to the imperfectly flat and smooth surfaces of the components; the compound has far greater thermal conductivity than air (but far less than metal). In electronics, it is often used to aid a component's thermal dissipation via a heat sink.

Thermal conductor types

Thermal greases use one or more different thermally conductive substances:

- Ceramic-based thermal grease has generally good thermal conductivity and is usually composed of a ceramic powder suspended in a liquid or gelatinous silicone compound, which may be described as 'silicone paste' or 'silicone thermal compound'. The most commonly used ceramics and their thermal conductivities (in units of W/(m ·K)) are: beryllium oxide (218), aluminum nitride (170), aluminum oxide (39), zinc oxide (21), and silicon dioxide (1). Thermal grease is usually white in colour since these ceramics are all white in powder form.

- Metal-based thermal grease contain solid metal particles (usually silver or aluminum). It has a better thermal conductivity (and is more expensive) than ceramic-based grease.

- Carbon based. There are products based on with carbon-based conductors, using diamond powder, or short carbon fibers , they have the best thermal conductivity and are generally more expensive than metal-based thermal grease.

- Liquid metal based. Some thermal pastes are made of liquid metal alloys of gallium. These are rare and expensive.

All but the last classification of compound usually use silicone grease as a medium, a heat conductor in itself, though some manufacturers prefer use of fractions of mineral oil.

All these compounds conduct heat far better than air, but far worse than metal. They are intended to fill gaps that would otherwise hold air, not to create a layer between component and heatsink—this will decrease the effectiveness of the heatsink. Ideally perfectly smooth and flat metallic surfaces would not need heatsink compound.

Purpose

Thermal grease is primarily used in the electronics and computer industries to assist a heat sink to draw heat away from a semiconductor component such as an integrated circuit or transistor.

Thermally conductive paste improves the efficiency of a heatsink by filling air gaps that occur when the imperfectly flat and smooth surface of a heat generating component is pressed against the similar surface of a heatsink, air being approximately 8000 times less efficient at conducting heat than, for example, aluminum, a good heatsink material. Surface imperfections and departure from perfect flatness inherently arise from limitations in manufacturing technology and range in size from visible and tactile flaws such as machining marks or casting irregularities to sub-microscopic ones not visible to the naked eye.

Thermal conductivity and "conformability" (i.e., the ability of the material to conform to irregular surfaces) are the important characteristics of thermal grease.

Both high-power handling transistors, such as those in an audio amplifier, and high-speed integrated circuits, such as the central processing unit (CPU) of a personal computer, generate sufficient heat to benefit from the use of thermal grease to improve the effectiveness of a heatsink.

The need for heatsink compound can be minimised or removed by lapping the surfaces of the hot component and the matching heatsink face so that they are virtually perfectly flat and mirror-smooth. Computer overclockers, who increase computer speed by measures which increase heat production, resort to lapping and other extreme cooling methods such as water-cooling.

Properties

The metal oxide and nitride particles suspended in silicone thermal compounds have thermal conductivities of up to 220 W/(m·K). (In comparison, the thermal conductivity of metals used particle additions, copper is 380 W/(m·K), silver 429 and aluminum 237.) The typical thermal conductivities of the silicone compounds are 0.7 to 3 W/(m·K). Silver thermal compounds may have a conductivity of 3 to 8 W/(m·K) or more.

In compounds containing suspended particles, the properties of the fluid may well be the most important. As seen by the thermal conductivity measures above, the conductivity is closer to that of the fluid components rather than the ceramic or metal components. Other properties of fluid components that are important for thermal grease might be:

1. How well it fills the gaps and conforms to both the component's and the heat sink's uneven surfaces.
2. How well it adheres to those surfaces
3. How well it maintains its consistency over the required temperature range
4. How well it resists drying out or flaking over time
5. Whether it degrades with oxidation or breaks down over time

The compound must have a suitable consistency to apply easily and remove all excess to leave only the minimum needed.

Application and removal

Computer processor heatsinks utilize a variety of designs to promote better thermal transfer between components. Some thermal greases have a durability up to at least 8 years. Flat and smooth surfaces may use a small line method to apply material, and exposed heat-pipe surfaces will be best prepared with multiple lines.

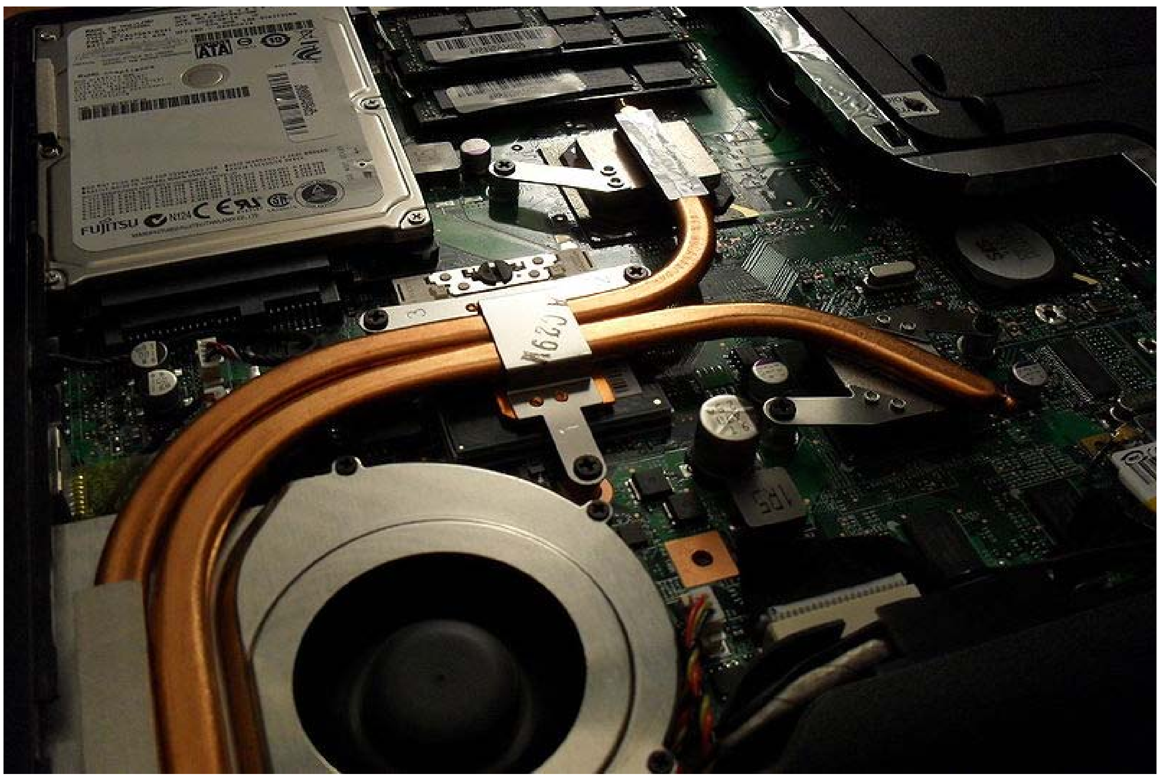
Excess grease separating the metal surfaces more than the minimum necessary to exclude air gaps will only degrade conductivity, increasing the risk of overheating. Silver-based thermal grease can also be either slightly electrically conductive or capacitive; if some flows onto the circuits it can cause malfunctioning and damage.

Over time, some thermal greases may dry out, have reduced heat transferring capabilities, or set like glue and make it difficult to remove the heat sink. If too much force is applied the processor may be damaged. Heating the grease by turning the processor on for a short period often softens the adhesion. It is recommended that thermal grease be re-applied with each removal of the heatsink.

Silicone oil-based thermal grease can be removed from a component or heatsink with an alcohol (such as rubbing alcohol) or acetone. Special-purpose cleaners are made for removing heatsink grease and cleaning the surfaces.

Chapter 4

Heat Pipe



A laptop heat pipe system

A **heat pipe** is a heat transfer mechanism that combines the principles of both thermal conductivity and phase transition to efficiently manage the transfer of heat between two solid interfaces.

At the hot interface within a heat pipe, which is typically at a very low pressure, a liquid in contact with a thermally conductive solid surface turns into a vapor by absorbing heat from that surface. The vapor condenses back into a liquid at the cold interface, releasing the latent heat. The liquid then returns to the hot interface through either capillary action or gravity action where it evaporates once more and repeats the cycle. In addition, the internal pressure of the heat pipe can be set or adjusted to facilitate the phase change depending on the demands of the working conditions of the thermally managed system.

Structure, design and construction

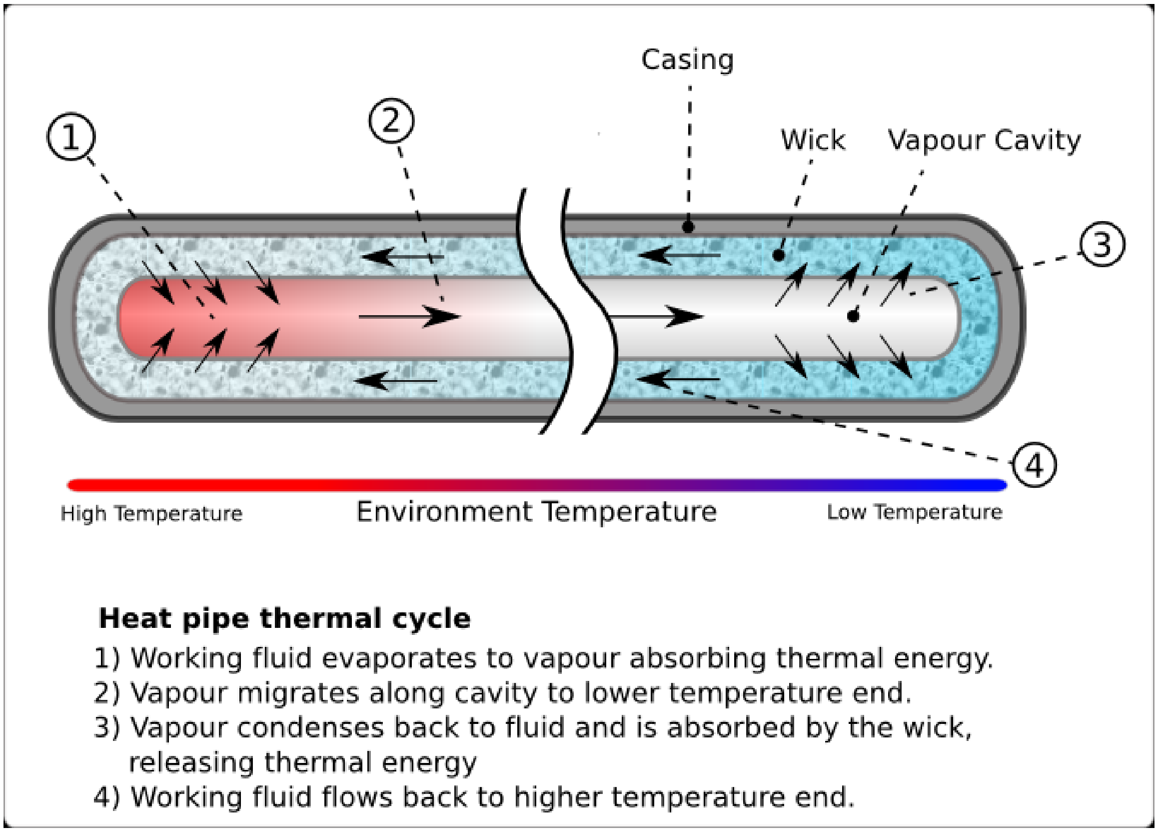
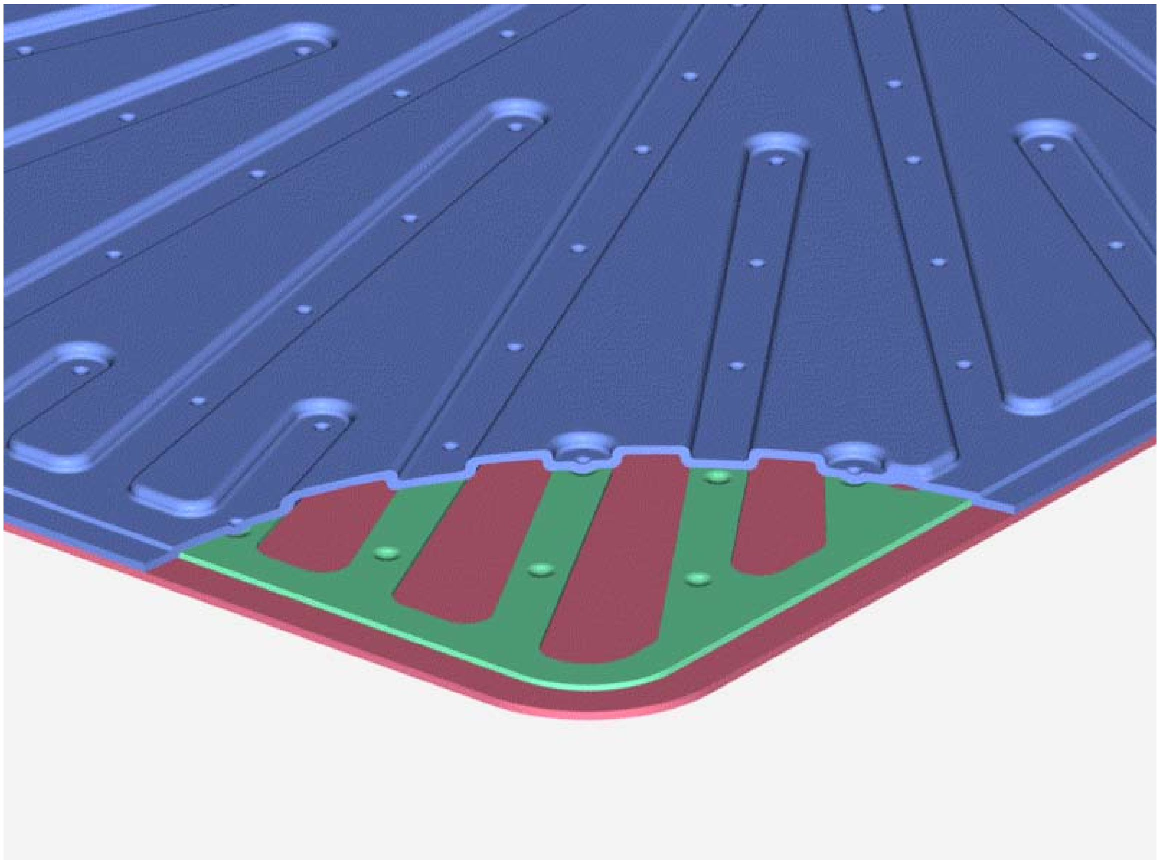
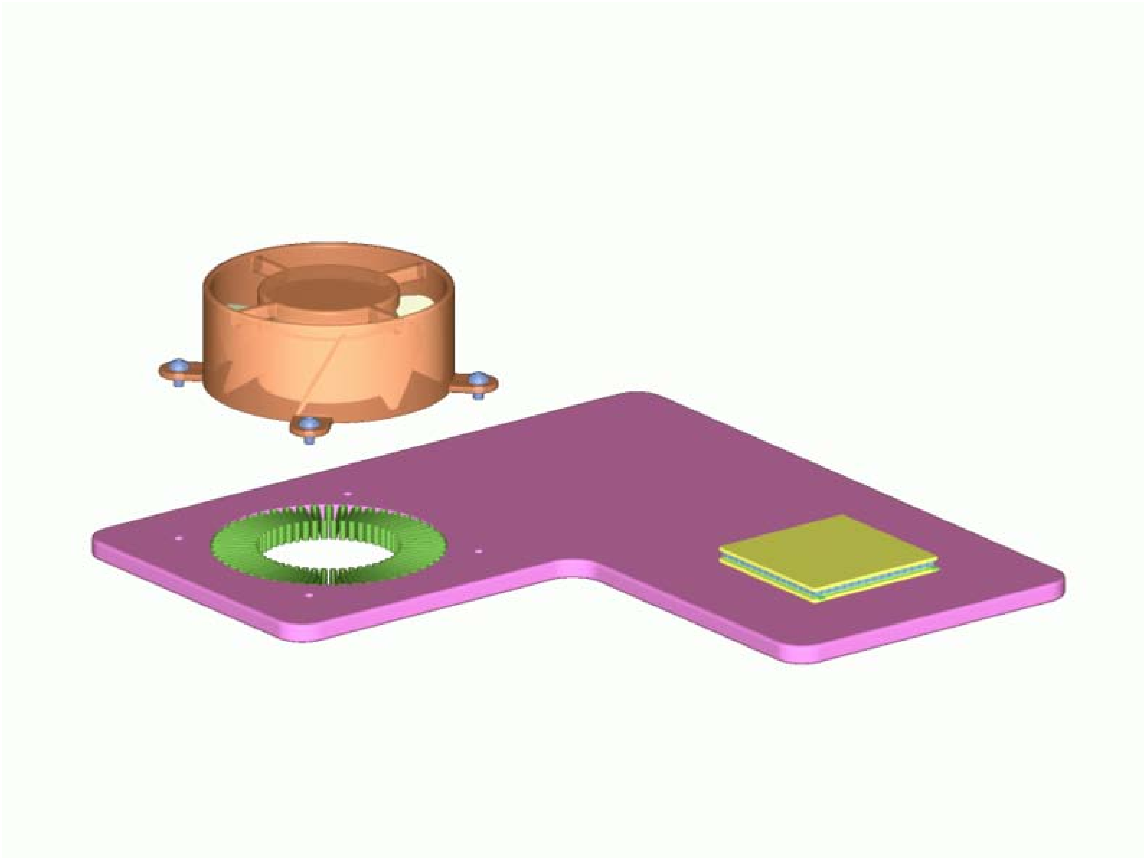


Diagram showing components and mechanism for a heat pipe containing a wick



Cut-away view of a 500 µm thick flat heat pipe, with a thin planar capillary (aqua colored)



Thin flat heat pipe (heat spreader) with remote heat sink and fan

A typical heat pipe consists of a sealed pipe or tube made of a material with high thermal conductivity such as copper or aluminium at both hot and cold ends. A vacuum pump is used to remove all air from the empty heat pipe, and then the pipe is filled with a fraction of a percent by volume of *working fluid* (or coolant) chosen to match the operating temperature. Examples of such fluids include water, ethanol, acetone, sodium, or mercury. Due to the partial vacuum that is near or below the vapor pressure of the fluid, some of the fluid will be in the liquid phase and some will be in the gas phase. The use of a vacuum eliminates the need for the working gas to diffuse through any other gas and so the bulk transfer of the vapor to the cold end of the heat pipe is at the speed of the moving molecules. In this sense, the only practical limit to the rate of heat transfer is the speed with which the gas can be condensed to a liquid at the cold end.

Inside the pipe's walls, an optional wick structure exerts a capillary pressure on the liquid phase of the working fluid. This is typically a sintered metal powder or a series of grooves parallel to the pipe axis, but it may be any material capable of exerting capillary pressure on the condensed liquid to wick it back to the heated end. The heat pipe may not need a wick structure if gravity or some other source of acceleration is sufficient to overcome surface tension and cause the condensed liquid to flow back to the heated end.

A heat pipe is not a thermosiphon, because there is no siphon. Thermosiphons transfer heat by single-phase convection.

Heat pipes contain no mechanical moving parts and typically require no maintenance, though non-condensing gases (that diffuse through the pipe's walls, result from breakdown of the working fluid, or exist as impurities in the materials) may eventually reduce the pipe's effectiveness at transferring heat. This is significant when the working fluid's vapour pressure is low.

The materials chosen depend on the temperature conditions in which the heat pipe must operate, with coolants ranging from liquid helium for extremely low temperature applications (2–4 K) to mercury (523–923 K) & sodium (873–1473 K) and even indium (2000–3000 K) for extremely high temperatures. The vast majority of heat pipes for low temperature applications use some combination of ammonia (213–373 K), alcohol (methanol (283–403 K) or ethanol (273–403 K)) or water (303–473 K) as working fluid. Since the heat pipe contains a vacuum, the working fluid will boil and hence take up latent heat at well below its boiling point at atmospheric pressure. Water, for instance, will boil at just above 273 K (0 degrees Celsius) and so can start to effectively transfer latent heat at this low temperature.

The advantage of heat pipes over many other heat-dissipation mechanisms is their great efficiency in transferring heat. They are a fundamentally better heat conductor than an equivalent cross-section of solid copper (a heat sink alone, though simpler in design and construction, does not take advantage of the principle of matter phase transition). Some heat pipes have demonstrated a heat flux of more than 230 MW/m^2, nearly four times the heat flux at the surface of the sun.

Active control of heat flux can be effected by adding a variable volume liquid reservoir to the evaporator section. Variable conductance heat pipes employ a large reservoir of inert immiscible gas attached to the condensing section. Varying the gas reservoir pressure changes the volume of gas charged to the condenser which in turn limits the area available for vapor condensation. Thus a wider range of heat fluxes and temperature gradients can be accommodated with a single design.

A modified heat pipe with a reservoir having no capillary connection to the heat pipe wick at the evaporator end can also be used as a thermal diode. This heat pipe will transfer heat in one direction, acting as an insulator in the other.

Vapor Chamber or Flat heat pipes

Thin planar heat pipes (heat spreaders) have the same primary components as tubular heat pipes. These components are a hermetically sealed hollow vessel, a working fluid, and a closed-loop capillary recirculation system.

Compared to a one-dimensional tubular heat pipe, the width of a two-dimensional heat pipe allows an adequate cross section for heat flow even with a very thin device. These thin planar heat pipes are finding their way into "height sensitive" applications, such as notebook computers, and surface mount circuit board cores. It is possible to produce flat heat pipes as thin as 0.5 mm (thinner than a credit card).

Heat transfer

A heat sink (aluminium) with heat pipe (copper)

Heat pipes employ evaporative cooling to transfer thermal energy from one point to another by the evaporation and condensation of a working fluid or coolant. Heat pipes rely on a temperature difference between the ends of the pipe, and cannot lower temperatures at either end beyond the ambient temperature (hence they tend to equalise the temperature within the pipe).

When one end of the heat pipe is heated the working fluid inside the pipe at that end evaporates and increases the vapour pressure inside the cavity of the heat pipe. The latent heat of evaporation absorbed by the vaporisation of the working fluid reduces the temperature at the hot end of the pipe.

The vapour pressure over the hot liquid working fluid at the hot end of the pipe is higher than the equilibrium vapour pressure over condensing working fluid at the cooler end of the pipe, and this pressure difference drives a rapid mass transfer to the condensing end where the excess vapour condenses, releases its latent heat, and warms the cool end of the pipe. Non-condensing gases (caused by contamination for instance) in the vapour impede the gas flow and reduce the effectiveness of the heat pipe, particularly at low temperatures, where vapour pressures are low. The velocity of molecules in a gas is approximately the speed of sound and in the absence of non condensing gases, this is the upper velocity with which they could travel in the heat pipe. In practice, the speed of the vapour through the heat pipe is dependent on the rate of condensation at the cold end.

The condensed working fluid then flows back to the hot end of the pipe. In the case of vertically-oriented heat pipes the fluid may be moved by the force of gravity. In the case of heat pipes containing wicks, the fluid is returned by capillary action.

When making heat pipes, there is no need to create a vacuum in the pipe. One simply boils the working fluid in the heat pipe until the resulting vapour has purged the non condensing gases from the pipe and then seals the end.

An interesting property of heat pipes is the temperature over which they are effective. Initially, it might be suspected that a water charged heat pipe would only work when the hot end reached the boiling point (100 °C) and steam was transferred to the cold end. However, the boiling point of water is dependent on absolute pressure inside the pipe. In an evacuated pipe, water will boil just slightly above its melting point (0 °C). The heat pipe will operate, therefore, when the hot end is just slightly warmer than the melting point of the working fluid. Similarly, a heat pipe with water as a working fluid can work well above the boiling point (100 °C), if the cold end is low enough in temperature to condense the fluid.

The main reason for the effectiveness of heat pipes is the evaporation and condensation of the working fluid. The heat of vaporization greatly exceeds the sensible heat capacity. Using water as an example, the energy needed to evaporate one gram of water is equivalent to the amount of energy needed to raise the temperature of that same gram of water by 540 °C (hypothetically, if the water was under extremely high pressure so it didn't vaporize or freeze over this temperature range). Almost all of that energy is rapidly transferred to the "cold" end when the fluid condenses there, making a very effective heat transfer system with no moving parts.

Origins and research in the United States

The general principle of heat pipes using gravity (commonly classified as two phase thermosiphons) dates back to the steam age. The modern concept for a capillary driven heat pipe was first suggested by R.S. Gaugler of General Motors in 1942 who patented the idea. The benefits of employing capillary action were independently developed and first demonstrated by George Grover at Los Alamos National Laboratory in 1963 and subsequently published in the Journal of Applied Physics in 1964. Grover noted in his notebook:

"Heat transfer via capillary movement of fluids. The "pumping" action of surface tension forces may be sufficient to move liquids from a cold temperature zone to a high temperature zone (with subsequent return in vapor form using as the driving force, the difference in vapor pressure at the two temperatures) to be of interest in transferring heat from the hot to the cold zone. Such a closed system, requiring no external pumps, may be of particular interest in space reactors in moving heat from the reactor core to a radiating system. In the absence of gravity, the forces must only be such as to overcome the capillary and the drag of the returning vapor through its channels."

Between 1964 and 1966, RCA was the first corporation to undertake research and development of heat pipes for commercial applications (though their work was mostly funded by the US government). During the late 1960s NASA played a large role in heat pipe development by funding a significant amount of research on their applications and reliability in space flight following from Grover's suggestion. NASA's attraction to heat pipe cooling systems was understandable given their low weight, high heat flux, and zero power draw. Their primary interest however was based on the fact that the system wouldn't be adversely affected by operating in a zero gravity environment. The first application of heat pipes in the space program was in thermal equilibration of satellite transponders. As satellites orbit, one side is exposed to the direct radiation of the sun while the opposite side is completely dark and exposed to the deep cold of outer space. This causes severe discrepancies in the temperature (and thus reliability and accuracy) of the transponders. The heat pipe cooling system designed for this purpose managed the high heat fluxes and demonstrated flawless operation with and without the influence of gravity. The developed cooling system was the first description and usage of variable conductance heat pipes to actively regulate heat flow or evaporator temperature.

Corporate R&D

Publications in 1967 and 1968 by Feldman, Eastman, & Katzoff first discussed applications of heat pipes to areas outside of government concern and that did not fall under the high temperature classification such as; air conditioning, engine cooling, and electronics cooling. These papers also made the first mentions of flexible, arterial, and flat plate heat pipes. 1969 publications introduced the concepts of the rotational heat pipe with its applications to turbine blade cooling and the first discussions of heat pipe applications to cryogenic processes.

Starting in the 1980s Sony began incorporating heat pipes into the cooling schemes for some of its commercial electronic products in place of both forced convection and passive finned heat sinks. Initially they were used in tuners & amplifiers, soon spreading to other high heat flux electronics applications. During the late 1990s increasingly hot microcomputer CPUs spurred a threefold increase in the number of U.S. heat pipe patent applications. As heat pipes transferred from a specialized industrial heat transfer component to a consumer commodity most development and production moved from the U.S. to Asia. Modern CPU heat pipes are typically made from copper and use water as the working fluid.

Applications

Alaska pipeline support legs cooled by heat pipes to keep permafrost frozen.

Grover and his colleagues were working on cooling systems for nuclear power cells for space craft, where extreme thermal conditions are found. Heat pipes have since been used extensively in spacecraft as a means for managing internal temperature conditions.

Heat pipes are extensively used in many modern computer systems, where increased power requirements and subsequent increases in heat emission have resulted in greater demands on cooling systems. Heat pipes are typically used to move heat away from components such as CPUs and GPUs to heat sinks where thermal energy may be dissipated into the environment.

Solar Thermal

Heat pipes are also being widely used in solar thermal water heating applications in combination with evacuated tube solar collector arrays. In these applications, distilled water is commonly used as the heat transfer fluid inside a sealed length of copper tubing that is located within an evacuated glass tube and oriented towards the sun.

In solar thermal water heating applications, an evacuated tube collector can deliver up to 40% more efficiency compared to more traditional "flat plate" solar water heaters. Evacuated tube collectors eliminate the need for anti-freeze additives to be added as the vacuum helps prevent heat loss. These types of solar thermal water heaters are frost protected down to more than -3 °C and are being used in Antarctica to heat water.

Pipelines over permafrost

Heat pipes are used to dissipate heat on the Trans-Alaska Pipeline System. Without them residual ground heat remaining in the oil, as well as that produced by friction and turbulence in the moving oil would conduct down the pipe's support legs. This would likely melt the permafrost on which the supports are anchored. This would cause the pipeline to sink and possibly sustain damage. To prevent this each vertical support member has been mounted with 4 vertical heat pipes.

Cooking

Heat pipes have been designed to speed the cooking of roasts. The pipe is poked through the roast. One end of the pipe extends into the oven where it draws heat to the middle of the roast.

Ventilation heat recovery

In heating, ventilation and air-conditioning systems, HVAC, heat pipes are positioned within the supply and exhaust air streams of an air handling system, or in the exhaust gases of an industrial process, in order to recover the heat energy.

The device consists of a battery of multi-row finned heat pipe tubes located within both the supply and exhaust air streams. Within the exhaust air side of the heat pipe, the refrigerant evaporates, taking its heat from the extract air. The refrigerant vapour moves towards the cooler end of the tube, within the supply air side of the device, where it condenses and gives up its heat. The condensed refrigerant returns by a combination of gravity and capillary action in the wick. Thus heat is transferred from the exhaust air stream through the tube wall to the refrigerant, and then from the refrigerant through the tube wall to the supply air stream.

Because of the characteristics of the device, better efficiencies are obtained when the unit is positioned upright with the supply air side mounted over the exhaust air side, this allows the liquid refrigerant to flow quickly under gravity back to the evaporator. Generally, gross heat transfer efficiencies of up to 75% are claimed by manufacturers.

Limitations

Heat pipes must be tuned to particular cooling conditions. The choice of pipe material, size and coolant all have an effect on the optimal temperatures in which heat pipes work.

When heated above a certain temperature, all of the working fluid in the heat pipe will vaporize and the condensation process will cease to occur; in such conditions, the heat pipe's thermal conductivity is effectively reduced to the heat conduction properties of its solid metal casing alone. As most heat pipes are constructed of copper (a metal with high heat conductivity), an overheated heatpipe will generally continue to conduct heat at around 1/80 of the original conductivity.

In addition, below a certain temperature, the working fluid will not undergo phase change, and the thermal conductivity will be reduced to that of the solid metal casing. One of the key criteria for the selection of a working fluid is the desired operational temperature range of the application. The lower temperature limit typically occurs a few degrees above the freezing point of the working fluid.

Most manufacturers cannot make a traditional heat pipe smaller than 3mm in diameter due to material limitations (though 1.6mm thin sheets can be fabricated). Experiments have been conducted with micro heat pipes, which use piping with sharp edges, such as triangular or rhombus-like tubing. In these cases, the sharp edges transfer the fluid through capillary action, and no wick is necessary.

Chapter 5

Thermal Conductivity

In physics, **thermal conductivity**, *k*, is the property of a material describing its ability to conduct heat. It appears primarily in Fourier's Law for heat conduction. Thermal conductivity is measured in watts per kelvin-meter ($W \cdot K^{-1} \cdot m^{-1}$, i.e. W/(K·m) or in IP units ($Btu \cdot hr^{-1} \cdot ft^{-1} \cdot F^{-1}$, i.e. Btu/(hr·ft · F). Multiplied by a temperature difference (in kelvins, K) and an area (in square meters, m^2), and divided by a thickness (in meters, m), the thermal conductivity predicts the rate of energy loss (in watts, W) through a piece of material. In the window building industry "thermal conductivity" is expressed as the U-Factor, which measures the rate of heat transfer and tells you how well the window insulates. U-factor values generally range from 0.15 to 1.25 and are measured in Watts per square meter per Kelvin - ($W/m^2 \cdot K$). The lower the U-factor, the better the window insulates.

The reciprocal of thermal conductivity is *thermal resistivity*.

Measurement

There are a number of ways to measure thermal conductivity. Each of these is suitable for a limited range of materials, depending on the thermal properties and the medium temperature. There is a distinction between steady-state and transient techniques.

In general, steady-state techniques are useful when the temperature of the material does not change with time. This makes the signal analysis straightforward (steady state implies constant signals). The disadvantage is that a well-engineered experimental setup is usually needed. The Divided Bar (various types) is the most common device used for consolidated rock samples.

The transient techniques perform a measurement during the process of heating up. Their advantage is quicker measurements. Transient methods are usually carried out by needle probes.

Standards

- IEEE Standard 442-1981, "IEEE guide for soil thermal resistivity measurements", ISBN 0-7381-0794-8..
- IEEE Standard 98-2002, "Standard for the Preparation of Test Procedures for the Thermal Evaluation of Solid Electrical Insulating Materials", ISBN 0-7381-3277-2
- ASTM Standard D5334-08, "Standard Test Method for Determination of Thermal Conductivity of Soil and Soft Rock by Thermal Needle Probe Procedure"
- ASTM Standard D5470-06, "Standard Test Method for Thermal Transmission Properties of Thermally Conductive Electrical Insulation Materials"
- ASTM Standard E1225-04, "Standard Test Method for Thermal Conductivity of Solids by Means of the Guarded-Comparative-Longitudinal Heat Flow Technique"
- ASTM Standard D5930-01, "Standard Test Method for Thermal Conductivity of Plastics by Means of a Transient Line-Source Technique"
- ASTM Standard D2717-95, "Standard Test Method for Thermal Conductivity of Liquids"
- ISO 22007-2:2008 "Plastics -- Determination of thermal conductivity and thermal diffusivity -- Part 2: Transient plane heat source (hot disc) method"
- Note: What is called the k-value of construction materials (e.g. window glass) in the U.S., is called λ-value in Europe. What is called U-value (= the inverse of R-value) in the U.S., used to be called k-value in Europe, but is now also called U-value in Europe.

Definitions

The reciprocal of thermal conductivity is *thermal resistivity*, usually measured in kelvin-meters per watt ($K \cdot m \cdot W^{-1}$). When dealing with a known amount of material, its *thermal conductance* and the reciprocal property, *thermal resistance*, can be described. Unfortunately, there are differing definitions for these terms.

Conductance

For general scientific use, *thermal conductance* is the quantity of heat that passes in unit time through a plate of *particular area and thickness* when its opposite faces differ in temperature by one kelvin. For a plate of thermal conductivity k, area A and thickness L this is kA/L, measured in $W \cdot K^{-1}$ (equivalent to: W/°C). Thermal conductivity and conductance are analogous to electrical conductivity ($A \cdot m^{-1} \cdot V^{-1}$) and electrical conductance ($A \cdot V^{-1}$).

There is also a measure known as heat transfer coefficient: the quantity of heat that passes in unit time through *unit area* of a plate of particular thickness when its opposite faces differ in temperature by one kelvin. The reciprocal is *thermal insulance*. In summary:

- *thermal conductance* = kA/L, measured in $W \cdot K^{-1}$
  - *thermal resistance* = $L/(kA)$, measured in $K \cdot W^{-1}$ (equivalent to: °C/W)
- *heat transfer coefficient* = k/L, measured in $W \cdot K^{-1} \cdot m^{-2}$
  - *thermal insulance* = L/k, measured in $K \cdot m^2 \cdot W^{-1}$.

The heat transfer coefficient is also known as *thermal admittance*

Resistance

When thermal resistances occur in series, they are additive. So when heat flows through two components each with a resistance of 1 °C/W, the total resistance is 2 °C/W.

A common engineering design problem involves the selection of an appropriate sized heat sink for a given heat source. Working in units of thermal resistance greatly simplifies the design calculation. The following formula can be used to estimate the performance:

$$R_{hs} = \frac{\Delta T}{P_{th}} - R_s$$

where:

- R_{hs} is the maximum thermal resistance of the heat sink to ambient, in °C/W
- ΔT is the temperature difference (temperature drop), in °C
- P_{th} is the thermal power (heat flow), in watts
- R_s is the thermal resistance of the heat source, in °C/W

For example, if a component produces 100 W of heat, and has a thermal resistance of 0.5 °C/W, what is the maximum thermal resistance of the heat sink? Suppose the maximum temperature is 125 °C, and the ambient temperature is 25 °C; then the ΔT is 100 °C. The heat sink's thermal resistance to ambient must then be 0.5 °C/W or less.

Transmittance

A third term, *thermal transmittance*, incorporates the thermal conductance of a structure along with heat transfer due to convection and radiation. It is measured in the same units as thermal conductance and is sometimes known as the *composite thermal conductance*. The term *U-value* is another synonym.

Summary

In summary, for a plate of thermal conductivity *k* (the *k value*), area *A* and thickness *t*:

- *thermal conductance* = k/t, measured in $W \cdot K^{-1} \cdot m^{-2}$;
- *thermal resistance* (*R-value*) = t/k, measured in $K \cdot m^2 \cdot W^{-1}$;
- *thermal transmittance* (*U-value*) = $1/(\Sigma(t/k))$ + convection + radiation, measured in $W \cdot K^{-1} \cdot m^{-2}$.
- *K-value* refers in Europe to the total insulation value of a building. K-value is obtained by multiplying the form factor of the building (= the total inward surface of the outward walls of the building divided by the total volume of the building) with the average U-value of the outward walls of the building. K value is therefore expressed as $(m^2 \cdot m^{-3}) \cdot (W \cdot K^{-1} \cdot m^{-2}) = W \cdot K^{-1} \cdot m^{-3}$. A house with a volume of 400 m^3 and a K-value of 0.45 (the new European norm. It is commonly referred to as K45) will therefore theoretically require 180 W to maintain its interior temperature 1 K above exterior temperature. So, to maintain the house at 20 °C when it is freezing outside (0 °C), 3600 W of continuous heating is required.

Examples

In metals, thermal conductivity approximately tracks electrical conductivity according to the Wiedemann-Franz law, as freely moving valence electrons transfer not only electric current but also heat energy. However, the general correlation between electrical and thermal conductance does not hold for other materials, due to the increased importance of phonon carriers for heat in non-metals. As shown in the table below, highly electrically conductive silver is less thermally conductive than diamond, which is an electrical insulator.

Thermal conductivity depends on many properties of a material, notably its structure and temperature. For instance, pure crystalline substances exhibit very different thermal conductivities along different crystal axes, due to differences in phonon coupling along a given crystal axis. Sapphire is a notable example of variable thermal conductivity based on orientation and temperature, with 35 W/(m·K) along the c-axis and 32 W/(m·K) along the a-axis.

Air and other gases are generally good insulators, in the absence of convection. Therefore, many insulating materials function simply by having a large number of gas-filled pockets which prevent large-scale convection. Examples of these include expanded and extruded polystyrene (popularly referred to as "styrofoam") and silica aerogel. Natural, biological insulators such as fur and feathers achieve similar effects by dramatically inhibiting convection of air or water near an animal's skin.

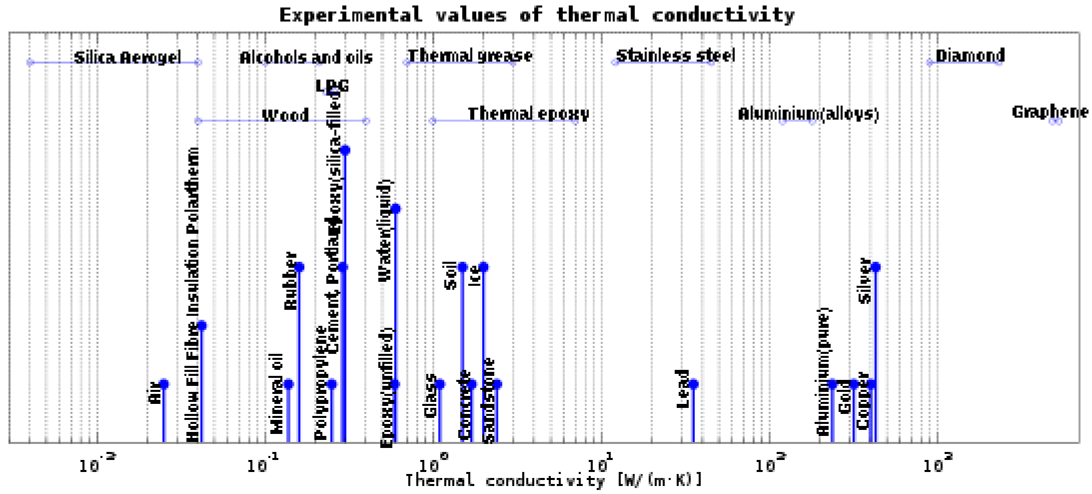
Ceramic is used for its low thermal conductivity on exhaust systems to prevent heat from reaching sensitive components

Light gases, such as hydrogen and helium typically have high thermal conductivity. Dense gases such as xenon and dichlorodifluoromethane have low thermal conductivity. An exception, sulfur hexafluoride, a dense gas, has a relatively high thermal conductivity due to its high heat capacity. Argon, a gas denser than air, is often used in insulated glazing (double paned windows) to improve their insulation characteristics.

Thermal conductivity is important in building insulation and related fields. However, materials used in such trades are rarely subjected to chemical purity standards. Several construction materials' *k* values are listed below. These should be considered approximate due to the uncertainties related to material definitions.

The following table is meant as a small sample of data to illustrate the thermal conductivity of various types of substances.

Experimental values



Experimental values of thermal conductivity.

This is a list of approximate values of thermal conductivity, *k*, for some common materials. Please consult the list of thermal conductivities for more accurate values, references and detailed information.

Material	Thermal conductivity W/(m·K)
Silica Aerogel	0.004 - 0.04
Air	0.025
Wood	0.04 - 0.4
Hollow Fill Fibre Insulation	0.042
Alcohols and oils	0.1 - 0.21
Polypropylene	0.25
Mineral oil	0.138
Rubber	0.16
LPG	0.23 - 0.26
Cement, Portland	0.29
Epoxy (silica-filled)	0.30
Epoxy (unfilled)	0.59
Water (liquid)	0.6
Thermal grease	0.7 - 3
Thermal epoxy	1 - 7
Glass	1.1
Soil	1.5
Concrete, stone	1.7

Ice	2
Sandstone	2.4
Stainless steel	12.11 ~ 45.0
Lead	35.3
Aluminium	237 (pure) 120—180 (alloys)
Gold	318
Copper	401
Silver	429
Diamond	900 - 2320
Graphene	(4840±440) - (5300±480)

Physical origins

Heat flux is exceedingly difficult to control and isolate in a laboratory setting. Thus at the atomic level, there are no simple, correct expressions for thermal conductivity. Atomically, the thermal conductivity of a system is determined by how atoms composing the system interact. There are two different approaches for calculating the thermal conductivity of a system.

- The first approach employs the Green-Kubo relations. Although this employs analytic expressions which in principle can be solved, in order to calculate the thermal conductivity of a dense fluid or solid using this relation requires the use of molecular dynamics computer simulation.

- The second approach is based upon the relaxation time approach. Due to the anharmonicity within the crystal potential, the phonons in the system are known to scatter. There are three main mechanisms for scattering:
  - Boundary scattering, a phonon hitting the boundary of a system;
  - Mass defect scattering, a phonon hitting an impurity within the system and scattering;
  - Phonon-phonon scattering, a phonon breaking into two lower energy phonons or a phonon colliding with another phonon and merging into one higher energy phonon.

Lattice waves

Heat transport in both glassy and crystalline dielectric solids occurs through elastic vibrations of the lattice (phonons). This transport is limited by elastic scattering of acoustic phonons by lattice defects. These predictions were confirmed by the experiments of Chang and Jones on commercial glasses and glass ceramics, where mean free paths were limited by "internal boundary scattering" to length scales of 10^{-2} cm to 10^{-3} cm.

The phonon mean free path has been associated directly with the effective relaxation length for processes without directional correlation. Thus, if V_g is the group velocity of a phonon wave packet, then the relaxation length l is defined as:

$$l = V_g t$$

where *t* is the characteristic relaxation time. Since longitudinal waves have a much greater phase velocity than transverse waves, V_{long} is much greater than V_{trans}, and the relaxation length or mean free path of longitudinal phonons will be much greater. Thus, thermal conductivity will be largely determined by the speed of longitudinal phonons.

Regarding the dependence of wave velocity on wavelength or frequency (dispersion), low-frequency phonons of long wavelength will be limited in relaxation length by elastic Rayleigh scattering. This type of light scattering form small particles is proportional to the fourth power of the frequency. For higher frequencies, the power of the frequency will decrease until at highest frequencies scattering is almost frequency independent. Similar arguments were subsequently generalized to many glass forming substances using Brillouin scattering.

Equations

First, we define heat conduction, *H*:

$$H = \frac{\Delta Q}{\Delta t} = kA\frac{\Delta T}{x}$$

where $\frac{\Delta Q}{\Delta t}$ is the rate of heat flow, *k* is the thermal conductivity, *A* is the total cross sectional area of conducting surface, ΔT is temperature difference, and *x* is the thickness of conducting surface separating the 2 temperatures. Dimension of thermal conductivity = $M^1L^1T^{-3}K^{-1}$

Rearranging the equation gives thermal conductivity:

$$k = \frac{\Delta Q}{\Delta t}\frac{1}{A}\frac{x}{\Delta T}$$

(Note: $\Delta T / x$ is the temperature gradient)

I.E. It is defined as the quantity of heat, ΔQ, transmitted during time Δt through a thickness *x*, in a direction normal to a surface of area *A*, per unit area of A, due to a temperature difference ΔT, under steady state conditions and when the heat transfer is dependent only on the temperature gradient.

Alternatively, it can be thought of as a flux of heat (energy per unit area per unit time) divided by a temperature gradient (temperature difference per unit length)

$$k = \frac{\Delta Q}{A \Delta t} \frac{x}{\Delta T}$$

Typical units are SI: W/(m·K) and English units: Btu/(h·ft·°F). To convert between the two, use the relation 1 Btu/(h·ft·°F) = 1.730735 W/(m·K). [Perry's Chemical Engineers' Handbook, 7th Edition, Table 1-4]

In the textile industry, a tog value may be quoted as a measure of thermal resistance in place of a measure in SI units.

Chapter 6

Thermal Contact Conductance

In physics, **thermal contact conductance** is the study of heat conduction between solid bodies in thermal contact. The **thermal contact conductance coefficient**, h_c, is a property indicating the thermal conductivity, or ability to conduct heat, between two bodies in contact. The inverse of this property is termed **thermal contact resistance**.

Definition

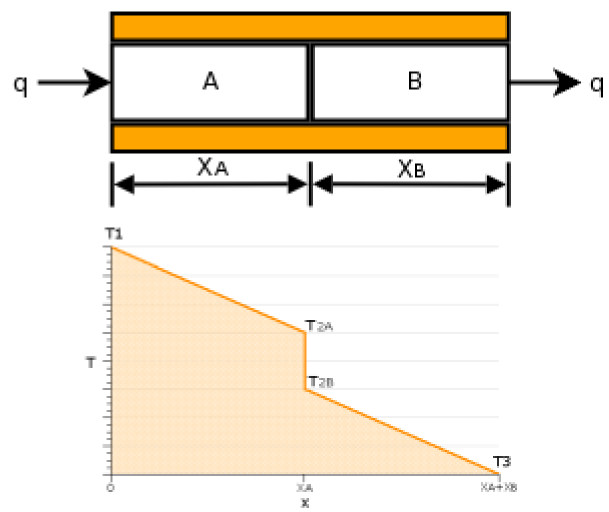


Fig. 1: Heat flow between two solids in contact and the temperature distribution.

When two solid bodies come in contact, such as A and B in Figure 1, heat flows from the hotter body to the colder body. From experience, the temperature profile along the two bodies varies, approximately, as shown in the figure. A temperature drop is observed at the interface between the two surfaces in contact. This phenomenon is said to be a result of a *thermal contact resistance* existing between the contacting surfaces. Thermal contact resistance is defined as the ratio between this temperature drop and the average heat flow across the interface.

According to **Fourier's law**, the heat flow between the bodies is found by the relation:

$$q = -kA\frac{dT}{dx} \quad (1)$$

where *q* is the heat flow, *k* is the thermal conductivity, *A* is the cross sectional area and *dT* / *dx* is the temperature gradient in the direction of flow.

From considerations of energy conservation, the heat flow between the two bodies in contact, bodies A and B, is found as:

$$q = \frac{T_1 - T_3}{\Delta x_A/(k_A A) + 1/(h_c A) + \Delta x_B/(k_B A)} \quad (2)$$

One may observe that the heat flow is directly related to the thermal conductivities of the bodies in contact, k_A and k_B, the contact area, *A* and the thermal contact resistance, $1 / h_c$, which, as previously noted, is the inverse of the thermal conductance coefficient, h_c.

Importance

Thermal contact conductance is an important factor in a variety of applications, largely because many physical systems contain a mechanical combination of two materials. Some of the fields where contact conductance is of importance are:

- Electronics
    - Electronic packaging
    - Heat sinks
    - Brackets
- Industry
    - Nuclear reactor cooling
    - Gas turbine cooling
    - Internal combustion engines
    - Heat exchangers
    - Thermal insulation
- Flight
    - Hypersonic flight vehicles
    - Thermal supervision for space vehicles

Factors influencing contact conductance

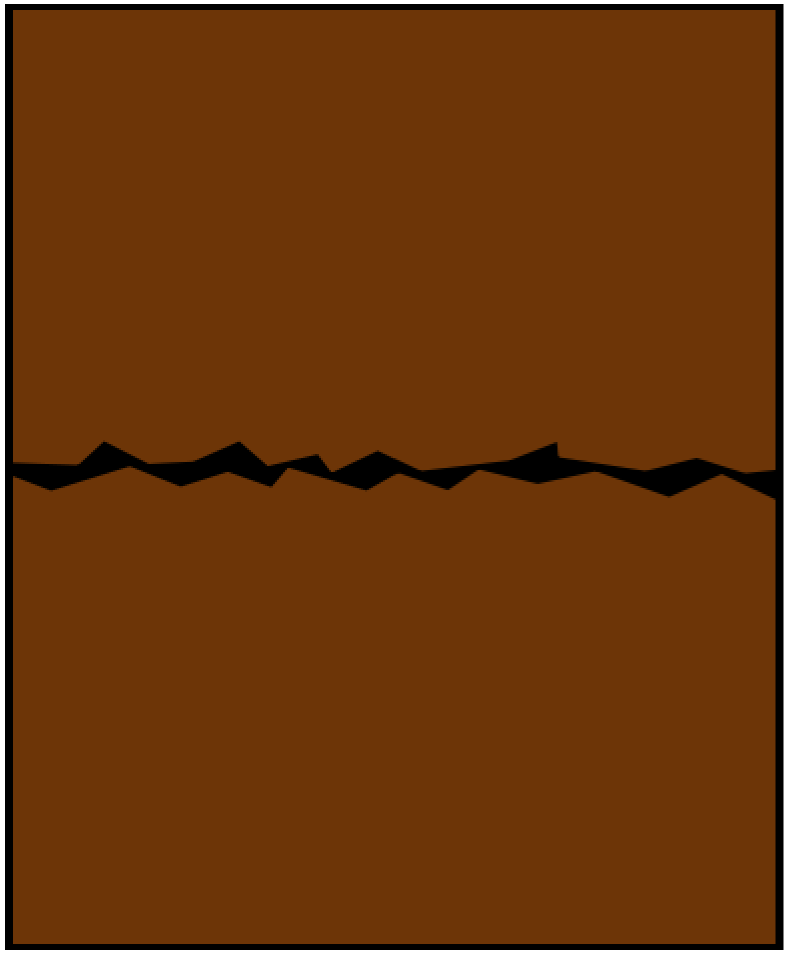


Fig. 2: An enlargement of the interface between two contacting surfaces. The finish quality is exaggerated for the sake of the argument.

Thermal contact conductance is a complicated phenomenon, influenced by many factors. Experience shows that the most important ones are as follows:

Contact pressure

The contact pressure is the factor of most influence on contact conductance. As contact pressure grows, contact conductance grows (And consequentially, contact resistance becomes smaller). This is attributed to the fact that the contact surface between the bodies grows as the contact pressure grows.

Since the contact pressure is the most important factor, most studies, correlations and mathematical models for measurement of contact conductance are done as a function of this factor.

The thermal contact resistance of certain sandwich kinds of materials that are manufactured by rolling under high temperatures may sometimes be ignored because the decrease in thermal conductivity between them is negligible.

Interstitial materials

No truly smooth surfaces really exist, and surface imperfections are visible under a microscope. As a result, when two bodies are pressed together, contact is only performed in a finite number of points, separated by relatively large gaps, as can be shown in Fig. 2. Since the actual contact area is reduced, another resistance for heat flow exists. The gasses/fluids filling these gaps may largely influence the total heat flow across the interface. The thermal conductivity of the interstitial material and its pressure are the two properties governing its influence on contact conductance.

In the absence of interstitial materials, as in a vacuum, the contact resistance will be much larger, since flow through the intimate contact points is dominant.

Surface roughness, waviness and flatness

One can characterise a surface that has undergone certain finishing operations by three properties: roughness, waviness and flatness. Among these, roughness is of most importance, and is usually indicated by an rms value, σ.

Surface deformations

When the two bodies come in contact, surface deformation may occur on both bodies. This deformation may either be plastic or elastic, depending on the material properties and the contact pressure. When a surface undergoes plastic deformation, contact resistance is lowered, since the deformation causes the actual contact area to increase

Surface cleanliness

The presence of dust particles, acids, etc., can also influence the contact conductance.

Measurement of thermal contact conductance

Going back to Formula 2, calculation of the thermal contact conductance may prove difficult, even impossible, due to the difficulty in measuring the contact area, A (A product of surface characteristics, as explained earlier). Because of this, contact conductance/resistance is usually found experimentally, by using a standard apparatus.

The results of such experiments are usually published in Engineering literature, on magazines such as *Journal of Heat Transfer*, *International Journal of Heat and Mass Transfer*, etc. Unfortunately, a centralized database of contact conductance coefficients does not exist, a situation which sometimes causes companies to use outdated, irrelevant data, or not taking contact conductance as a consideration at all.

CoCoE (Contact Conductance Estimator), a project founded to solve this problem and create a centralized database of contact conductance data and a computer program that uses it, was started in 2006.

Thermal boundary conductance

While a finite thermal contact conductance is due to voids at the interface, surface waviness, and surface roughness, etc., a finite conductance exists even at near ideal interfaces as well. This conductance, known as thermal boundary conductance, is due to the differences in electronic and vibrational properties between the contacting materials. This conductance is generally much higher than thermal contact conductance, but becomes important in nanoscale material systems.

Chapter 7

Heat Transfer Coefficient

The **heat transfer coefficient**, in thermodynamics and in mechanical and chemical engineering, is used in calculating the heat transfer, typically by convection or phase change between a fluid and a solid:

$$h = \frac{q}{A \cdot \Delta T}$$

where

q = heat flow in input or lost heat flow , J/s = W
h = heat transfer coefficient, W/(m^{2}K)
A = heat transfer surface area, m^2
ΔT = difference in temperature between the solid surface and surrounding fluid area, K

From the above equation, the heat transfer coefficient is the proportionality coefficient between the heat flux that is a heat flow per unit area, q/A, and the thermodynamic driving force for the flow of heat (i.e., the temperature difference, ΔT).

The heat transfer coefficient has SI units in watts per meter squared-kelvin: W/(m^{2}K).

Heat transfer coefficient is the inverse of thermal insulance.

There are numerous methods for calculating the heat transfer coefficient in different heat transfer modes, different fluids, flow regimes, and under different thermohydraulic conditions. Often it can be estimated by dividing the thermal conductivity of the convection fluid by a length scale. The heat transfer coefficient is often calculated from the Nusselt number (a dimensionless number).

Dittus–Boelter correlation (1930): forced convection inside tubes

A common and particularly simple correlation useful for many applications is the Dittus–Boelter heat transfer correlation for fluids in turbulent flow. This correlation is applicable when forced convection is the only mode of heat transfer; i.e., there is no boiling, condensation, significant radiation, etc. The accuracy of this correlation is anticipated to be ±15%.

For a liquid flowing in a straight circular pipe with a Reynolds number between 10 000 and 120 000 (in the turbulent pipe flow range), when the liquid's Prandtl number is between 0.7 and 120, for a location far from the pipe entrance (more than 10 pipe diameters; more than 50 diameters according to many authors) or other flow disturbances, and when the pipe surface is hydraulically smooth, the heat transfer coefficient between the bulk of the fluid and the pipe surface can be expressed as:

$$h = \frac{k_w}{D_H} Nu$$

where

> k_w - thermal conductivity of the liquid (i.e.water)
> D_H - D_i - Hydraulic diameter
> Nu - Nusselt number
> $Nu = 0.023 \cdot Re^{0.8} \cdot Pr^n$ (Dittus-Boelter correlation)
> Pr - Prandtl number
> Re - Reynolds number
> n = 0.4 for heating (wall hotter than the bulk fluid) and 0.33 for cooling (wall cooler than the bulk fluid) .

The fluid properties necessary for the application of this equation are evaluated at the bulk temperature thus avoiding iteration

Thom correlation

There exist simple fluid-specific correlations for heat transfer coefficient in boiling. The Thom correlation is for flow boiling of water (subcooled or saturated at pressures up to about 20 MPa) under conditions where the nucleate boiling contribution predominates over forced convection. This correlation is useful for rough estimation of expected temperature difference given the heat flux:

$$\Delta T_{sat} = 22.5 \cdot q^{0.5} \exp(-P/8.7)$$

where:

> ΔT_{sat} is the wall temperature elevation above the saturation temperature, K

q is the heat flux, MW/m^2
P is the pressure of water, MPa

Note that this empirical correlation is specific to the units given.

Heat transfer coefficient of pipe wall

The resistance to the flow of heat by the material of pipe wall can be expressed as a "heat transfer coefficient of the pipe wall". However, one needs to select if the heat flux is based on the pipe inner or the outer diameter.

Selecting to base the heat flux on the pipe inner diameter, and assuming that the pipe wall thickness is small in comparison with the pipe inner diameter, then the heat transfer coefficient for the pipe wall can be calculated as if the wall were not curved:

$$h_{wall} = \frac{k}{x}$$

where k is the effective thermal conductivity of the wall material and x is the wall thickness.

If the above assumption does not hold, then the wall heat transfer coefficient can be calculated using the following expression:

$$h_{wall} = \frac{2k}{d_i \ln(d_o/d_i)}$$

where d_i and d_o are the inner and outer diameters of the pipe, respectively.

The thermal conductivity of the tube material usually depends on temperature; the mean thermal conductivity is often used.

Combining heat transfer coefficients

For two or more heat transfer processes acting in parallel, heat transfer coefficients simply add:

$$h = h_1 + h_2 + \ldots$$

For two or more heat transfer processes connected in series, heat transfer coefficients add inversely:

$$\frac{1}{h} = \frac{1}{h_1} + \frac{1}{h_2} + \ldots$$

For example, consider a pipe with a fluid flowing inside. The rate of heat transfer between the bulk of the fluid inside the pipe and the pipe external surface is:

$$q = \left(\frac{1}{\frac{1}{h} + \frac{t}{k}} \right) \cdot A \cdot \Delta T$$

where

q = heat transfer rate (W)
h = heat transfer coefficient ($W/(m^2 \cdot K)$)
t = wall thickness (m)
k = wall thermal conductivity (W/m·K)
A = area (m^2)
ΔT = difference in temperature.

Overall heat transfer coefficient

The **overall heat transfer coefficient** U is a measure of the overall ability of a series of conductive and convective barriers to transfer heat. It is commonly applied to the calculation of heat transfer in heat exchangers, but can be applied equally well to other problems.

For the case of a heat exchanger, U can be used to determine the total heat transfer between the two streams in the heat exchanger by the following relationship:

$$q = UA\Delta T_{LM}$$

where

q = heat transfer rate (W)
U = overall heat transfer coefficient ($W/(m^2 \cdot K)$)
A = heat transfer surface area (m^2)
ΔT_{LM} = log mean temperature difference (K)

The overall heat transfer coefficient takes into account the individual heat transfer coefficients of each stream and the resistance of the pipe material. It can be calculated as the reciprocal of the sum of a series of thermal resistances (but more complex relationships exist, for example when heat transfer takes place by different routes in parallel):

$$\frac{1}{UA} = \sum \frac{1}{hA} + \sum R$$

where

R = Resistance(s) to heat flow in pipe wall (K/W)
Other parameters are as above.

The heat transfer coefficient is the heat transferred per unit area per kelvin. Thus *area* is included in the equation as it represents the area over which the transfer of heat takes place. The areas for each flow will be different as they represent the contact area for each fluid side.

The *thermal resistance* due to the pipe wall is calculated by the following relationship:

$$R = \frac{x}{k \cdot A}$$

where

x = the wall thickness (m)
k = the thermal conductivity of the material (W/(m·K))
A = the total area of the heat exchanger (m^2)

This represents the heat transfer by conduction in the pipe.

The *thermal conductivity* is a characteristic of the particular material. Values of thermal conductivities for various materials are listed in the list of thermal conductivities.

As mentioned earlier, the *convection heat transfer coefficient* for each stream depends on the type of fluid, flow properties and temperature properties.

Some typical heat transfer coefficients include:

- Air - h = 10 to 100 W/(m^2K)
- Water - h = 500 to 10,000 W/(m^2K)

Thermal resistance due to fouling deposits

Surface coatings can build on heat transfer surfaces during heat exchanger operation due to fouling. These add extra thermal resistance to the wall and may noticeably decrease the overall heat transfer coefficient and thus performance. (Fouling can also cause other problems.)

The additional thermal resistance due to fouling can be found by comparing the overall heat transfer coefficient determined from laboratory readings with calculations based on theoretical correlations. They can also be evaluated from the development of the overall heat transfer coefficient with time (assuming the heat exchanger operates under otherwise identical conditions). This is commonly applied in practice, e.g.. The following relationship is often used:

$$\frac{1}{U_{exp}} = \frac{1}{U_{pre}} + R_f$$

where

U_{exp} = overall heat transfer coefficient based on experimental data for the heat exchanger in the "fouled" state, $\frac{W}{m^2K}$

U_{pre} = overall heat transfer coefficient based on calculated or measured ("clean heat exchanger") data, $\frac{W}{m^2K}$

R_f = thermal resistance due to fouling, $\frac{m^2K}{W}$

Chapter 8

Lumped Capacitance Model

A **lumped capacitance model**, also called **lumped system analysis**, reduces a thermal system to a number of discrete “lumps” and assumes that the temperature difference inside each lump is negligible. This approximation is useful to simplify otherwise complex differential heat equations. It was developed as a mathematical analog of electrical capacitance.

This is a common approximation in transient conduction, which may be used whenever heat conduction within an object is much faster than heat conduction across the boundary of the object. This is a method of approximation that suitably reduces one aspect of the transient conduction system (that within the object) to an equivalent steady state system (that is, it is assumed that the temperature within the object is completely uniform, although its value may be changing in time).

Method

To determine the number of lumps the Biot number (Bi), a dimensionless parameter of the system, is used. Bi is defined as the ratio of the conductive heat resistance within the object to the convective heat transfer resistance across the object's boundary with a uniform bath of different temperature. When the thermal resistance to heat transferred into the object is larger than the resistance to heat being diffused completely within the object, the Biot number is less than 1. In this case, particularly for Biot numbers which are even smaller, the approximation of *spatially uniform temperature within the object* can begin to be used, since it can be presumed that heat transferred into the object has time to uniformaly distribute itself, due to the lower resistance to doing so, as compared with the resistance to heat entering the object.

If the Biot number is less than 0.1 for a solid object, then the entire material will be nearly the same temperature with the dominant temperature difference will be at the surface. It may be regarded as being "thermally thin". The Biot number must generally be less than 0.1 for usefully accurate approximation and heat transfer analysis. The mathematical solution to the lumped system approximation gives Newton's law of cooling.

A Biot number greater than 0.1 (a "thermally thick" substance) indicates that one cannot make this assumption, and more complicated heat transfer equations for "transient heat conduction" will be required to describe the time-varying and non-spatially-uniform temperature field within the material body.

The single capacitance approach can be expanded to involve many resistive and capacitive elements, with Bi < 0.1 for each lump. As the Biot number is calculated based upon a characteristic length of the system, the system can often be broken into a sufficient number of sections, or lumps, so that the Biot number is acceptably small.

Some characteristic lengths of thermal systems are:

- Plate: thickness
- Fin: thickness/2
- Long cylinder: diameter/4
- Sphere: diameter/6

For arbitrary shapes, it may be useful to consider the characteristic length to be volume / surface area.

Thermal circuits

A very useful concept used in heat transfer applications is the representation of thermal transfer by what is known as thermal circuits. A thermal circuit is the representation of the resistance to heat flow as though it were an electrical resistor. The heat transferred is analogous to the electrical current and the thermal resistance is analogous to the electrical resistor. The values of the thermal resistance for the different modes of heat transfer are calculated as the denominators of the developed equations. The thermal resistances of the different modes of heat transfer are used in analyzing combined modes of heat transfer.

The equations describing the three heat transfer modes and their thermal resistances, as discussed previously, are summarized in the table below:

Equations for different heat transfer modes and their thermal resistances.

Transfer Mode	Rate of Heat Transfer	Thermal Resistance
Conduction	$\dot{Q} = \frac{T_1 - T_2}{\left(\frac{L}{kA}\right)}$	$\frac{L}{kA}$
Convection	$\dot{Q} = \frac{T_{surf} - T_{envr}}{\left(\frac{1}{h_{conv}A_{surf}}\right)}$	$\frac{1}{h_{conv}A_{surf}}$
Radiation	$\dot{Q} = \frac{T_{surf} - T_{surr}}{\left(\frac{1}{h_r A_{surf}}\right)}$	$\frac{1}{h_r A}$, where $h_r = \epsilon\sigma(T_{surf} + T_{surr})(T_{surf}^2 + T_{surr}^2)$

In cases where there is heat transfer through different media (for example, through a composite material), the equivalent resistance is the sum of the resistances of the components that make up the composite. Likely, in cases where there are different heat transfer modes, the total resistance is the sum of the resistances of the different modes. Using the thermal circuit concept, the amount of heat transferred through any medium is the quotient of the temperature change and the total thermal resistance of the medium.

As an example, consider a composite wall of cross-sectional area A. The composite is made of an L_1 long cement plaster with a thermal coefficient k_1 and L_2 long paper faced fiber glass, with thermal coefficient k_2. The left surface of the wall is at T_i and exposed to air with a convective coefficient of h_i. The right surface of the wall is at T_o and exposed to air with convective coefficient h_o.

Using the thermal resistance concept heat flow through the composite is as follows:

$$\dot{Q} = \frac{T_i - T_o}{R_i + R_1 + R_2 + R_o} = \frac{T_i - T_1}{R_i} = \frac{T_i - T_2}{R_i + R_1} = \frac{T_i - T_3}{R_i + R_1 + R_2} = \frac{T_1 - T_2}{R_1} = \frac{T_3 - T_o}{R_0}$$

where

$$R_i = \frac{1}{h_i A}, \quad R_o = \frac{1}{h_o A}, \quad R_1 = \frac{L_1}{k_1 A}, \text{ and } \quad R_2 = \frac{L_2}{k_2 A}$$

Example: Newton's law of cooling

Many situations in which a object has a large thermal capacity and large conductivity, is immersed in a uniform bath which conducts heat relatively poorly, are described by **Newton's law of cooling.** This law stated in non-mathematical form is that *the rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings.* For the law to be correct, the temperatures at all points inside the body must be approximately the same, including the temperature at its surface. Thus, the temperature difference between the body and surroundings does not depend on which part of the body is chosen, since all parts of the body have effectively the same temperature. In these situations, the material of the body does not act to "insulate" other parts of the body from heat flow, and all of the significant insulation (or "thermal resistance") controlling the rate of heat flow in the situation resides in the area of contact between the body and its surroundings. Across this boundary, the temperature-value jumps in a discontinuous fashion.

In such situations, heat can be transferred from the exterior to the interior of a body, across the insulating boundary, by convection, conduction, or diffusion, so long as the boundary serves as a relatively poor conductor with regard to the object's interior. The presence of a physical insulator is not required, so long as the process which serves to pass heat across the boundary is "slow" in comparison to the conductive transfer of heat inside the body (or inside the region of interest—the "lump" described i the introduction).

Newton's law is mathematically stated by the simple first-order differential equation:

$$\frac{dQ}{dt} = h \cdot A(T_{\text{env}} - T(t)) = -h \cdot A\Delta T(t)$$

Q = Thermal energy in joules
h = Heat transfer coefficient
A = Surface area of the heat being transferred
T = Temperature of the object's surface and interior (since these are the same in this approximation)
T_{env} = Temperature of the environment
$\Delta T(t) = T(t) - T_{\text{env}}$ is the time-dependent thermal gradient between environment and object

Putting heat transfers into this form is sometimes not a very good approximation, depending on ratios of heat conductances in the system. If the differences are not large, an accurate formulation of heat transfers in the system may require analysis of heat flow based on the (transient) heat transfer equation in nonhomogeneous, or poorly conductive mediums.

Solution in terms of object heat capacity

If the entire body is treated as lumped capacitance heat reservoir, with total heat content which is proportional to simple total heat capacity C, and T, the temperature of the body, or $Q = CT$. It is expected that the system will experience exponential decay with time in the temperature of a body.

From the definition of heat capacity C comes the relation $C = dQ / dT$. Differentiating this equation with regard to time gives the identity (valid so long as temperatures in the object are uniform at any given time): $dQ / dt = C(dT / dt)$. This expression may be used to replace dQ / dt in the first equation which begins this section, above. Then, if $T(t)$ is the temperature of such a body at time t, and T_{env} is the temperature of the environment around the body:

$$\frac{dT(t)}{dt} = -r(T(t) - T_{\text{env}}) = -r\Delta T(t)$$

where

$r = hA / C$ is a positive constant characteristic of the system, which must be in units of s^{-1}, and is therefore sometimes expressed in terms of a characteristic time constant t_0 given by: $r = 1 / t_0 = \Delta T / (dT(t) / dt)$. Thus, in thermal systems, $t_0 = C / hA$. (The total heat capacity C of a system may be further represented by its mass-specific heat capacity c_p multiplied by its mass m, so that the time constant t_0 is also given by mc_p / hA).

The solution of this differential equation, by standard methods of integration and substitution of boundary conditions, gives:

$$T(t) = T_{\mathrm{env}} + (T(0) - T_{\mathrm{env}})\ e^{-rt}.$$

If:

$\Delta T(t)$ is defined as : $T(t) - T_{\mathrm{env}}$,where $\Delta T(0)$ is the initial temperature difference at time 0,

then the Newtonian solution is written as:

$$\Delta T(t) = \Delta T(0)\ e^{-rt} = \Delta T(0)\ e^{-t/t_0}.$$

This same solution is almost immediately apparent if the initial differential equation is written in terms of $\Delta T(t)$, as the single function to be solved for. '

$$\frac{dT(t)}{dt} = \frac{d\Delta T(t)}{dt} = -\frac{1}{t_0}\Delta T(t)$$

Applications

This mode of analysis has been applied to forensic sciences to analyze the time of death of humans. Also it can be applied to HVAC (heating, ventilating and air-conditioning, or building climate control), to ensure more nearly instantaneous effects of a change in comfort level setting.

Chapter 9

Relativistic Heat Conduction

The theory of **Relativistic Heat Conduction (RHC)** claims to be the only model for heat conduction (and similar diffusion processes) that is compatible with the theory of special relativity, the second law of thermodynamics, electrodynamics, and quantum mechanics, simultaneously. The main features of RHC are:

1. It admits a finite speed of heat propagation, and allows for relativistic effects when heat flux transients approach that speed.
2. It removes the possibility of paradoxical situations that may violate the second law of thermodynamics.
3. It, implicitly, admits the wave–particle duality of the heat-carrying "phonon".

These outcomes are achieved by (1) upgrading the Fourier equation of heat conduction to the form of a Telegraph equation of electrodynamics, and (2) introducing a new definition of the heat flux vector. Consequently, RHC gives rise to a number of interesting phenomena, such as thermal resonance and thermal shock waves, which are possible during high-frequency pulsed laser heating of thermal insulators. The main appealing feature of the theory is its mathematical *elegance and simplicity*.

Background

Classical model

For most of the last two centuries, heat conduction has been modelled by the well-known Fourier equation:

$$\frac{\partial \theta}{\partial t} = \alpha \nabla^2 \theta,$$

where θ is temperature, t is time, $\alpha = k/(\rho c)$ is thermal diffusivity, k is thermal conductivity, ρ is density, and c is specific heat capacity. The Laplace operator, ∇^2, is defined in Cartesian coordinates as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

This Fourier equation can be derived by substituting Fourier's linear approximation of the heat flux vector, $\boldsymbol{q}$, as a function of temperature gradient,

$$\mathbf{q} = -k \, \nabla\theta,$$

into the first law of thermodynamics

$$\rho \, c \, \frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{q} = 0,$$

where the del operator, ∇, is defined in 3D as

$$\nabla = \frac{\partial}{\partial x} \, \mathbf{i} + \frac{\partial}{\partial y} \, \mathbf{j} + \frac{\partial}{\partial z} \, \mathbf{k}.$$

It can be shown that this definition of the heat flux vector also satisfies the second law of thermodynamics,

$$\nabla \cdot \left(\frac{\mathbf{q}}{\theta}\right) + \rho \, \frac{\partial s}{\partial t} = \sigma,$$

where s is specific entropy and σ is entropy production. Alternatively, the second law can be written as

$$\sigma = \frac{-1}{\theta^2} \, \mathbf{q} \cdot \nabla\theta,$$

which leads to the condition

$$\sigma = \frac{k}{\theta^2} \left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 + \left(\frac{\partial \theta}{\partial z}\right)^2 \right],$$

which is always true, because k is a non-negative material property.

Hyperbolic model

For most of the last century, it was recognized that Fourier equation (and its more general Fick's law of diffusion) is in contradiction with the theory of relativity, for *at least* one reason: it admits infinite speed of propagation of heat signals within the continuum field. For example, consider a pulse of heat at the origin; then according to Fourier equation, it is felt (i.e. temperature changes) at infinity, instantaneously. The speed of information propagation is faster than the speed of light in vacuum, which is physically inadmissible within the framework of relativity.

To overcome this contradiction, workers such as Cattaneo, Vernotte, Chester, and others proposed that Fourier equation should be upgraded from the parabolic to a hyperbolic form,

$$\frac{1}{C^2}\,\frac{\partial^2\theta}{\partial t^2} \;+\; \frac{1}{\alpha}\,\frac{\partial\theta}{\partial t} \;=\; \nabla^2\theta,$$

also known as the Telegrapher's equation. Interestingly, the form of this equation traces its origins to Maxwell’s equations of electrodynamics; hence, the wave nature of heat is implied. In this equation, *C* is called the speed of second sound (i.e. the fictitious quantum particles, phonons) and this equation is known as the Hyperbolic Heat Conduction (HHC) equation.

For the HHC equation to remain compatible with the first law of thermodynamics, it is necessary to modify the definition of heat flux vector, ***q***, to

$$\tau_0\,\frac{\partial \mathbf{q}}{\partial t} \;+\; \mathbf{q} \;=\; -\,k\,\nabla\theta,$$

where $\tau_0$is a relaxation time, such that $C = \alpha/\tau_0$.

The most important implication of the hyperbolic equation is that by switching from a parabolic (dissipative) to a hyperbolic (includes a conservative term) partial differential equation, there is the possibility of phenomena such as thermal resonance and thermal shock waves.

Criticism to the HHC model

- The relaxation time, τ_0, is justified based on microscopic aspects of lattice vibration and electron transport; is an extension of kinetic theory calculations and Boltzmann equation for rarefied gases to the case of solids; and is calculated from statistical Newtonian mechanics.. Further, the speed *C* is only a collection of terms, α and τ_0, and has no physical reality or significance similar to that associated with the speed of light. Hence, the hyperbolic equation is compatible with relativity artificially (in form only), but is still fundamentally classical Newtonian.
- The new definition of heat flux vector is an *ad hoc* mathematical approximation of a far more complicated expression; this raises some doubts about the whole approach.
- The most serious criticism is that the hyperbolic equation can violate the second law of thermodynamics. For example, consider an infinitely long wire conductor,

with a heat source at the origin, and measure temperature at distances significantly remote from origin. If the heat source at origin varies with a frequency much higher than the relaxation time (i.e. faster than the speed of second sound) then the hyperbolic equation admits a temperature field in which heat would appear to be moving from cold to hot, in violation of the second law. This contradiction was demonstrated in more mathematically rigorous form.

The theory of RHC attempts to resolve the controversies surrounding the hyperbolic equation, while maintaining the form of that equation. This is achieved by:

- Deriving the hyperbolic equation starting from space-time duality of a Minkowski space, and simple Lorentz transformations, that are basic to the theory of special relativity. This is done without any reference to microstructure or statistical mechanics.
- Treating the speed of second sound, *C*, as a fundamental property of the temperature field, although still fundamentally inferior to the speed of light.
- Modifying the definition of the heat flux vector so that it is simpler, more elegant, and bring it in compliance with the second law of thermodynamics.

Derivation of the RHC equation

Transformations

In a Euclidean space, distance between any two points, *ds*, is measured by

$$ds^2 = dx^2 + dy^2 + dz^2,$$

where *dx*, *dy*, and *dz* are displacements along three orthogonal axes.

In a Minkowski space, distance between two events, *ds*, is measured by

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2,$$

where, τ, is space-like-time and is related to real time, *t*, by

$$\tau = i\,C\,t,$$

where *C* is speed of light in vacuum and $i = \sqrt{-1}$. Hence,

$$ds^2 = dx^2 + dy^2 + dz^2 - C^2\,dt^2.$$

Consequently, the 3D del, ∇ ,operator is upgraded to the 4D quad, $\Box$, operator (also known as the Four-gradient)

$$\Box = \frac{\partial}{\partial \tau}\mathbf{o} + \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} = \frac{\partial}{\partial \tau}\mathbf{o} + \nabla = \frac{-i}{C}\frac{\partial}{\partial t}\mathbf{o} + \nabla.$$

Likewise, the 3D Laplacian, ∇^2, operator is upgraded to the 4D d'Alembert operator, $\Box^2$,

$$\Box^2 = \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \tau^2} + \nabla^2 = \frac{-1}{C^2}\frac{\partial^2}{\partial t^2} + \nabla^2.$$

Any physical quantity that is Galilean invariant in Euclidean space can be made Lorentz invariant in a Minkowski space, by upgrading from 3D to 4D operators. Consequently, Fourier's equation can be upgraded to 4D as

$$\frac{\partial \theta}{\partial t} = \alpha\,\Box^2\theta = \frac{-\alpha}{C^2}\frac{\partial^2\theta}{\partial t^2} + \alpha\,\nabla^2\theta,$$

which is called the Relativistic Heat Conduction equation. Likewise, the definition of the heat-flux vector, $\boldsymbol{q}$, is upgraded to the 4D form as

$$\mathbf{q} = -k\,\Box\theta = -k\,\nabla\theta + \frac{i\,k}{C}\frac{\partial \theta}{\partial t}\mathbf{o}.$$

Implications

It can be shown that this definition of $\boldsymbol{q}$ is compatible with the first law of thermodynamics,

$$\rho\,c\,\frac{\partial \theta}{\partial t} + \Box\cdot\mathbf{q} = 0 = \rho\,c\,\frac{\partial \theta}{\partial t} + \nabla\cdot\mathbf{q} + \frac{-i}{C}\frac{\partial \mathbf{q}}{\partial t}\cdot\mathbf{o},$$

as well as the second law of thermodynamics,

$$\Box\cdot\left(\frac{\mathbf{q}}{\theta}\right) + \rho\,\frac{\partial s}{\partial t} = \sigma,$$

in their 4D upgraded form. The imaginary terms in these equations are direct manifestation of the wave nature of heat, and are essential for the heat equation to become compatible with all laws of physics. The real terms in these equations are identical to those in the classical heat model.

The most interesting observation about RHC is that it reduces the second law of thermodynamics to a statement of the form

$$\left(\frac{dt}{dx}\right)^2 + \left(\frac{dt}{dy}\right)^2 + \left(\frac{dt}{dz}\right)^2 \geqslant \frac{1}{C^2},$$

which is the “no action at a distance” principle of special relativity. Essentially, the RHC asserts that relativity and the second law of thermodynamics are two alternative, but equal statements about the nature of time. Both physical principles are mutually derivable from each other and are complementary.

Criticism to RHC

As far as heat conduction is concerned, the RHC equation is identical in form to the hyperbolic equation, and all analytical and experimental results that are relevant to one are equally applicable to the other. The definition of heat flux vector, however, is different; but the RHC definition is merely a 4D upgrade of the original linear Fourier approximation. The mathematics of RHC is much simpler and more elegant. However, RHC raises some significant conceptual challenges:

1. This weak interpretation of relativity, in which the speed of second sound plays a role similar to that of the speed of light, can be viewed as downgrading or degrading to the universality of the theory of relativity. Notice how the symbol *c* in standard relativity theory is replaced with *C* without much interpretation.
2. The implied wave nature of heat is controversial. Some workers reject the wave nature of heat on dogmatic grounds. Moreover, RHC implies that a phonon is a full-fledged objective quantum particle whose physical reality is no lesser than that of a photon. Existing experimental evidences are not enough to support for or against such views.
3. Heat quantities become complex numbers, with values including "imaginary temperature", which are hard to interpret experimentally.
4. The equivalence of relativity and the second law is shocking, because it implies that one of them can be a derivative of the other.

In summary, while the RHC is mathematically simple and elegant, and experimentally practical and relevant, it raises a number of conceptual issues that are highly controversial.

Applications

The RHC theory is applicable for any physical problem in which the hyperbolic equation is relevant: when speed of heat propagation is small, e.g. thermal insulators, or when speed of heat-flux variation is very large, e.g. pulsed-laser heating. Applications for those types of problems are abundant, and there is plenty of published work. Most of these results remain relevant to RHC, but because the definition of heat flux vector is different, final closed-form solutions may not be the same. In many cases, RHC provides closed-form solutions that are not possible using the HHC model. A number of useful fundamental solutions for 1D and 2D relativistic moving heat sources are available in closed-form.

Chapter 10

Heat Kernel & Thermal Diffusivity

Heat Kernel

In the mathematical study of heat conduction and diffusion, a **heat kernel** is the fundamental solution to the heat equation on a particular domain with appropriate boundary conditions. It is also one of the main tools in the study of the spectrum of the Laplace operator, and is thus of some auxiliary importance throughout mathematical physics. The heat kernel represents the evolution of temperature in a region whose boundary is held fixed at a particular temperature (typically zero), such that an initial unit of heat energy is placed at a point at time $t = 0$.

The most well-known heat kernel is the heat kernel of d-dimensional Euclidean space $\mathbf{R}^d$, which has the form

$$K(t,x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t}.$$

This solves the heat equation

$$K_t(t,x,y) = \Delta_x K(t,x,y)$$

for all $t > 0$ and $x,y \in \mathbf{R}^d$, with the initial condition

$$\lim_{t\downarrow 0} K(t,x,y) = \delta(x-y) = \delta_x(y)$$

where δ is a Dirac delta distribution and the limit is taken in the sense of distributions. To wit, for every smooth function φ of compact support,

$$\lim_{t\downarrow 0} \int_{\mathbf{R}^n} K(t,x,y)\phi(y)\,dy = \phi(x).$$

On a more general domain Ω in $\mathbf{R}^d$, such an explicit formula is not generally possible. The next simplest cases of a disc or square involve, respectively, Bessel functions and Jacobi theta functions. Nevertheless, the heat kernel (for, say, the Dirichlet problem) still exists and is smooth for $t > 0$ on arbitrary domains and indeed on any Riemannian manifold with boundary, provided the boundary is sufficiently regular. More precisely, in these more general domains, the heat kernel for the Dirichlet problem is the solution of the initial boundary value problem

$$\begin{aligned}
&K_t(t,x,y) = \Delta K(t,x,y) \quad \text{for all t} > 0 \text{ and x, y} \in \Omega \\
&\lim_{t\downarrow 0} K(t,x,y) = \delta_x(y) \quad \text{for all x, y} \in \Omega \\
&K(t,x,y) = 0, \quad x \in \partial\Omega \text{ or y} \in \partial\Omega.
\end{aligned}$$

It is not difficult to derive a formal expression for the heat kernel on an arbitrary domain. Consider the Dirichlet problem in a connected domain (or manifold with boundary) U. Let λ_n be the eigenvalues for the Dirichlet problem of the Laplacian

$$\begin{cases} \Delta\phi + \lambda\phi = 0 & \text{in} \ \ U \\ \phi = 0 & \text{on} \ \ \partial U. \end{cases}$$

Let φ_n denote the associated eigenfunctions, normalized to be orthonormal in $L^2(U)$. The inverse Dirichlet Laplacian Δ^{-1} is a compact and selfadjoint operator, and so the spectral theorem implies that the eigenvalues satisfy

$$0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots, \quad \lambda_n \to \infty.$$

The heat kernel has the following expression:

$$K(t,x,y) = \sum_{n=0}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y). \qquad \textbf{(}\textit{1}\textbf{)}$$

Formally differentiating the series under the sign of the summation shows that this should satisfy the heat equation. However, convergence and regularity of the series are quite delicate.

The heat kernel is also sometimes identified with the associated integral transform, defined for compactly supported smooth φ by

$$T\phi = \int_\Omega K(t,x,y)\phi(y)\,dy.$$

The spectral mapping theorem gives a representation of T in the form

$$T = e^{t\Delta}.$$

Thermal Diffusivity

In heat transfer analysis, **thermal diffusivity** (symbol: α, but note that the symbols κ, *D*, and *k* are all commonly used) is the thermal conductivity divided by the volumetric heat capacity. It has the SI unit of m^2/s.

$$\alpha = \frac{k}{\rho c_p}$$

where:

- k : thermal conductivity (SI units: W/(m·K))
- ρ : density (kg/m^3)
- c_p : specific heat capacity (J/(kg·K))

The denominator of the thermal diffusivity expression above, ρc_p, can be identified as the volumetric heat capacity with the SI unit of J/(m^3·K).

Substances with high thermal diffusivity rapidly adjust their temperature to that of their surroundings because they conduct heat quickly in comparison to their volumetric heat capacity or 'thermal bulk' and they generally do not require much energy from their surroundings to reach thermal equilibrium.

Thermal diffusivity of selected materials and substances

Material	**Thermal diffusivity (m^2/s)**	**Thermal diffusivity (m^2/s) × 10^{-6}**
Pyrolytic graphite, parallel to layers	1.22×10^{-3}	1220
Silver, pure (99.9%)	1.6563×10^{-4}	165.63
Gold	1.27×10^{-4}	127.
Copper	1.1234×10^{-4}	112.34
Aluminium	8.418×10^{-5}	84.18
Aluminum 6061-T6 Alloy	6.4×10^{-5}	64.
Water vapour (1 atm, 400 K)	2.338×10^{-5}	23.38
Air (1 atm, 300 K)	2.2160×10^{-5}	22.16
Aluminium oxide (polycrystalline)	1.20×10^{-5}	12.0
Steel, carbon (1%)	1.172×10^{-5}	11.72
Steel, stainless 304A	4.2×10^{-6}	4.2
Iron	2.3×10^{-5}	23.
Silicon	8.8×10^{-5}	88
Quartz	1.4×10^{-6}	1.4

Silicon Dioxide (Polycrystalline)	8.3×10^{-7}	0.83
Water	1.4×10^{-7}	0.14
Polyvinyl Chloride (PVC)	8×10^{-8}	0.08
Alcohol	7×10^{-8}	0.07
Air	1.9×10^{-5}	19
Pyrolytic graphite, normal to layers	3.6×10^{-6}	3.6
Sandstone	$1.12–1.19 \times 10^{-6}$	1.15
Tin	4.0×10^{-5}	40.
Brick, common	5.2×10^{-7}	0.52
Glass, window	3.4×10^{-7}	0.34
Rubber	1.3×10^{-7}	0.13
Nylon	9×10^{-8}	0.09
Wood (Yellow Pine)	8.2×10^{-8}	0.082
Oil, engine (saturated liquid, 100 °C)	7.38×10^{-8}	0.0738

Chapter 11

Heat Transfer

Heat transfer is a discipline of thermal engineering that concerns the transfer of thermal energy from one physical system to another. Heat transfer is classified into various mechanisms, such as heat conduction, convection, thermal radiation, and phase-change transfer. Engineers also consider the transfer of mass of differing chemical species, either cold or hot, to achieve heat transfer. All forms of heat transfer may occur in some systems (for example, in transparent fluids like the Earth's atmosphere) at the same time.

Heat conduction, also called diffusion, is the direct microscopic exchange of kinetic energy of particles through the boundary between two systems. When an object is at a different temperature from another body or its surroundings, heat flows so that the body and the surroundings reach the same temperature at thermal equilibrium. Conduction happens in both fluids and solids.

Heat convection only occurs in fluids. It occurs when bulk flow of a fluid (gas or liquid) carries heat along with the flow of matter in the fluid. The flow of fluid may be forced by external processes, or sometimes (in gravitational fields) by buoyancy forces caused when thermal energy expands the fluid (for example in a fire plume), thus influencing its own transfer. The latter process is sometimes called "natural convection". All convective processes also move heat partly by diffusion, as well.

The final major form of heat transfer is by radiation, which occurs in any transparent medium (solid or fluid) but may also even occur across vacuum (as when the Sun heats the Earth). Heat transfer by thermal radiation is the transfer of energy by transmission of electromagnetic radiation described by black body theory.

Overview

Heat is defined in physics as the transfer of thermal energy across a well-defined boundary around a thermodynamic system. It is a characteristics of a process and is not statically contained in matter. In engineering contexts, however, the term *heat transfer* has acquired a specific usage, despite its literal redundancy of the characterization of transfer. In these contexts, *heat* is taken as synonymous to thermal energy. This usage has its origin in the historical interpretation of heat as a fluid (*caloric*) that can be transferred by various causes, and that is also common in the language of laymen and everyday life.

Fundamental methods of heat transfer in engineering include conduction, convection, and radiation. Physical laws describe the behavior and characteristics of each of these methods. Real systems often exhibit a complicated combination of them. Heat transfer methods are used in numerous disciplines, such as automotive engineering, thermal management of electronic devices and systems, climate control, insulation, materials processing, and power plant engineering.

Various mathematical methods have been developed to solve or approximate the results of heat transfer in systems. Heat transfer is a path function (or process quantity), as opposed to a state quantity; therefore, the amount of heat transferred in a thermodynamic process that changes the state of a system depends on how that process occurs, not only the net difference between the initial and final states of the process. Heat flux is a quantitative, vectorial representation of the heat flow through a surface.

Heat transfer is typically studied as part of a general chemical engineering or mechanical engineering curriculum. Typically, thermodynamics is a prerequisite for heat transfer courses, as the laws of thermodynamics are essential to the mechanism of heat transfer. Other courses related to heat transfer include energy conversion, thermofluids, and mass transfer.

The transport equations for thermal energy (Fourier's law), mechanical momentum (Newton's law for fluids), and mass transfer (Fick's laws of diffusion) are similar and analogies among these three transport processes have been developed to facilitate prediction of conversion from any one to the others.

Mechanisms

The fundamental modes of heat transfer are:

Conduction or diffusion
: The transfer of energy between objects that are in physical contact

Convection
: The transfer of energy between an object and its environment, due to fluid motion

Radiation
: The transfer of energy to or from a body by means of the emission or absorption of electromagnetic radiation

Mass transfer
: The transfer of energy from one location to another as a side effect of physically moving an object containing that energy

Conduction

On a microscopic scale, heat conduction occurs as hot, rapidly moving or vibrating atoms and molecules interact with neighboring atoms and molecules, transferring some of their energy (heat) to these neighboring particles. In other words, heat is transferred by conduction when adjacent atoms vibrate against one another, or as electrons move from one atom to another. Conduction is the most significant means of heat transfer within a solid or between solid objects in thermal contact. Fluids—especially gases—are less conductive. Thermal contact conductance is the study of heat conduction between solid bodies in contact.

Steady state conduction is a form of conduction that happens when the temperature difference driving the conduction is constant, so that after an equilibration time, the spatial distribution of temperatures in the conducting object does not change any further. In steady state conduction, the amount of heat entering a section is equal to amount of heat coming out.

Transient conduction occurs when the temperature within an object changes as a function of time. Analysis of transient systems is more complex and often calls for the application of approximation theories or numerical analysis by computer.

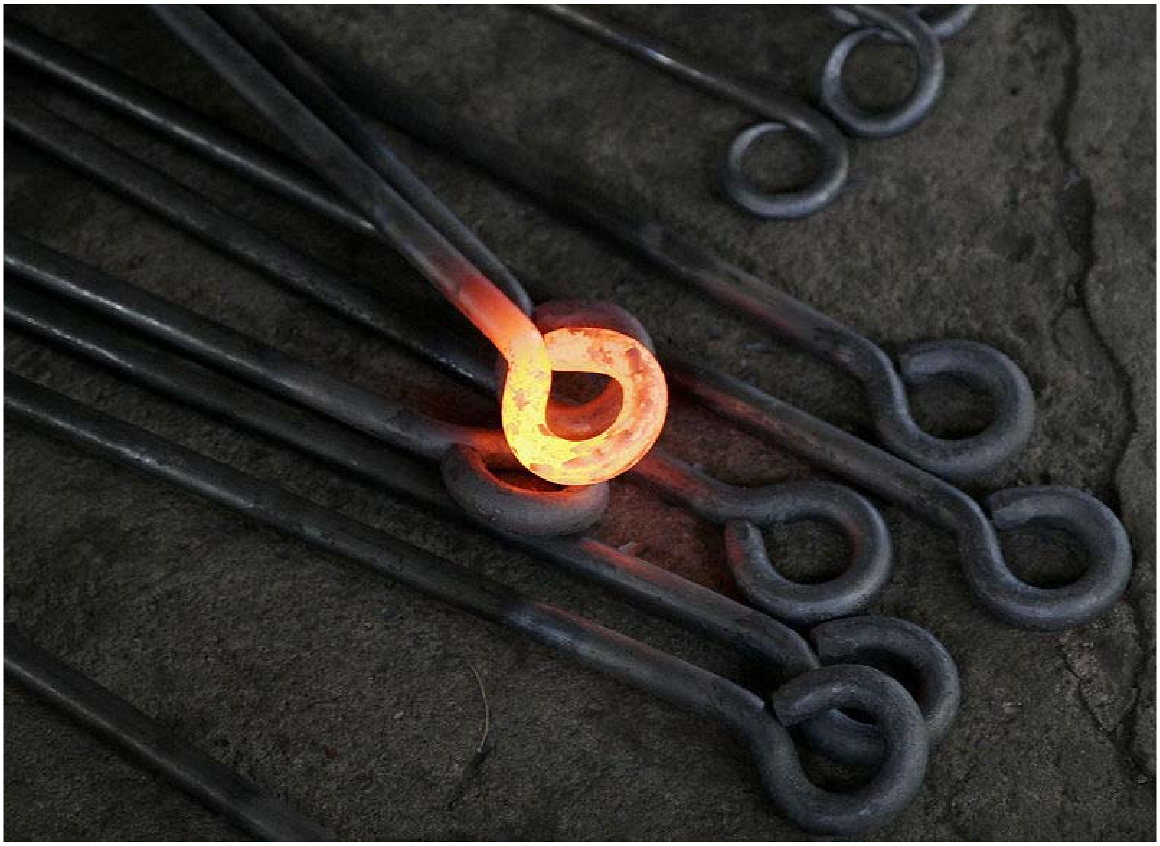
Convection

Convective heat transfer, or convection, is the transfer of heat from one place to another by the movement of fluids. (In physics, the term *fluid* means any substance that deforms under shear stress; it includes liquids, gases, plasmas, and some plastic solids.) Bulk motion of the fluid enhances the heat transfer between the solid surface and the fluid. Convection is usually the dominant form of heat transfer in liquids and gases. Although often discussed as a third method of heat transfer, convection actually describes the combined effects of conduction and fluid flow.

Free, or natural, convection occurs when the fluid motion is caused by buoyancy forces that result from density variations due to variations of temperature in the fluid. *Forced* convection is when the fluid is forced to flow over the surface by external means—such as fans, stirrers, and pumps—creating an artificially induced convection current.

Convection is described by Newton's law of cooling: "The rate of heat loss of a body is proportional to the difference in temperatures between the body and its surroundings."

Radiation



A red-hot iron object, transferring heat to the surrounding environment primarily through thermal radiation.

Thermal radiation is energy emitted by matter as electromagnetic waves due to the pool of thermal energy that all matter possesses that has a temperature above absolute zero. Thermal radiation propagates without the presence of matter through the vacuum of space.

Thermal radiation is a direct result of the random movements of atoms and molecules in matter. Since these atoms and molecules are composed of charged particles (protons and electrons), their movement results in the emission of electromagnetic radiation, which carries energy away from the surface.

Unlike conductive and convective forms of heat transfer, thermal radiation can be concentrated in a small spot by using reflecting mirrors, which is exploited in concentrating solar power generation. For example, the sunlight reflected from mirrors heats the PS10 solar power tower and during the day it can heat water to 285 °C (545 °F).

Mass Transfer

In mass transfer, energy—including thermal energy—is moved by the physical transfer of a hot or cold object from one place to another. This can be as simple as placing hot water in a bottle and heating a bed, or the movement of an iceberg in changing ocean currents. A practical example is thermal hydraulics.

Convection vs. conduction

In a body of fluid that is heated from underneath its container, conduction and convection can be considered to compete for dominance. If heat conduction is too great, fluid moving down by convection is heated by conduction so fast that its downward movement will be stopped due to its buoyancy, while fluid moving up by convection is cooled by conduction so fast that its driving buoyancy will diminish. On the other hand, if heat conduction is very low, a large temperature gradient may be formed and convection might be very strong.

The Rayleigh number (*Ra*) is a measure determining the result of this competition.

$$Ra = \frac{g\Delta\rho L^3}{\mu\alpha} = \frac{g\beta\Delta T L^3}{\nu\alpha}$$

where

- g is acceleration due to gravity
- ρ is the density with $\Delta\rho$ being the density difference between the lower and upper ends
- μ is the dynamic viscosity
- α is the Thermal diffusivity
- β is the volume thermal expansivity (sometimes denoted α elsewhere)
- T is the temperature and
- ν is the kinematic viscosity.

The Rayleigh number can be understood as the ratio between the rate of heat transfer by convection to the rate of heat transfer by conduction; or, equivalently, the ratio between the corresponding timescales (i.e. conduction timescale divided by convection timescale), up to a numerical factor. This can be seen as follows, where all calculations are up to numerical factors depending on the geometry of the system.

The buoyancy force driving the convection is roughly $g\Delta\rho L^3$, so the corresponding pressure is roughly $g\Delta\rho L$. In steady state, this is canceled by the shear stress due to viscosity, and therefore roughly equals $\mu V / L = \mu / T_{conv}$, where V is the typical fluid velocity due to convection and T_{conv} the order of its timescale. The conduction timescale, on the other hand, is of the order of $T_{cond} = L^2 / \alpha$.

Convection occurs when the Rayleigh number is above 1,000–2,000. For example, the Earth's mantle, exhibiting non-stable convection, has Rayleigh number of the order of 1,000, and T_{conv} as calculated above is around 100 million years.

Phase changes

Transfer of heat through a phase transition in the medium—such as water-to-ice, water-to-steam, steam-to-water, or ice-to-water—involves significant energy and is exploited in many ways: steam engines, refrigerators, etc. For example, the Mason equation is an approximate analytical expression for the growth of a water droplet based on the effects of heat transport on evaporation and condensation.

Boiling

Heat transfer in boiling fluids is complex, but of considerable technical importance. It is characterized by an *S*-shaped curve relating heat flux to surface temperature difference.

At low driving temperatures, no boiling occurs and the heat transfer rate is controlled by the usual single-phase mechanisms. As the surface temperature is increased, local boiling occurs and vapor bubbles nucleate, grow into the surrounding cooler fluid, and collapse. This is *sub-cooled nucleate boiling*, and is a very efficient heat transfer mechanism. At high bubble generation rates, the bubbles begin to interfere and the heat flux no longer increases rapidly with surface temperature (this is the departure from nucleate boiling, or DNB). At higher temperatures still, a maximum in the heat flux is reached (the critical heat flux, or CHF). The regime of falling heat transfer that follows is not easy to study, but is believed to be characterized by alternate periods of nucleate and film boiling. Nucleate boiling slows the heat transfer due to gas bubbles on the heater's surface; as mentioned, gas-phase thermal conductivity is much lower than liquid-phase thermal conductivity, so the outcome is a kind of "gas thermal barrier".

At higher temperatures still, the hydrodynamically-quieter regime of film boiling is reached. Heat fluxes across the stable vapor layers are low, but rise slowly with temperature. Any contact between fluid and the surface that may be seen probably leads to the extremely rapid nucleation of a fresh vapor layer ("spontaneous nucleation").

Condensation

Condensation occurs when a vapor is cooled and changes its phase to a liquid. Condensation heat transfer, like boiling, is of great significance in industry. During condensation, the latent heat of vaporization must be released. The amount of the heat is the same as that absorbed during vaporization at the same fluid pressure.

There are several types of condensation:

- Homogeneous condensation, as during a formation of fog.
- Condensation in direct contact with subcooled liquid.

- Condensation on direct contact with a cooling wall of a heat exchanger: This is the most common mode used in industry:
    - Filmwise condensation is when a liquid film is formed on the subcooled surface, and usually occurs when the liquid wets the surface.
    - Dropwise condensation is when liquid drops are formed on the subcooled surface, and usually occurs when the liquid does not wet the surface.

  Dropwise condensation is difficult to sustain reliably; therefore, industrial equipment is normally designed to operate in filmwise condensation mode.

Modeling approaches

Complex heat transfer phenomena can be modeled in different ways.

Heat equation

The heat equation is an important partial differential equation that describes the distribution of heat (or variation in temperature) in a given region over time. In some cases, exact solutions of the equation are available; in other cases the equation must be solved numerically using computational methods. For example, simplified climate models may use Newtonian cooling, instead of a full (and computationally expensive) radiation code, to maintain atmospheric temperatures.

Lumped system analysis

System analysis by the lumped capacitance model is a common approximation in transient conduction that may be used whenever heat conduction within an object is much faster than heat conduction across the boundary of the object.

This is a method of approximation that reduces one aspect of the transient conduction system—that within the object—to an equivalent steady state system. That is, the method assumes that the temperature within the object is completely uniform, although its value may be changing in time.

In this method, the ratio of the conductive heat resistance within the object to the convective heat transfer resistance across the object's boundary, known as the *Biot number*, is calculated. For small Biot numbers, the approximation of *spatially uniform temperature within the object* can be used: it can be presumed that heat transferred into the object has time to uniformly distribute itself, due to the lower resistance to doing so, as compared with the resistance to heat entering the object.

Applications and techniques

Heat transfer has broad application to the functioning of numerous devices and systems. Heat-transfer principles may be used to preserve, increase, or decrease temperature in a wide variety of circumstances.

Insulation and radiant barriers

Heat exposure as part of a fire test for firestop products.

Thermal insulators are materials specifically designed to reduce the flow of heat by limiting conduction, convection, or both. Radiant barriers are materials that reflect radiation, and therefore reduce the flow of heat from radiation sources. Good insulators are not necessarily good radiant barriers, and vice versa. Metal, for instance, is an excellent reflector and a poor insulator.

The effectiveness of an insulator is indicated by its **R-value**, or resistance value. The R-value of a material is the inverse of the conduction coefficient (*k*) multiplied by the thickness (*d*) of the insulator. In most of the world, R-values are measured in SI units: square-meter kelvins per watt ($m^2 \cdot K/W$). In the United States, R-values are customarily given in units of British thermal units per hour per square-foot degrees Fahrenheit ($Btu/h \cdot ft^2 \cdot °F$).

$$R = \frac{d}{k}$$

$$C = \frac{Q}{m \Delta T}$$

Rigid fiberglass, a common insulation material, has an R-value of four per inch, while poured concrete, a poor insulator, has an R-value of 0.08 per inch.

The tog is a measure of thermal resistance, commonly used in the textile industry, and often seen quoted on, for example, duvets and carpet underlay.

The effectiveness of a radiant barrier is indicated by its **reflectivity**, which is the fraction of radiation reflected. A material with a high reflectivity (at a given wavelength) has a low emissivity (at that same wavelength), and vice versa. At any specific wavelength, reflectivity = 1 - emissivity. An ideal radiant barrier would have a reflectivity of 1, and would therefore reflect 100 percent of incoming radiation. Vacuum flasks, or Dewars, are silvered to approach this ideal. In the vacuum of space, satellites use multi-layer insulation, which consists of many layers of aluminized (shiny) Mylar to greatly reduce radiation heat transfer and control satellite temperature.

Critical insulation thickness

Low thermal conductivity (*k*) materials reduce heat fluxes. The smaller the *k* value, the larger the corresponding thermal resistance (*R*) value. Thermal conductivity is measured in watts-per-meter per kelvin ($W{\cdot}m^{-1}{\cdot}K^{-1}$), represented as *k*. As the thickness of insulating material increases, the thermal resistance—or R-value—also increases.

However, adding layers of insulation has the potential of increasing the surface area, and hence the thermal convection area.

For example, as thicker insulation is added to a cylindrical pipe, the outer radius of the pipe-and-insulation system increases, and therefore surface area increases. The point where the added resistance of increasing insulation thickness becomes overshadowed by the effect of increased surface area is called the critical insulation thickness. In simple cylindrical pipes, this is calculated as a radius:

$$R_{critical} = \frac{k}{h}$$

Heat exchangers

A heat exchanger is a tool built for efficient heat transfer from one fluid to another, whether the fluids are separated by a solid wall so that they never mix, or the fluids are in direct contact. Heat exchangers are widely used in refrigeration, air conditioning, space heating, power generation, and chemical processing. One common example of a heat exchanger is a car's radiator, in which the hot coolant fluid is cooled by the flow of air over the radiator's surface.

Common types of heat exchanger flows include parallel flow, counter flow, and cross flow. In parallel flow, both fluids move in the same direction while transferring heat; in counter flow, the fluids move in opposite directions; and in cross flow, the fluids move at right angles to each other. Common constructions for heat exchanger include shell and tube, double pipe, extruded finned pipe, spiral fin pipe, u-tube, and stacked plate.

When engineers calculate the theoretical heat transfer in a heat exchanger, they must contend with the fact that the driving temperature difference between the two fluids varies with position. To account for this in simple systems, the log mean temperature difference (LMTD) is often used as an "average" temperature. In more complex systems, direct knowledge of the LMTD is not available, and the number of transfer units (NTU) method can be used instead.

Heat dissipation

A heat sink is a component that transfers heat generated within a solid material to a fluid medium, such as air or a liquid. Examples of heat sinks are the heat exchangers used in refrigeration and air conditioning systems, and the radiator in a car (which is also a heat exchanger). Heat sinks also help to cool electronic and optoelectronic devices such as CPUs, higher-power lasers, and light-emitting diodes (LEDs). A heat sink uses its extended surfaces to increase the surface area in contact with the cooling fluid.

Buildings

In cold climates, houses with their heating systems form dissipative systems. In spite of efforts to insulate houses to reduce heat losses via their exteriors, considerable heat is lost, which can make their interiors uncomfortably cool or cold. For the comfort of the inhabitants, the interiors must be maintained out of thermal equilibrium with the external surroundings. In effect, these domestic residences are oases of warmth in a sea of cold, and the thermal gradient between the inside and outside is often quite steep. This can lead to problems such as condensation and uncomfortable air currents, which—if left unaddressed—can cause cosmetic or structural damage to the property. Such issues can be prevented by use of insulation techniques for reducing heat loss.

Thermal transmittance is the rate of transfer of heat through a structure divided by the difference in temperature across the structure. It is expressed in watts per square meter per kelvin, or W/m^2K. Well-insulated parts of a building have a low thermal transmittance, whereas poorly-insulated parts of a building have a high thermal transmittance.

A thermostat is a device capable of starting the heating system when the house's interior falls below a set temperature, and of stopping that same system when another (higher) set temperature has been achieved. Thus, the thermostat controls the flow of energy into the house, that energy eventually being dissipated to the exterior.

Thermal energy storage

Thermal energy storage refers to technologies that store energy in a thermal reservoir for later use. They can be employed to balance energy demand between daytime and nighttime. The thermal reservoir may be maintained at a temperature above (hotter) or below (colder) than that of the ambient environment. Applications include later use in space heating, domestic or process hot water, or to generate electricity. Most practical active solar heating systems have storage for a few hours to a day's worth of heat collected.

Evaporative cooling

Evaporative cooling is a physical phenomenon in which evaporation of a liquid, typically into surrounding air, cools an object or a liquid in contact with it. Latent heat describes the amount of heat that is needed to evaporate the liquid; this heat comes from the liquid itself and the surrounding gas and surfaces. The greater the difference between the two temperatures, the greater the evaporative cooling effect. When the temperatures are the same, no net evaporation of water in air occurs; thus, there is no cooling effect. A simple example of natural evaporative cooling is perspiration, or sweat, which the body secretes in order to cool itself. An evaporative cooler is a device that cools air through the simple evaporation of water.

Radiative cooling

Radiative cooling is the process by which a body loses heat by radiation. It is an important effect in the Earth's atmosphere. In the case of the Earth-atmosphere system, it refers to the process by which long-wave (infrared) radiation is emitted to balance the absorption of short-wave (visible) energy from the Sun. Convective transport of heat and evaporative transport of latent heat both remove heat from the surface and redistribute it in the atmosphere, making it available for radiative transport at higher altitudes.

Laser cooling

Laser cooling refers to techniques in which atomic and molecular samples are cooled through the interaction with one or more laser light fields. The most common method of laser cooling is Doppler cooling. In Doppler cooling, the frequency of the laser light is tuned slightly below an electronic transition in the atom. Thus, the atoms would absorb more photons if they moved towards the light source, due to the Doppler effect. If an excited atom then emits a photon spontaneously, it will be accelerated. The result of the absorption and emission process is to reduce the speed of the atom. Eventually the mean velocity, and therefore the kinetic energy of the atoms, will be reduced. Since the temperature of an ensemble of atoms is a measure of the random internal kinetic energy, this is equivalent to cooling the atoms.

Sympathetic cooling is a process in which particles of one type cool particles of another type. Typically, atomic ions that can be directly laser-cooled are used to cool nearby ions or atoms. This technique allows cooling of ions and atoms that cannot be laser cooled directly.

Magnetic cooling

Magnetic evaporative cooling is a technique for lowering the temperature of a group of atoms. The process confines atoms using a magnetic field. Over time, individual atoms will become much more energetic than the others due to random collisions, and will escape—removing energy from the system and reducing the temperature of the remaining group. This process is similar to the familiar process by which standing water becomes water vapor.

Other

A heat pipe is a passive device constructed in such a way that it acts as though it has extremely high thermal conductivity. Heat pipes use latent heat and capillary action to move heat, and can carry many times as much heat as a similar-sized copper rod. Originally invented for use in satellites, they have applications in personal computers.

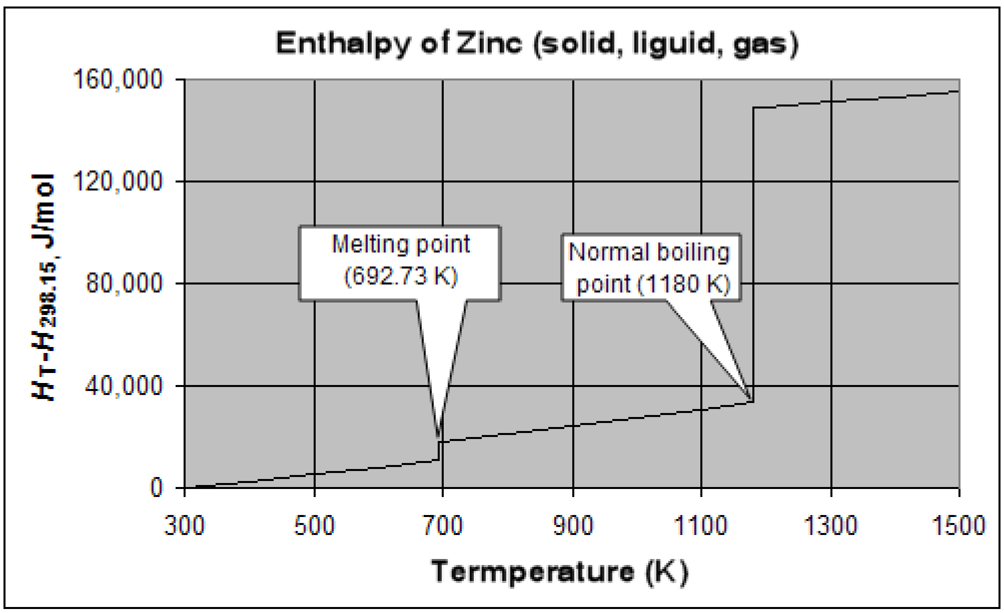
A thermocouple is a junction between two different metals that produces a voltage related to a temperature difference. Thermocouples are a widely used type of temperature sensor for measurement and control, and can also be used to convert heat into electric power.

A thermopile is an electronic device that converts thermal energy into electrical energy. It is composed of thermocouples. Thermopiles do not measure the absolute temperature, but generate an output voltage proportional to a temperature difference. Thermopiles are widely used, e.g., they are the key component of infrared thermometers, such as those used to measure body temperature via the ear.

A thermal diode or thermal rectifier is a device that preferentially passes heat in one direction: a "one-way valve" for heat.

Chapter 12

Enthalpy of Fusion



Molar enthalpy of zinc above 298.15 K and at 1 atm pressure, showing discontinuities at the melting and boiling points. The enthalpy of melting ($\Delta H°m$) of zinc is 7323J/mol and the enthalpy of vaporization ($\Delta H°v$) is 115330J/mol.

The **enthalpy of fusion**, also known as the **heat of fusion** or **specific melting heat**, is the change in enthalpy resulting from the addition or removal of heat from 1 mole of a substance to change its state from a solid to a liquid (melting) or the reverse processes of freezing. It is also called the **latent heat of fusion**, and the temperature at which it occurs is called the melting point.

When thermal energy is withdrawn from a liquid or solid, the temperature falls. When thermal energy is added to a liquid or solid, the temperature rises. However, at the transition point between solid and liquid (the melting point), extra energy is required (the heat of fusion).

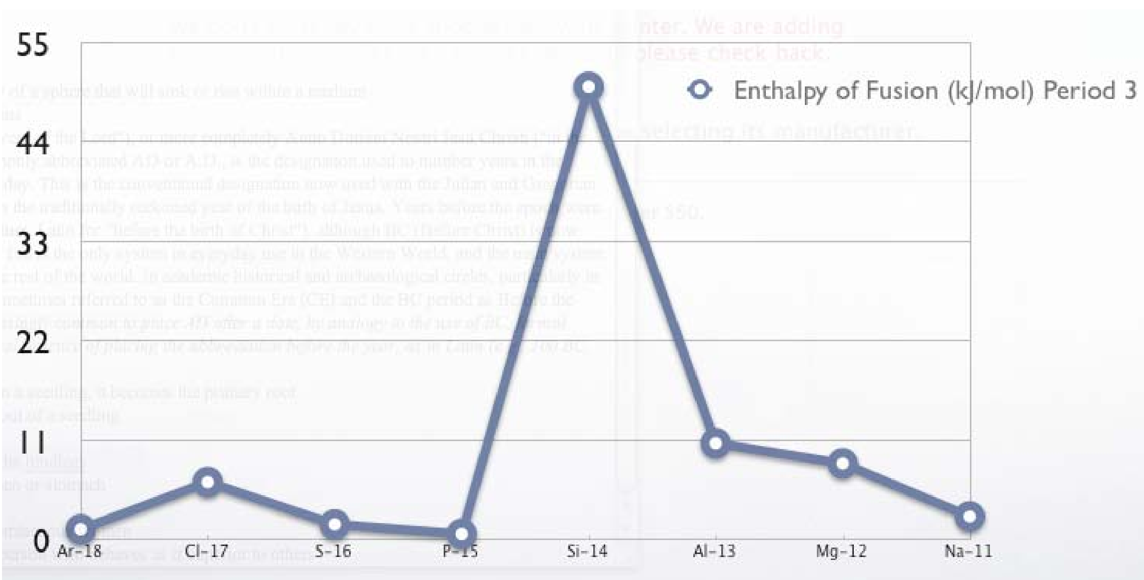
In going from liquid to solid (freezing), the molecules of a substance become arranged in a more ordered state. For them to attain the order of a solid, slightly less heat is withdrawn at the point of crystallization. That not withdrawn heat is stored in the form of primarily potential energy to build the solid lattice. In going from solid to liquid (melting), the molecules of a substance become arranged in a less ordered state. To create the relative disorder from the solid crystal to liquid, slightly more heat is added at the point of decrystallization. That energy from heat is utilized to break the solid lattice. This heat does not result in a temperature change, and is called a *latent* (or hidden) heat.

The heat of fusion can be observed by measuring the temperature of water as it freezes. If a closed container of room temperature water is plunged into a very cold environment (say −20 °C), the temperature will fall steadily until it drops just below the freezing point (0 °C). The temperature then will rebound and hold steady while the water crystallizes. Once the water is completely frozen, its temperature will fall steadily again.

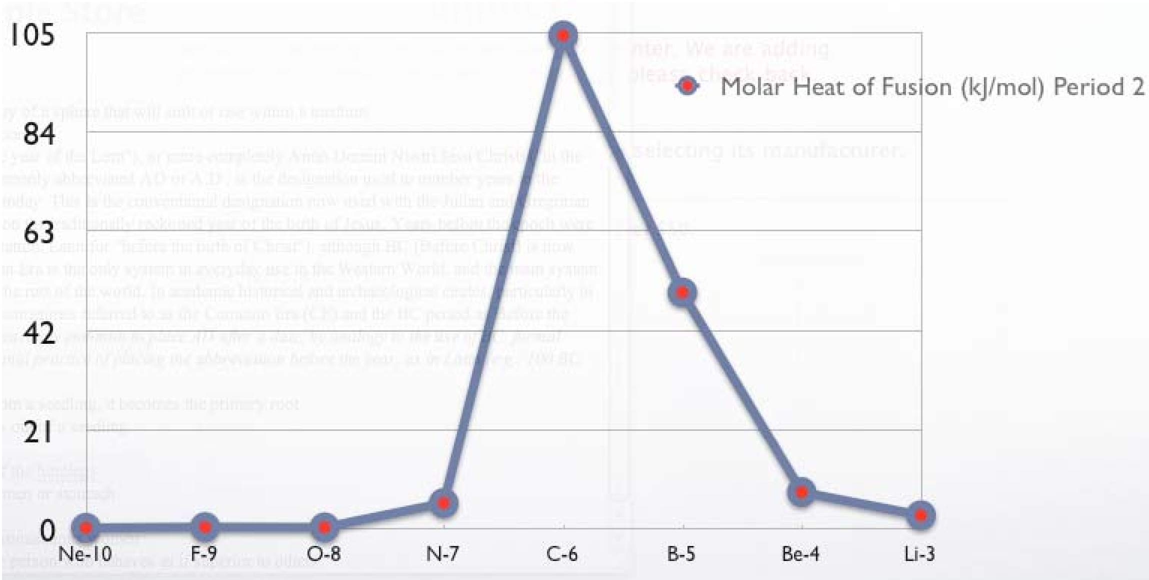
The units of heat of fusion are usually expressed as:

- kilojoules per mole (the SI units)
- calories per gram (old metric units now little used, except for a different, larger calorie used in nutritional contexts)
- British thermal units per pound or Btu per pound-mole

Reference values of common substances



Standard enthalpy change of fusion of period three.



Standard enthalpy change of fusion of period two of the periodic table of elements.

Substance	Heat of fusion (cal/g)	Heat of fusion (kJ/kg)
water	79.72	334
methane	13.96	58.41
ethane	22.73	95.10
propane	19.11	79.96
methanol	23.70	99.16
ethanol	26.05	108.99
glycerol	47.95	200.62
formic acid	66.05	276.35
acetic acid	45.91	192.09
acetone	23.42	97.99
benzene	30.45	127.40
myristic acid	47.49	198.70
palmitic acid	39.18	163.93
stearic acid	47.54	198.91
Paraffin wax ($C_{25}H_{52}$)	47.8-52.6	200–220

These values are from the CRC *Handbook of Chemistry and Physics*, 62nd edition. The conversion between cal/g and kJ/kg in the above table uses the thermochemical calorie (cal_{th}) = 4.184 joules rather than the International Steam Table calorie (cal_{INT}) = 4.1868 joules.

Applications

To heat one kilogram (about 1 litre) of water from 283.15 K to 303.15 K (10 °C to 30 °C) requires 83.6 kJ.
However, to melt ice and raise the resulting water temperature by 20 K requires extra energy. To heat ice from 273.15 K to water at 293.15 K (0 °C to 20 °C) requires:

> (1) 333.55 J/g (heat of fusion of ice) = 333.55 kJ/kg = 333.55 kJ for 1 kg of ice to melt
> PLUS
> (2) 4.18 J/(g·K) = 4.18 kJ/(kg·K) = 83.6 kJ for 1kg of water to go up 20 K
> = 417.15 kJ

Solubility prediction

The heat of fusion can also be used to predict solubility for solids in liquids. Provided an ideal solution is obtained the mole fraction (x_2) of solute at saturation is a function of the heat of fusion, the melting point of the solid (T_{fus}) and the temperature (T) of the solution:

$$\ln x_2 = -\frac{\Delta H^\circ_{fus}}{R}\left(\frac{1}{T} - \frac{1}{T_{fus}}\right)$$

Here, R is the gas constant. For example the solubility of paracetamol in water at 298 K is predicted to be:

$$x_2 = \exp\left(-\frac{28100 \text{ J mol}^{-1}}{8.314 \text{ J K}^{-1} \text{ mol}^{-1}}\left(\frac{1}{298} - \frac{1}{442}\right)\right) = 0.0248$$

This equals to a solubility in grams per liter of:

$$\frac{0.0248 * \frac{1000 \text{ g}}{18.053 \text{ mol}^{-1}}}{1 - 0.0248} * 151.17 \text{ mol}^{-1} = 213.4$$

which is a deviation from the real solubility (240 g/L) of 11%. This error can be reduced when an additional heat capacity parameter is taken into account

Proof

At equilibrium the chemical potentials for the pure solvent and pure solid are identical:

$$\mu^\circ_{solid} = \mu^\circ_{solution}$$

or

$$\mu^\circ_{solid} = \mu^\circ_{liquid} + RT\ln X_2$$

with $R$the gas constant and $T$the temperature.

Rearranging gives:

$$RT\ln X_2 = -(\mu^\circ_{liquid} - \mu^\circ_{solid})$$

and since

$$\Delta G^\circ_{fus} = -(\mu^\circ_{liquid} - \mu^\circ_{solid})$$

the heat of fusion being the difference in chemical potential between the pure liquid and the pure solid, it follows that

$$RT\ln X_2 = -(\Delta G^\circ_{fus})$$

Application of the Gibbs-Helmholtz equation:

$$\left(\frac{\partial(\frac{\Delta G^\circ_{fus}}{T})}{\partial T}\right)_p = \frac{\Delta H^\circ_{fus}}{T^2}$$

ultimately gives:

$$\left(\frac{\partial(\ln X_2)}{\partial T}\right) = \frac{\Delta H^\circ_{fus}}{RT^2}$$

or:

$$\partial\ln X_2 = \frac{\Delta H^\circ_{fus}}{RT^2} * \delta T$$

and with integration:

$$\int_{x2=1}^{x2=x2} \delta\ln X_2 = \ln x_2 = \int_{T_{fus}}^{T} \frac{\Delta H^\circ_{fus}}{RT^2} * \Delta T$$

the end result is obtained:

$$\ln x_2 = -\frac{\Delta H^\circ_{fus}}{R}\left(\frac{1}{T} - \frac{1}{T_{fus}}\right)$$