Mechanical Vibrations

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Table of Contents

Chapter 1

Vibration

Vibration refers to mechanical oscillations about an equilibrium point. The oscillations may be periodic such as the motion of a pendulum or random such as the movement of a tire on a gravel road.

Vibration is occasionally "desirable". For example the motion of a tuning fork, the reed in a woodwind instrument or harmonica, or the cone of a loudspeaker is desirable vibration, necessary for the correct functioning of the various devices.

More often, vibration is undesirable, wasting energy and creating unwanted sound – noise. For example, the vibrational motions of engines, electric motors, or any mechanical device in operation are typically unwanted. Such vibrations can be caused by imbalances in the rotating parts, uneven friction, the meshing of gear teeth, etc. Careful designs usually minimize unwanted vibrations.

The study of sound and vibration are closely related. Sound, or "pressure waves", are generated by vibrating structures (e.g. vocal cords); these pressure waves can also induce the vibration of structures (e.g. ear drum). Hence, when trying to reduce noise it is often a problem in trying to reduce vibration.
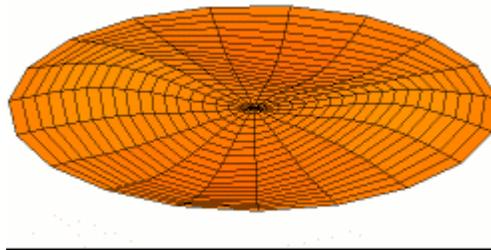
One of the possible modes of vibration of a circular drum.
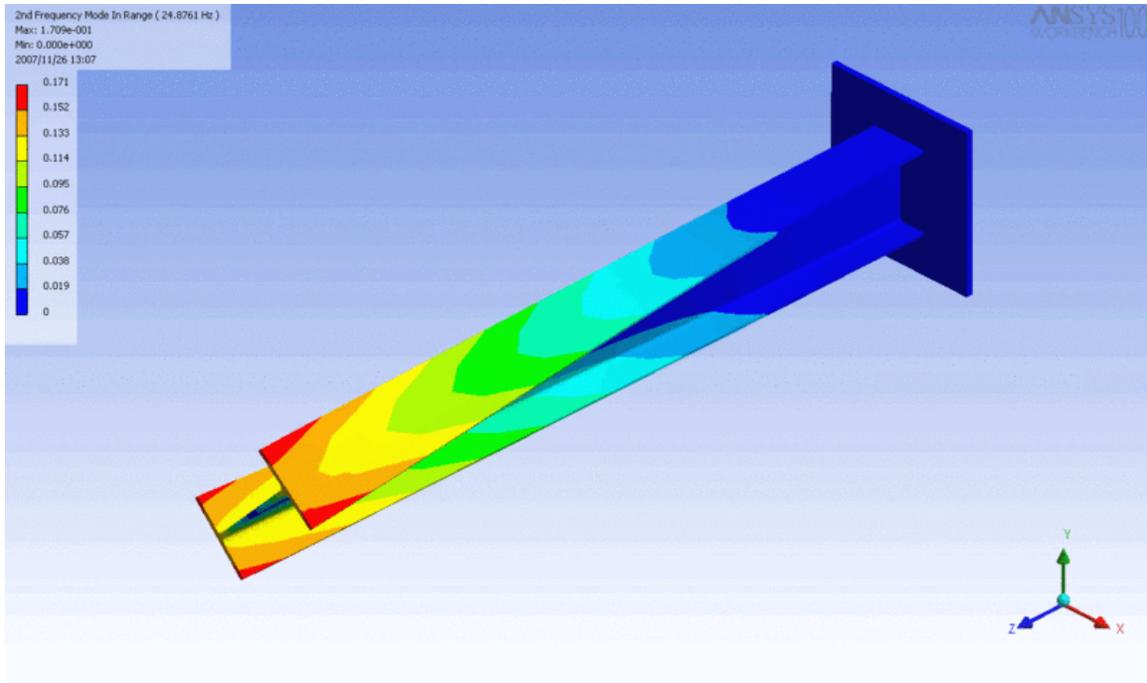
One of the possible modes of vibration of a cantilevered I-beam.

Types of vibration

Free vibration occurs when a mechanical system is set off with an initial input and then allowed to vibrate freely. Examples of this type of vibration are pulling a child back on a swing and then letting go or hitting a tuning fork and letting it ring. The mechanical system will then vibrate at one or more of its "natural frequency" and damp down to zero.

Forced vibration is when an alternating force or motion is applied to a mechanical system. Examples of this type of vibration include a shaking washing machine due to an imbalance, transportation vibration (caused by truck engine, springs, road, etc.), or the vibration of a building during an earthquake. In forced vibration the frequency of the vibration is the frequency of the force or motion applied, with order of magnitude being dependent on the actual mechanical system.

Vibration testing

Vibration testing is accomplished by introducing a forcing function into a structure, usually with some type of shaker. Alternately, a DUT (device under test) is attached to the "table" of a shaker. For relatively low frequency forcing, servohydraulic (electrohydraulic) shakers are used. For higher frequencies, electrodynamic shakers are used. Generally, one or more "input" or "control" points located on the DUT-side of a fixture is kept at a specified acceleration. Other "response" points experience maximum vibration level (resonance) or minimum vibration level (anti-resonance).

Two typical types of vibration tests performed are random- and sine test. Sine (one-frequency-at-a-time) tests are performed to survey the structural response of the device under test (DUT). A random (all frequencies at once) test is generally considered to more closely replicate a real world environment, such as road inputs to a moving automobile.

Most vibration testing is conducted in a single DUT axis at a time, even though most real-world vibration occurs in various axes simultaneously. MIL-STD-810G, released in late 2008, Test Method 527, calls for multiple exciter testing.

Vibration analysis

The fundamentals of vibration analysis can be understood by studying the simple mass–spring–damper model. Indeed, even a complex structure such as an automobile body can be modeled as a "summation" of simple mass–spring–damper models. The mass–spring–damper model is an example of a simple harmonic oscillator. The mathematics used to describe its behavior is identical to other simple harmonic oscillators such as the RLC circuit.

Free vibration without damping
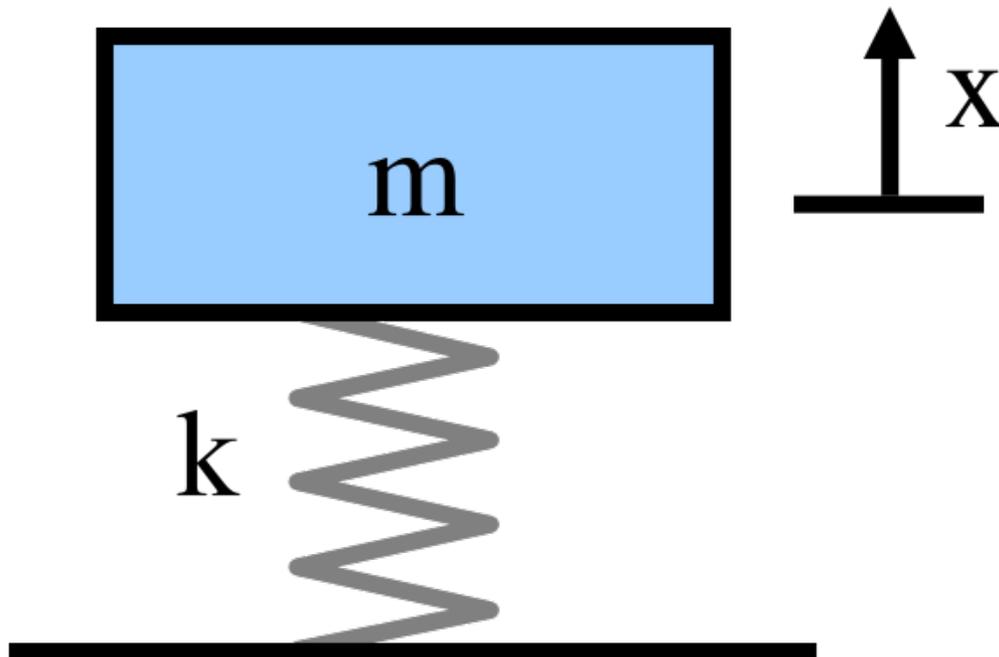
To start the investigation of the mass–spring–damper we will assume the damping is negligible and that there is no external force applied to the mass (i.e. free vibration).

The force applied to the mass by the spring is proportional to the amount the spring is stretched "x" (we will assume the spring is already compressed due to the weight of the mass). The proportionality constant, k, is the stiffness of the spring and has units of force/distance (e.g. lbf/in or N/m)

$$F_s = -kx.$$

The force generated by the mass is proportional to the acceleration of the mass as given by Newton's second law of motion.

$$\Sigma F = ma = m\ddot{x} = m\frac{d^2x}{dt^2}.$$

The sum of the forces on the mass then generates this ordinary differential equation:

$$m\ddot{x} + kx = 0.$$
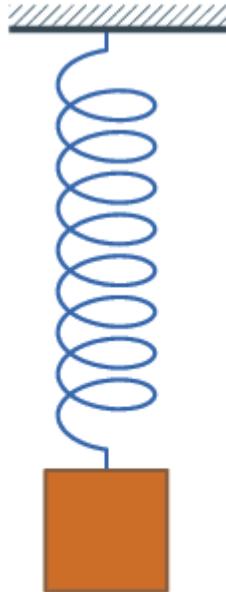
Simple harmonic motion of the mass–spring system

If we assume that we start the system to vibrate by stretching the spring by the distance of A and letting go, the solution to the above equation that describes the motion of mass is:

$$x(t) = A\cos(2\pi f_n t).$$

This solution says that it will oscillate with simple harmonic motion that has an amplitude of A and a frequency of f_n. The number f_n is one of the most important quantities in

vibration analysis and is called the **undamped natural frequency**. For the simple mass–spring system, f_n is defined as:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}}.$$

Note: Angular frequency ω ($\omega = 2\pi f$) with the units of radians per second is often used in equations because it simplifies the equations, but is normally converted to "standard" frequency (units of Hz or equivalently cycles per second) when stating the frequency of a system.

If you know the mass and stiffness of the system you can determine the frequency at which the system will vibrate once it is set in motion by an initial disturbance using the above stated formula. Every vibrating system has one or more natural frequencies that it will vibrate at once it is disturbed. This simple relation can be used to understand in general what will happen to a more complex system once we add mass or stiffness. For example, the above formula explains why when a car or truck is fully loaded the suspension will feel "softer" than unloaded because the mass has increased and therefore reduced the natural frequency of the system.

What causes the system to vibrate: from conservation of energy point of view

Vibrational motion could be understood in terms of conservation of energy. In the above example we have extended the spring by a value of x and therefore have stored some potential energy ($\frac{1}{2}kx^2$) in the spring. Once we let go of the spring, the spring tries to return to its un-stretched state (which is the minimum potential energy state) and in the process accelerates the mass. At the point where the spring has reached its un-stretched state all the potential energy that we supplied by stretching it has been transformed into kinetic energy ($\frac{1}{2}mv^2$). The mass then begins to decelerate because it is now compressing the spring and in the process transferring the kinetic energy back to its potential. Thus oscillation of the spring amounts to the transferring back and forth of the kinetic energy into potential energy.

In our simple model the mass will continue to oscillate forever at the same magnitude, but in a real system there is always something called **damping** that dissipates the energy and therefore the system eventually bringing it to rest.

Free vibration with damping
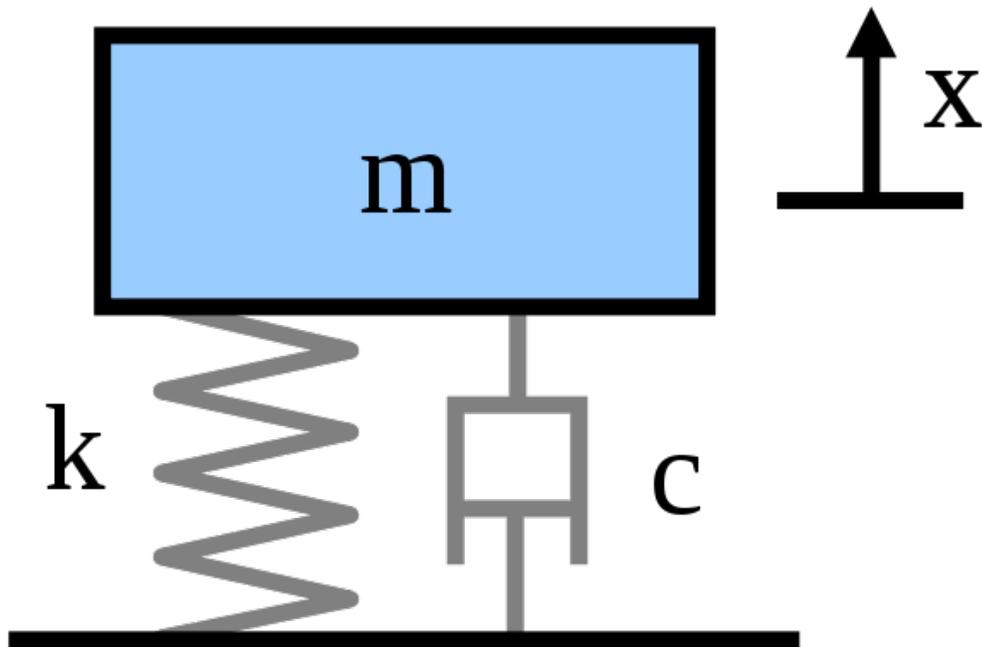
Mass Spring Damper Model

We now add a "viscous" damper to the model that outputs a force that is proportional to the velocity of the mass. The damping is called viscous because it models the effects of an object within a fluid. The proportionality constant c is called the damping coefficient and has units of Force over velocity (lbf s/ in or N s/m).

$$F_d = -cv = -c\dot{x} = -c\frac{dx}{dt}.$$

By summing the forces on the mass we get the following ordinary differential equation:

$$m\ddot{x} + c\dot{x} + kx = 0.$$

The solution to this equation depends on the amount of damping. If the damping is small enough the system will still vibrate, but eventually, over time, will stop vibrating. This case is called underdamping – this case is of most interest in vibration analysis. If we increase the damping just to the point where the system no longer oscillates we reach the point of **critical damping** (if the damping is increased past critical damping the system is

called overdamped). The value that the damping coefficient needs to reach for critical damping in the mass spring damper model is:

$$c_c = 2\sqrt{km}.$$

To characterize the amount of damping in a system a ratio called the damping ratio (also known as damping factor and % critical damping) is used. This damping ratio is just a ratio of the actual damping over the amount of damping required to reach critical damping. The formula for the damping ratio (ζ) of the mass spring damper model is:

$$\zeta = \frac{c}{2\sqrt{km}}.$$

For example, metal structures (e.g. airplane fuselage, engine crankshaft) will have damping factors less than 0.05 while automotive suspensions in the range of 0.2–0.3.

The solution to the underdamped system for the mass spring damper model is the following:

$$x(t) = Xe^{-\zeta \omega_n t} \cos(\sqrt{1 - \zeta^2}\omega_n t - \phi), \qquad \omega_n = 2\pi f_n.$$

$$f_n = 1$$
$$\omega_n = 2\pi f_n$$
$$\zeta = 0.1$$

$$Xe^{-\zeta \omega_n t}$$

Time →

$$f_n = 1$$
$$\omega_n = 2\pi f_n$$
$$\zeta = 0.3$$

$$Xe^{-\zeta \omega_n t}$$

Time →
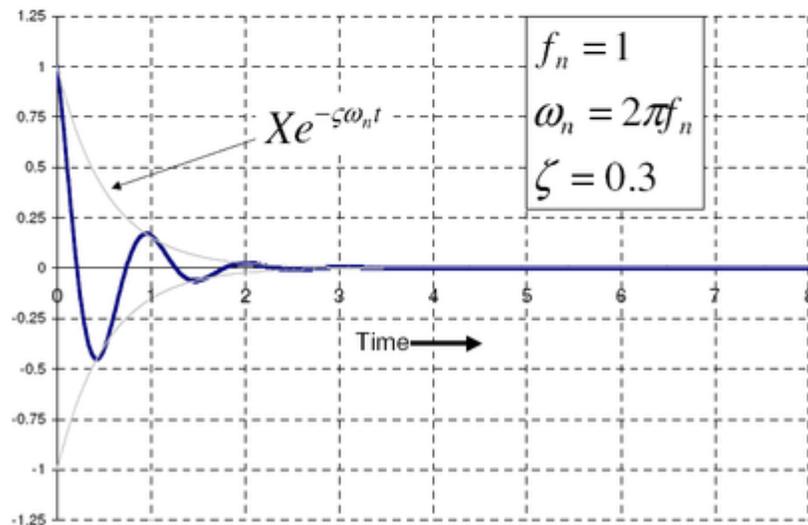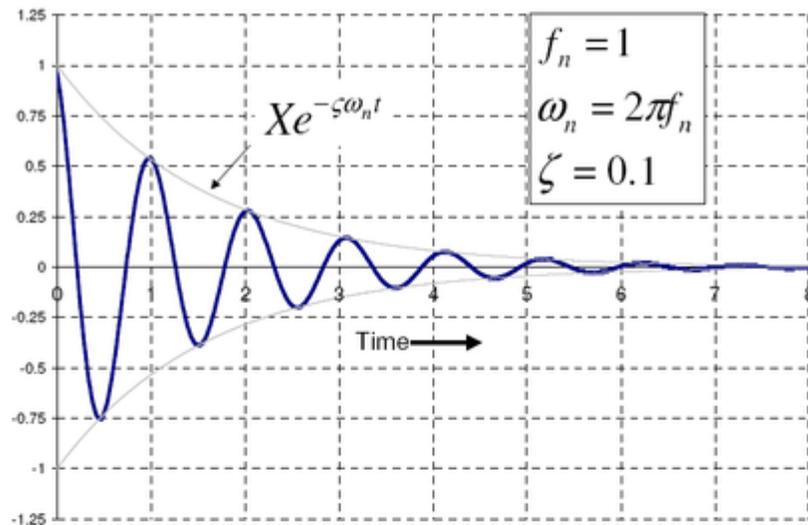
The value of X, the initial magnitude, and φ, the phase shift, are determined by the amount the spring is stretched. The formulas for these values can be found in the references.

Damped and undamped natural frequencies

The major points to note from the solution are the exponential term and the cosine function. The exponential term defines how quickly the system "damps" down – the larger the damping ratio, the quicker it damps to zero. The cosine function is the oscillating portion of the solution, but the frequency of the oscillations is different from the undamped case.

The frequency in this case is called the "damped natural frequency", f_d, and is related to the undamped natural frequency by the following formula:

$$f_d = \sqrt{1 - \zeta^2} f_n.$$

The damped natural frequency is less than the undamped natural frequency, but for many practical cases the damping ratio is relatively small and hence the difference is negligible. Therefore the damped and undamped description are often dropped when stating the natural frequency (e.g. with 0.1 damping ratio, the damped natural frequency is only 1% less than the undamped).

The plots to the side present how 0.1 and 0.3 damping ratios effect how the system will "ring" down over time. What is often done in practice is to experimentally measure the free vibration after an impact (for example by a hammer) and then determine the natural frequency of the system by measuring the rate of oscillation as well as the damping ratio by measuring the rate of decay. The natural frequency and damping ratio are not only important in free vibration, but also characterize how a system will behave under forced vibration.

Forced vibration with damping

Here we will see the behavior of the spring mass damper model when we add a harmonic force in the form below. A force of this type could, for example, be generated by a rotating imbalance.

$$F = F_0 \cos(2\pi ft).$$

If we again sum the forces on the mass we get the following ordinary differential equation:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos(2\pi ft).$$

The steady state solution of this problem can be written as:

$$x(t) = X \cos(2\pi ft - \phi).$$

The result states that the mass will oscillate at the same frequency, f, of the applied force, but with a phase shift φ.

The amplitude of the vibration "X" is defined by the following formula.
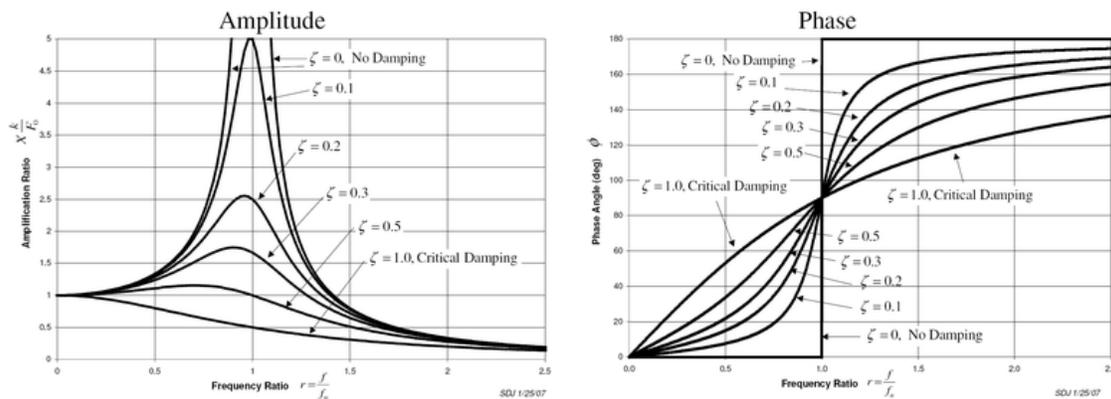
$$X = \frac{F_0}{k} \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}.$$

Where "r" is defined as the ratio of the harmonic force frequency over the undamped natural frequency of the mass–spring–damper model.

$$r = \frac{f}{f_n}.$$

The phase shift , φ, is defined by the following formula.

$$\phi = \arctan\left(\frac{2\zeta r}{1 - r^2}\right).$$

The plot of these functions, called "the frequency response of the system", presents one of the most important features in forced vibration. In a lightly damped system when the forcing frequency nears the natural frequency ($r \approx 1$) the amplitude of the vibration can get extremely high. This phenomenon is called **resonance** (subsequently the natural frequency of a system is often referred to as the resonant frequency). In rotor bearing systems any rotational speed that excites a resonant frequency is referred to as a critical speed.

If resonance occurs in a mechanical system it can be very harmful – leading to eventual failure of the system. Consequently, one of the major reasons for vibration analysis is to predict when this type of resonance may occur and then to determine what steps to take to prevent it from occurring. As the amplitude plot shows, adding damping can significantly reduce the magnitude of the vibration. Also, the magnitude can be reduced if the natural frequency can be shifted away from the forcing frequency by changing the stiffness or mass of the system. If the system cannot be changed, perhaps the forcing frequency can be shifted (for example, changing the speed of the machine generating the force).

The following are some other points in regards to the forced vibration shown in the frequency response plots.

- At a given frequency ratio, the amplitude of the vibration, X, is directly proportional to the amplitude of the force F_0 (e.g. if you double the force, the vibration doubles)
- With little or no damping, the vibration is in phase with the forcing frequency when the frequency ratio $r < 1$ and 180 degrees out of phase when the frequency ratio $r > 1$
- When $r \ll 1$ the amplitude is just the deflection of the spring under the static force F_0. This deflection is called the static deflection δ_{st}. Hence, when $r \ll 1$ the effects of the damper and the mass are minimal.
- When $r \gg 1$ the amplitude of the vibration is actually less than the static deflection δ_{st}. In this region the force generated by the mass ($F = ma$) is dominating because the acceleration seen by the mass increases with the frequency. Since the deflection seen in the spring, X, is reduced in this region, the force transmitted by the spring ($F = kx$) to the base is reduced. Therefore the mass–spring–damper system is isolating the harmonic force from the mounting base – referred to as vibration isolation. Interestingly, more damping actually reduces the effects of vibration isolation when $r \gg 1$ because the damping force ($F = cv$) is also transmitted to the base.

What causes resonance?

Resonance is simple to understand if you view the spring and mass as energy storage elements – with the mass storing kinetic energy and the spring storing potential energy. As discussed earlier, when the mass and spring have no force acting on them they transfer energy back and forth at a rate equal to the natural frequency. In other words, if energy is to be efficiently pumped into both the mass and spring the energy source needs to feed the energy in at a rate equal to the natural frequency. Applying a force to the mass and spring is similar to pushing a child on swing, you need to push at the correct moment if you want the swing to get higher and higher. As in the case of the swing, the force applied does not necessarily have to be high to get large motions; the pushes just need to keep adding energy into the system.

The damper, instead of storing energy, dissipates energy. Since the damping force is proportional to the velocity, the more the motion, the more the damper dissipates the energy. Therefore a point will come when the energy dissipated by the damper will equal the energy being fed in by the force. At this point, the system has reached its maximum amplitude and will continue to vibrate at this level as long as the force applied stays the same. If no damping exists, there is nothing to dissipate the energy and therefore theoretically the motion will continue to grow on into infinity.

Applying "complex" forces to the mass–spring–damper model

In a previous section only a simple harmonic force was applied to the model, but this can be extended considerably using two powerful mathematical tools. The first is the Fourier transform that takes a signal as a function of time (time domain) and breaks it down into its harmonic components as a function of frequency (frequency domain). For example, let

us apply a force to the mass–spring–damper model that repeats the following cycle – a force equal to 1 newton for 0.5 second and then no force for 0.5 second. This type of force has the shape of a 1 Hz square wave.

Square Wave: Generated by Harmonics

Frequency Spectrum: Square Wave
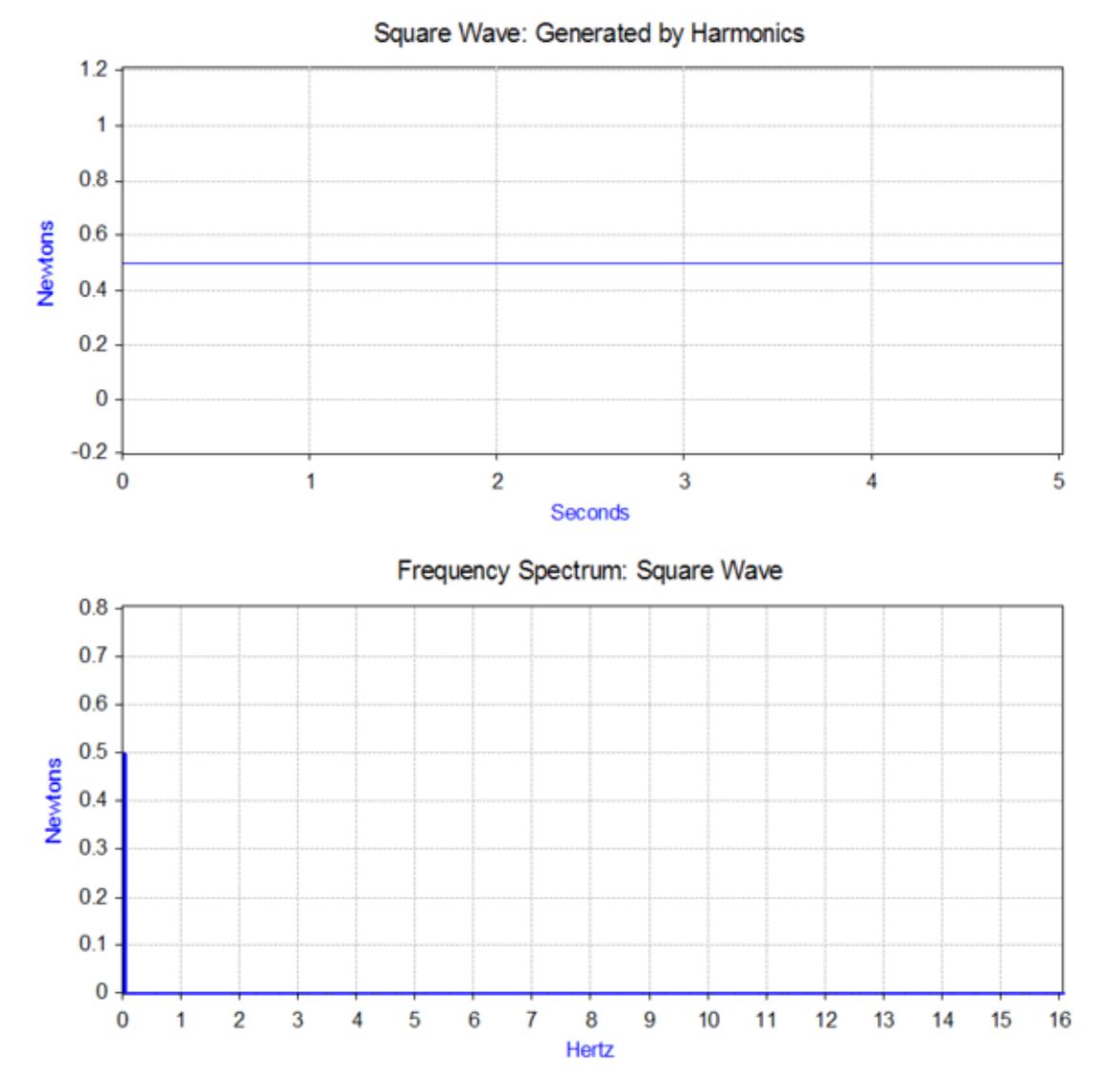
How a 1 Hz square wave can be represented as a summation of sine waves(harmonics) and the corresponding frequency spectrum.

The Fourier transform of the square wave generates a frequency spectrum that presents the magnitude of the harmonics that make up the square wave (the phase is also generated, but is typically of less concern and therefore is often not plotted). The Fourier transform can also be used to analyze non-periodic functions such as transients (e.g. impulses) and random functions. With the advent of the modern computer the Fourier transform is almost always computed using the Fast Fourier Transform (FFT) computer algorithm in combination with a window function.

In the case of our square wave force, the first component is actually a constant force of 0.5 newton and is represented by a value at "0" Hz in the frequency spectrum. The next component is a 1 Hz sine wave with an amplitude of 0.64. This is shown by the line at 1 Hz. The remaining components are at odd frequencies and it takes an infinite amount of sine waves to generate the perfect square wave. Hence, the Fourier transform allows you to interpret the force as a sum of sinusoidal forces being applied instead of a more "complex" force (e.g. a square wave).

In the previous section, the vibration solution was given for a single harmonic force, but the Fourier transform will in general give multiple harmonic forces. The second mathematical tool, "the principle of superposition", allows you to sum the solutions from multiple forces if the system is linear. In the case of the spring–mass–damper model, the system is linear if the spring force is proportional to the displacement and the damping is proportional to the velocity over the range of motion of interest. Hence, the solution to the problem with a square wave is summing the predicted vibration from each one of the harmonic forces found in the frequency spectrum of the square wave.

Frequency response model

We can view the solution of a vibration problem as an input/output relation – where the force is the input and the output is the vibration. If we represent the force and vibration in the frequency domain (magnitude and phase) we can write the following relation:

$$X(\omega) = H(\omega) \cdot F(\omega) \;\; or \;\; H(\omega) = \frac{X(\omega)}{F(\omega)}.$$

$H(\omega)$ is called the frequency response function (also referred to as the transfer function, but not technically as accurate) and has both a magnitude and phase component (if represented as a complex number, a real and imaginary component). The magnitude of the frequency response function (FRF) was presented earlier for the mass–spring–damper system.
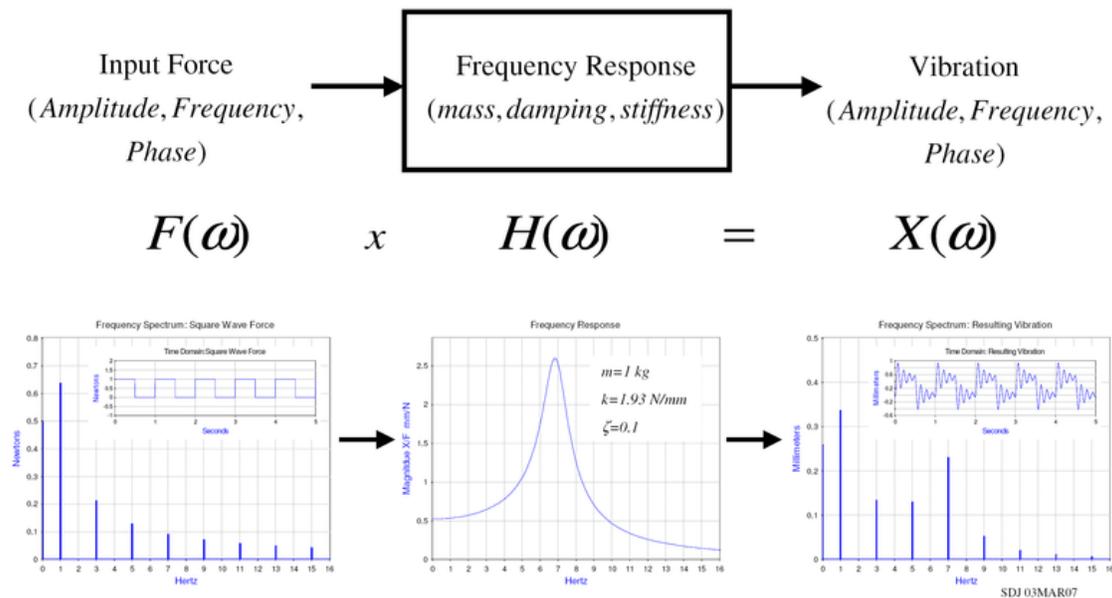
$$|H(\omega)| = \left|\frac{X(\omega)}{F(\omega)}\right| = \frac{1}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}, \;\; \text{where}$$

$$r = \frac{f}{f_n} = \frac{\omega}{\omega_n}.$$

The phase of the FRF was also presented earlier as:

$$\angle H(\omega) = \arctan\left(\frac{2\zeta r}{1 - r^2}\right).$$

For example, let us calculate the FRF for a mass–spring–damper system with a mass of 1 kg, spring stiffness of 1.93 N/mm and a damping ratio of 0.1. The values of the spring and mass give a natural frequency of 7 Hz for this specific system. If we apply the 1 Hz square wave from earlier we can calculate the predicted vibration of the mass. The figure illustrates the resulting vibration. It happens in this example that the fourth harmonic of the square wave falls at 7 Hz. The frequency response of the mass–spring–damper therefore outputs a high 7 Hz vibration even though the input force had a relatively low 7 Hz harmonic. This example highlights that the resulting vibration is dependent on both the forcing function and the system that the force is applied to.

Input Force	Frequency Response	Vibration
(*Amplitude, Frequency, Phase*)	(*mass, damping, stiffness*)	(*Amplitude, Frequency, Phase*)

$$F(\omega) \quad x \quad H(\omega) \quad = \quad X(\omega)$$

Frequency response model

The figure also shows the time domain representation of the resulting vibration. This is done by performing an inverse Fourier Transform that converts frequency domain data to time domain. In practice, this is rarely done because the frequency spectrum provides all the necessary information.

The frequency response function (FRF) does not necessarily have to be calculated from the knowledge of the mass, damping, and stiffness of the system, but can be measured experimentally. For example, if you apply a known force and sweep the frequency and then measure the resulting vibration you can calculate the frequency response function and then characterize the system. This technique is used in the field of experimental modal analysis to determine the vibration characteristics of a structure.

Multiple degrees of freedom systems and mode shapes

The simple mass–spring damper model is the foundation of vibration analysis, but what about more complex systems? The mass–spring–damper model described above is called

a single degree of freedom (SDOF) model since we have assumed the mass only moves up and down. In the case of more complex systems we need to discretize the system into more masses and allow them to move in more than one direction – adding degrees of freedom. The major concepts of multiple degrees of freedom (MDOF) can be understood by looking at just a 2 degree of freedom model as shown in the figure.
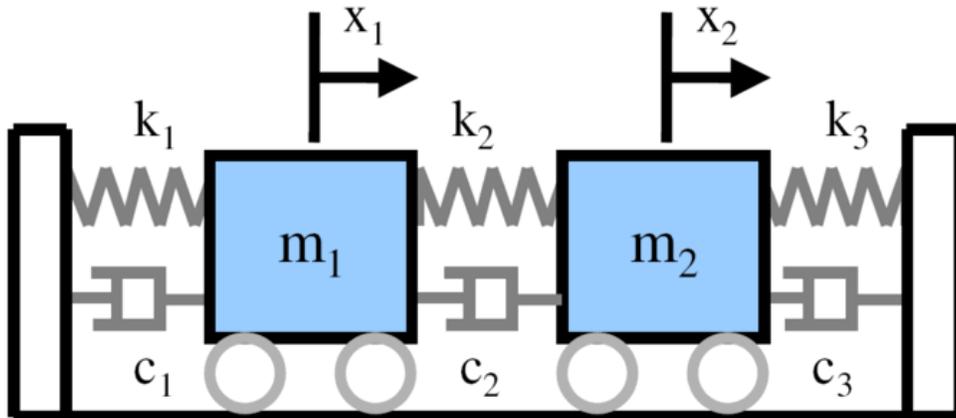
2 degree of freedom model

The equations of motion of the 2DOF system are found to be:

$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1,$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + (c_2 + c_3)\dot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = f_2.$$

We can rewrite this in matrix format:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}.$$

A more compact form of this matrix equation can be written as:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f\},$$

where $[M], [C],$ and $[K]$ are symmetric matrices referred respectively as the mass, damping, and stiffness matrices. The matrices are NxN square matrices where N is the number of degrees of freedom of the system.

In the following analysis we will consider the case where there is no damping and no applied forces (i.e. free vibration). The solution of a viscously damped system is somewhat more complicated.

$$[M]\{\ddot{x}\} + [K]\{x\} = 0.$$

This differential equation can be solved by assuming the following type of solution:

$$\{x\} = \{X\}e^{i\omega t}.$$

Note: Using the exponential solution of $\{X\}e^{i\omega t}$ is a mathematical trick used to solve linear differential equations. If we use Euler's formula and take only the real part of the solution it is the same cosine solution for the 1 DOF system. The exponential solution is only used because it easier to manipulate mathematically.

The equation then becomes:

$$[-\omega^2[M] + [K]]\{X\}e^{i\omega t} = 0.$$

Since $e^{i\omega t}$ cannot equal zero the equation reduces to the following.

$$[[K] - \omega^2[M]]\{X\} = 0.$$

Eigenvalue problem

This is referred to an eigenvalue problem in mathematics and can be put in the standard format by pre-multiplying the equation by $[M]^{-1}$

$$\left[[M]^{-1}[K] - \omega^2[M]^{-1}[M]\right]\{X\} = 0$$

and if we let $[M]^{-1}[K] = [A]$ and $\lambda = \omega^2$

$$[[A] - \lambda[I]]\{X\} = 0.$$

The solution to the problem results in N **eigenvalues** (i.e. $\omega_1^2, \omega_2^2, \cdots \omega_N^2$), where N corresponds to the number of degrees of freedom. The eigenvalues provide the natural frequencies of the system. When these eigenvalues are substituted back into the original set of equations, the values of $\{X\}$ that correspond to each eigenvalue are called the **eigenvectors**. These eigenvectors represent the mode shapes of the system. The solution of an eigenvalue problem can be quite cumbersome (especially for problems with many

degrees of freedom), but fortunately most math analysis programs have eigenvalue routines.

The eigenvalues and eigenvectors are often written in the following matrix format and describe the modal model of the system:

$$[\backslash\omega_r^2\backslash] = \begin{bmatrix} \omega_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \omega_N^2 \end{bmatrix}_{and} [\Psi] = [\{\psi_1\}\{\psi_2\} \cdots \{\psi_N\}].$$

A simple example using our 2 DOF model can help illustrate the concepts. Let both masses have a mass of 1 kg and the stiffness of all three springs equal 1000 N/m. The mass and stiffness matrix for this problem are then:

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{and} [K] = \begin{bmatrix} 2000 & -1000 \\ -1000 & 2000 \end{bmatrix}.$$

Then $[A] = \begin{bmatrix} 2000 & -1000 \\ -1000 & 2000 \end{bmatrix}.$

The eigenvalues for this problem given by an eigenvalue routine will be:

$$[\backslash\omega_r^2\backslash] = \begin{bmatrix} 1000 & 0 \\ 0 & 3000 \end{bmatrix}.$$

The natural frequencies in the units of hertz are then (remembering $\omega=2\pi f$) $f_1=5.033$ Hz and $f_2=8.717$ Hz.

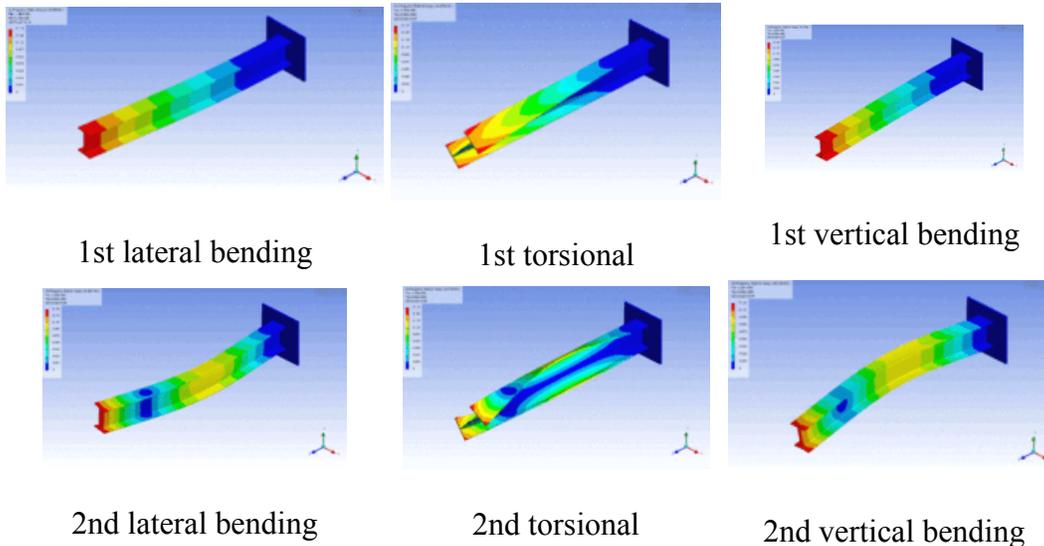The two mode shapes for the respective natural frequencies are given as:

$$[\Psi] = [\{\psi_1\}\{\psi_2\}] = \left[\left\{ \begin{matrix} -0.707 \\ -0.707 \end{matrix} \right\}_1 \left\{ \begin{matrix} 0.707 \\ -0.707 \end{matrix} \right\}_2 \right].$$

Since the system is a 2 DOF system, there are two modes with their respective natural frequencies and shapes. The mode shape vectors are not the absolute motion, but just describe relative motion of the degrees of freedom. In our case the first mode shape vector is saying that the masses are moving together in phase since they have the same value and sign. In the case of the second mode shape vector, each mass is moving in opposite direction at the same rate.

Illustration of a multiple DOF problem

When there are many degrees of freedom, the best method of visualizing the mode shapes is by animating them. An example of animated mode shapes is shown in the figure below for a cantilevered I-beam. In this case, a finite element model was used to generate the mass and stiffness matrices and solve the eigenvalue problem. Even this relatively simple model has over 100 degrees of freedom and hence as many natural frequencies and mode shapes. In general only the first few modes are important.

The mode shapes of a cantilevered I-beam

1st lateral bending   1st torsional   1st vertical bending

2nd lateral bending   2nd torsional   2nd vertical bending

Multiple DOF problem converted to a single DOF problem

The eigenvectors have very important properties called orthogonality properties. These properties can be used to greatly simplify the solution of multi-degree of freedom models. It can be shown that the eigenvectors have the following properties:

$$[\Psi]^T [M] [\Psi] = [\,\diagdown m_r \diagdown\,],$$
$$[\Psi]^T [K] [\Psi] = [\,\diagdown k_r \diagdown\,].$$

$[\,\diagdown m_r \diagdown\,]$ and $[\,\diagdown k_r \diagdown\,]$ are diagonal matrices that contain the modal mass and stiffness values for each one of the modes. (Note: Since the eigenvectors (mode shapes) can be arbitrarily scaled, the orthogonality properties are often used to scale the eigenvectors so the modal mass value for each mode is equal to 1. The modal mass matrix is therefore an identity matrix)

These properties can be used to greatly simplify the solution of multi-degree of freedom models by making the following coordinate transformation.

$$\{x\} = [\Psi]\{q\}.$$

If we use this coordinate transformation in our original free vibration differential equation we get the following equation.

$$[M][\Psi]\{\ddot{q}\} + [K][\Psi]\{q\} = 0.$$

We can take advantage of the orthogonality properties by premultiplying this equation by $[\Psi]^T$

$$[\Psi]^T[M][\Psi]\{\ddot{q}\} + [\Psi]^T[K][\Psi]\{q\} = 0.$$

The orthogonality properties then simplify this equation to:

$$[\diagdown m_r \diagdown]\{\ddot{q}\} + [\diagdown k_r \diagdown]\{q\} = 0.$$

This equation is the foundation of vibration analysis for multiple degree of freedom systems. A similar type of result can be derived for damped systems. The key is that the modal and stiffness matrices are diagonal matrices and therefore we have "decoupled" the equations. In other words, we have transformed our problem from a large unwieldy multiple degree of freedom problem into many single degree of freedom problems that can be solved using the same methods outlined above.

Instead of solving for x we are instead solving for q, referred to as the modal coordinates or modal participation factors.

It may be clearer to understand if we write $\{x\} = [\Psi]\{q\}$ as:

$$\{x_n\} = q_1\{\psi\}_1 + q_2\{\psi\}_2 + q_3\{\psi\}_3 + \cdots + q_N\{\psi\}_N.$$

Written in this form we can see that the vibration at each of the degrees of freedom is just a linear sum of the mode shapes. Furthermore, how much each mode "participates" in the final vibration is defined by q, its modal participation factor.

Chapter 2

Molecular Vibration

A **molecular vibration** occurs when atoms in a molecule are in periodic motion while the molecule as a whole has constant translational and rotational motion. The frequency of the periodic motion is known as a vibration frequency. In general, a molecule with N atoms has 3N-6 normal modes of vibration but *linear* molecules have only 3N-5 normal modes of vibration as rotation about its molecular axis cannot be observed.. A diatomic molecule has one normal mode of vibration. The normal modes of vibration of polyatomic molecules are independent of each other but each normal mode will involve simultaneous vibrations of different parts of the molecule such as different chemical bonds.

A molecular vibration is excited when the molecule absorbs a quantum of energy, E, corresponding to the vibration's frequency, ν, according to the relation $E=h\nu$, where h is Planck's constant. A fundamental vibration is excited when one such quantum of energy is absorbed by the molecule in its ground state. When two quanta are absorbed the first overtone is excited, and so on to higher overtones.

To a first approximation, the motion in a normal vibration can be described as a kind of simple harmonic motion. In this approximation, the vibrational energy is a quadratic function (parabola) with respect to the atomic displacements and the first overtone has twice the frequency of the fundamental. In reality, vibrations are anharmonic and the first overtone has a frequency that is slightly lower than twice that of the fundamental. Excitation of the higher overtones involves progressively less and less additional energy and eventually leads to dissociation of the molecule, as the potential energy of the molecule is more like a Morse potential.

The vibrational states of a molecule can be probed in a variety of ways. The most direct way is through infrared spectroscopy, as vibrational transitions typically require an amount of energy that corresponds to the infrared region of the spectrum. Raman

spectroscopy, which typically uses visible light, can also be used to measure vibration frequencies directly.

Vibrational excitation can occur in conjunction with electronic excitation (vibronic transition), giving vibrational fine structure to electronic transitions, particularly with molecules in the gas state.

Simultaneous excitation of a vibration and rotations gives rise to vibration-rotation spectra.

Vibrational coordinates

The coordinate of a normal vibration is a combination of *changes* in the positions of atoms in the molecule. When the vibration is excited the coordinate changes sinusoidally with a frequency ν, the frequency of the vibration.

Internal coordinates

Internal coordinates are of the following types, illustrated with reference to the planar molecule ethylene,
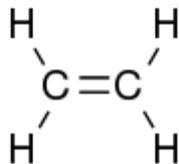

- Stretching: a change in the length of a bond, such as C-H or C-C
- Bending: a change in the angle between two bonds, such as the HCH angle in a methylene group
- Rocking: a change in angle between a group of atoms, such as a methylene group and the rest of the molecule.
- Wagging: a change in angle between the plane of a group of atoms, such as a methylene group and a plane through the rest of the molecule,
- Twisting: a change in the angle between the planes of two groups of atoms, such as a change in the angle between the two methylene groups.
- Out-of-plane: a change in the angle between any one of the C-H bonds and the plane defined by the remaining atoms of the ethylene molecule. Another example is in BF_3 when the boron atom moves in and out of the plane of the three fluorine atoms.

In a rocking, wagging or twisting coordinate the bond lengths within the groups involved do not change. The angles do. Rocking is distinguished from wagging by the fact that the atoms in the group stay in the same plane.

In ethene there are 12 internal coordinates: 4 C-H stretching, 1 C-C stretching, 2 H-C-H bending, 2 CH_2 rocking, 2 CH_2 wagging, 1 twisting. Note that the H-C-C angles cannot be used as internal coordinates as the angles at each carbon atom cannot all increase at the same time.

Vibrations of a Methylene group (-CH₂-) in a molecule for illustration

The atoms in a CH_2 group, commonly found in organic compounds, can vibrate in six different ways: **symmetric and antisymmetric stretching**, **scissoring**, **rocking**, **wagging** and **twisting** as shown here:

|Symmetrical stretching|Antisymmetrical stretching|Scissoring|
|Rocking|Wagging|Twisting|
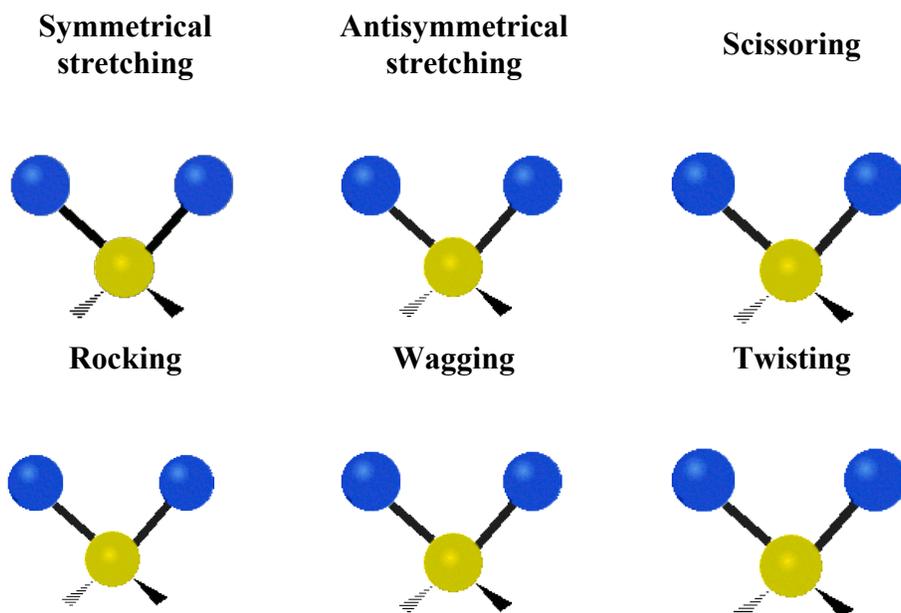
(These figures do not represent the "recoil" of the C atoms, which, though necessarily present to balance the overall movements of the molecule, are much smaller than the movements of the lighter H atoms).

Symmetry-adapted coordinates

Symmetry-adapted coordinates may be created by applying a projection operator to a set of internal coordinates. The projection operator is constructed with the aid of the character table of the molecular point group. For example, the four(un-normalised) C-H stretching coordinates of the molecule ethene are given by

$$Q_{s1} = q_1 + q_2 + q_3 + q_4$$
$$Q_{s2} = q_1 + q_2 - q_3 - q_4$$
$$Q_{s3} = q_1 - q_2 + q_3 - q_4$$
$$Q_{s4} = q_1 - q_2 - q_3 + q_4$$

where q_1 - q_4 are the internal coordinates for stretching of each of the four C-H bonds.

Illustrations of symmetry-adapted coordinates for most small molecules can be found in Nakamoto.

Normal coordinates

The normal coordinates, denoted as Q, refer to the positions of atoms away from their equilibrium positions, with respect to a normal mode of vibration. Each normal mode is assigned a single normal coordinate, and so the normal coordinate refers to the "progress" along that normal mode at any given time. Formally, normal modes are determined by solving a secular determinant, and then the normal coordinates (over the normal modes) can be expressed as a summation over the cartesian cordinates (over the atom positions). The advantage of working in normal modes is that they diagonalize the matrix governing the molecular vibrations, so each normal mode is an independent molecular vibration, associated with its own spectrum of quantum mechanical states. If the molecule possesses symmetries, it will belong to a point group, and the normal modes will "transform as" an irreducible representation under that group. The normal modes can then be qualitatively determined by applying group theory and projecting the irreducible representation onto the cartesian coordinates. For example, when this treatment is applied to CO_2, it is found that the C=O stretches are not independent, but rather there is a O=C=O symmetric stretch and an O=C=O asymmetric stretch.

- symmetric stretching: the sum of the two C-O stretching coordinates; the two C-O bond lengths change by the same amount and the carbon atom is stationary. $Q = q_1 + q_2$
- asymmetric stretching: the difference of the two C-O stretching coordinates; one C-O bond length increases while the other decreases. $Q = q_1 - q_2$

When two or more normal coordinates belong to the same irreducible representation of the molecular point group (colloquially, have the same symmetry) there is "mixing" and the coefficients of the combination cannot be determined *a priori*. For example, in the linear molecule hydrogen cyanide, HCN, The two stretching vibrations are

1. principally C-H stretching with a little C-N stretching; $Q_1 = q_1 + a\, q_2$ ($a \ll 1$)
2. principally C-N stretching with a little C-H stretching; $Q_2 = b\, q_1 + q_2$ ($b \ll 1$)

The coefficients a and b are found by performing a full normal coordinate analysis by means of the Wilson GF method.

Newtonian mechanics

Perhaps surprisingly, molecular vibrations can be treated using Newtonian mechanics to calculate the correct vibration frequencies. The basic assumption is that each vibration can be treated as though it corresponds to a spring. In the harmonic approximation the spring obeys Hooke's law: the force required to extend the spring is proportional to the extension. The proportionality constant is known as a *force constant, k*. The anharmonic oscillator is considered elsewhere.

$$Force = -kQ$$

By Newton's second law of motion this force is also equal to a reduced mass, μ, times acceleration.

$$Force = \mu \frac{d^2 Q}{dt^2}$$

Since this is one and the same force the ordinary differential equation follows.

$$\mu \frac{d^2 Q}{dt^2} + kQ = 0$$

The solution to this equation of simple harmonic motion is

$$Q(t) = A\cos(2\pi\nu t); \quad \nu = \frac{1}{2\pi}\sqrt{\frac{k}{\mu}}.$$

A is the maximum amplitude of the vibration coordinate Q. It remains to define the reduced mass, μ. In general, the reduced mass of a diatomic molecule, AB, is is expressed in terms of the atomic masses, m_A and m_B, as

$$\frac{1}{\mu} = \frac{1}{m_A} + \frac{1}{m_B}.$$

The use of the reduced mass ensures that the centre of mass of the molecule is not affected by the vibration. In the harmonic approximation the potential energy of the molecule is a quadratic function of the normal coordinate. It follows that the force-constant is equal to the second derivative of the potential energy.

$$k = \frac{\partial^2 V}{\partial Q^2}$$

When two or more normal vibrations have the same symmetry a full normal coordinate analysis must be performed. The vibration frequencies,ν_i are obtained from the eigenvalues,λ_i, of the matrix product GF. G is a matrix of numbers derived from the masses of the atoms and the geometry of the molecule. F is a matrix derived from force-constant values. Details concerning the determination of the eigenvalues can be found in .

Quantum mechanics

In the harmonic approximation the potential energy is a quadratic function of the normal coordinates. Solving the Schrödinger wave equation, the energy states for each normal coordinate are given by

$$E_n = h\nu = h\left(n + \frac{1}{2}\right)\frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where *n* is a quantum number that can take values of 0, 1, 2 ... The difference in energy when *n* changes by 1 are therefore equal to the energy derived using classical mechanics. See quantum harmonic oscillator for graphs of the first 5 wave functions. Knowing the wave functions, certain selection rules can be formulated. For example, for a harmonic oscillator transitions are allowed only when the quantum number *n* changes by one,

$$\Delta n = \pm 1$$

but this does not apply to an anharmonic oscillator; the observation of overtones is only possible because vibrations are anharmonic. Another consequence of anharmonicity is that transitions such as between states *n*=2 and *n*=1 have slightly less energy than transitions between the ground state and first excited state. Such a transition gives rise to a hot band.

Intensities

In an infrared spectrum the intensity of an absorption band is proportional to the derivative of the molecular dipole moment with respect to the normal coordinate. The intensity of Raman bands depends on polarizability.

Chapter 3

Damping
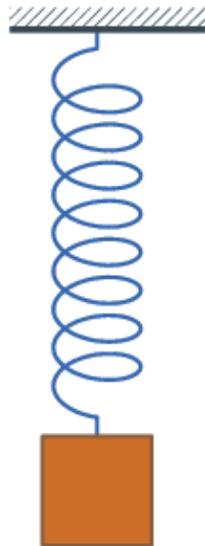
Damped spring-mass system

In physics, **damping** is any effect that tends to reduce the amplitude of oscillations in an oscillatory system, particularly the harmonic oscillator.

In mechanics, friction is one such damping effect. For many purposes the frictional force F_f can be modeled as being proportional to the velocity v of the object:

$$F_f = -cv \, ,$$

where c is the *viscous damping coefficient*, given in units of newton seconds per meter (N s/m).

Generally, damped harmonic oscillators satisfy the second-order differential equation:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0,$$

where ω_0 is the undamped angular frequency of the oscillator and ζ is a constant called the damping ratio. For a mass on a spring having a spring constant k and a damping coefficient c,

$$\omega_0 = \sqrt{\frac{k}{m}}$$

and

$$\zeta = \frac{c}{2m\omega_0}.$$

The value of the damping ratio ζ determines the behavior of the system. A damped harmonic oscillator can be:

- *Overdamped* ($\zeta > 1$): The system returns (exponentially decays) to equilibrium without oscillating. Larger values of the damping ratio ζ return to equilibrium slower.
- *Critically damped* ($\zeta = 1$): The system returns to equilibrium as quickly as possible without oscillating. This is often desired for the damping of systems such as doors.
- *Underdamped* ($0 < \zeta < 1$): The system oscillates (at reduced frequency compared to the *undamped* case) with the amplitude gradually decreasing to zero.
- *Undamped* ($\zeta = 0$): The system oscillates at its natural resonant frequency (ω_o).

The damped natural (angular) frequency ω_d, i.e., the frequency the oscillation occurs when the system is underdamped ($0 < \zeta < 1$) and under free vibration, with regards to the damping factor ζ and the undamped natural (angular) frequency ω_0 is given by:

$$\omega_d = \omega_0\sqrt{1 - \zeta^2}.$$

This is not to be confused with the resonant frequency ω_{peak}. This is the frequency at which a moderately underdamped ($\zeta < 1/\sqrt{2}$) simple 2nd order harmonic oscillator has a peak gain (or peak transmissibility) when driven by a sinusoidal input. The frequency at which this occurs is given by:

$$\omega_{peak} = \omega_0 \sqrt{1 - 2\zeta^2}.$$

Definition

In physics and engineering, damping may be mathematically modelled as a force synchronous with the velocity of the object but opposite in direction to it. If such force is also proportional to the velocity, as for a simple mechanical viscous damper (dashpot), the force F may be related to the velocity v by
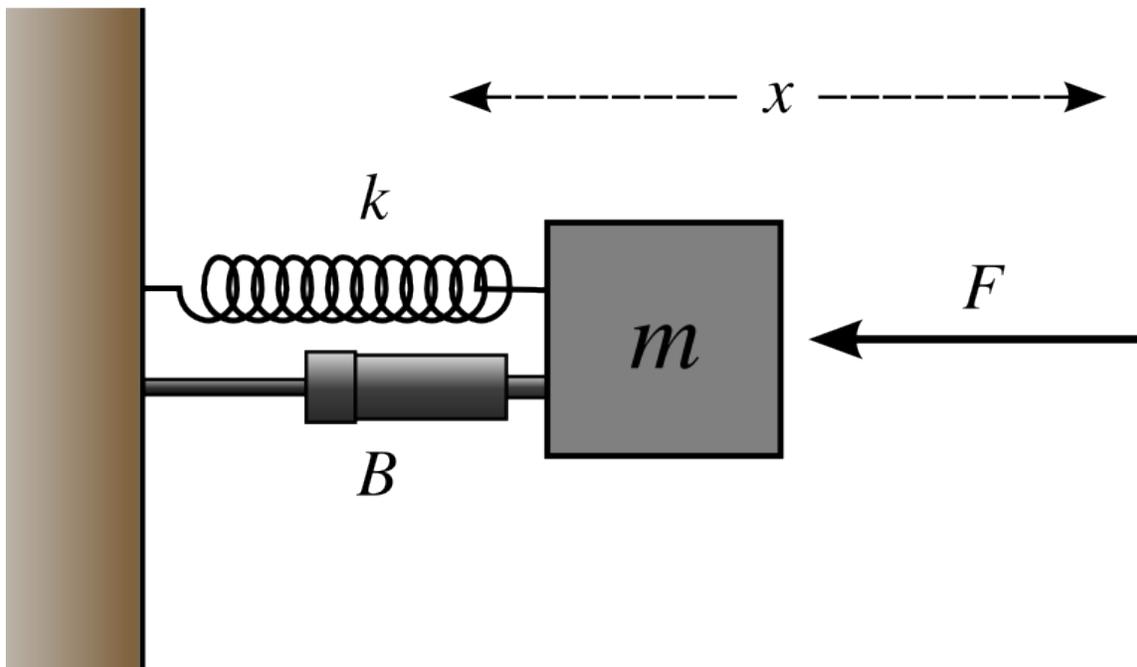
$$F = -cv \, ,$$

where c is the *viscous damping coefficient*, given in units of newton-seconds per meter.

This force is an approximation to the friction caused by drag.

This relationship is perfectly analogous to electrical resistance as described by Ohm's law.

Example: mass-spring-damper

A mass attached to a spring and damper. The damping coefficient, usually c, is represented by B in this case. The F in the diagram denotes an external force, which this example does not include.

An ideal mass-spring-damper system with mass m (in kilograms), spring constant k (in newtons per meter) and viscous damper of damping coefficient c (in newton-seconds per meter or kilograms per second) is subject to an oscillatory force

$$F_s = -kx$$

and a damping force

$$F_d = -cv = -c\frac{dx}{dt} = -c\dot{x}.$$

Treating the mass as a free body and applying Newton's second law, the total force F_{tot} on the body is

$$F_{tot} = ma = m\frac{d^2x}{dt^2} = m\ddot{x}.$$

where a is the acceleration (in metres per second squared) of the mass and x is the displacement (in metres) of the mass relative to a fixed point of reference.

Since $F_{tot} = F_s + F_d$,

$$m\ddot{x} = -kx + -c\dot{x}.$$

This differential equation may be rearranged into

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.$$

The following parameters are then defined:

$$\omega_0 = \sqrt{\frac{k}{m}}$$
$$\zeta = \frac{c}{2\sqrt{mk}}.$$

The first parameter, ω_0, is called the (undamped) natural frequency of the system . The second parameter, ζ, is called the *damping ratio*. The natural frequency represents an angular frequency, expressed in radians per second. The damping ratio is a dimensionless quantity.

The differential equation now becomes

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = 0.$$

Continuing, we can solve the equation by assuming a solution x such that:

$$x = e^{\gamma t}$$

where the parameter γ (gamma) is, in general, a complex number.

Substituting this assumed solution back into the differential equation gives

$$\gamma^2 + 2\zeta\omega_0\gamma + \omega_0^2 = 0 \,,$$
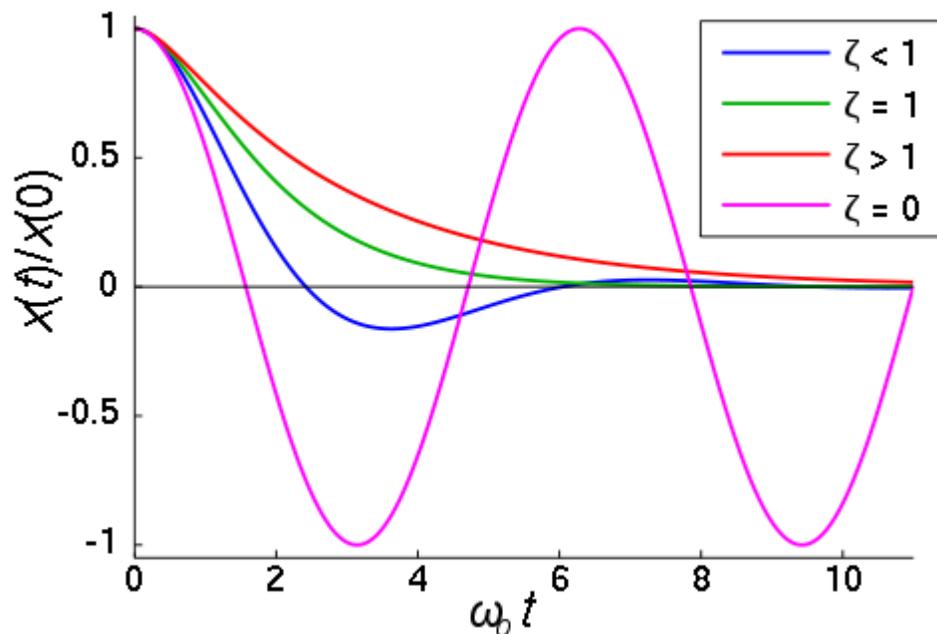
which is the characteristic equation.

Solving the characteristic equation will give two roots, γ_+ and γ_-. The solution to the differential equation is thus

$$x(t) = Ae^{\gamma_+ t} + Be^{\gamma_- t} \,,$$

where A and B are determined by the initial conditions of the system:

$$A = x(0) + \frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+}$$

$$B = -\frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+} \,.$$

System behavior

Dependence of the system behavior on the value of the damping ratio ζ, for under-damped *(blue)*, critically-damped *(green)*, over-damped *(red)*, and undamped cases *(pink)*, for zero-velocity initial condition.

The behavior of the system depends on the relative values of the two fundamental parameters, the natural frequency ω_0 and the damping ratio ζ. In particular, the qualitative behavior of the system depends crucially on whether the quadratic equation for γ has one real solution, two real solutions, or two complex conjugate solutions.

Critical damping ($\zeta = 1$)

When $\zeta = 1$, there is a double root γ (defined above), which is real. The system is said to be *critically damped*. A critically damped system converges to zero faster than any other, and without oscillating. An example of critical damping is the door closer seen on many hinged doors in public buildings. The recoil mechanisms in most guns are also critically damped so that they return to their original position, after the recoil due to firing, in the least possible time.

In this case, with only one root γ, there is in addition to the solution $x(t) = e^{\gamma t}$ a solution $x(t) = te^{\gamma t}$:

$$x(t) = (A + Bt)\,e^{-\omega_0 t}$$

where A and B are determined by the initial conditions of the system (usually the initial position and velocity of the mass):

$$A = x(0)$$
$$B = \dot{x}(0) + \omega_0 x(0)$$

Over-damping ($\zeta > 1$)

When $\zeta > 1$, the system is *over-damped* and there are two different real roots. An over-damped door-closer will take longer to close than a critically damped door would.

The solution to the motion equation is:

$$x(t) = Ae^{\gamma_+ t} + Be^{\gamma_- t}$$

where A and B are determined by the initial conditions of the system:

$$A = x(0) + \frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+}$$
$$B = -\frac{\gamma_+ x(0) - \dot{x}(0)}{\gamma_- - \gamma_+}.$$

Under-damping ($0 \leq \zeta < 1$)

Finally, when $0 \leq \zeta < 1$, γ is complex, and the system is *under-damped*. In this situation, the system will oscillate at the natural damped frequency ω_d, which is a function of the natural frequency and the damping ratio. To continue the analogy, an underdamped door closer would close quickly, but would hit the door frame with significant velocity, or would oscillate in the case of a swinging door.

In this case, the solution can be generally written as:

$$x(t) = e^{-\zeta \omega_0 t} (A \cos(\omega_d t) + B \sin(\omega_d t))$$

where

$$\omega_d = \omega_0 \sqrt{1 - \zeta^2}$$

represents the *natural damped frequency* of the system, and A and B are again determined by the initial conditions of the system:

$$A = x(0)$$
$$B = \frac{1}{\omega_d}(\zeta \omega_0 x(0) + \dot{x}(0)).$$

For an under-damped system, the value of ζ can be found by examining the logarithm of the ratio of succeeding amplitudes of a system. This is called the logarithmic decrement.

Alternative models

Viscous damping models, although widely used, are not the only damping models. A wide range of models can be found in specialized literature, but one of them should be referred here: the so called "hysteretic damping model" or "structural damping model".

When a metal beam is vibrating, the internal damping can be better described by a force proportional to the displacement but in phase with the velocity. In such case, the differential equation that describes the free movement of a single-degree-of-freedom system becomes:

$$m\ddot{x} + hxi + kx = 0$$

where h is the hysteretic damping coefficient and i denotes the imaginary unit; the presence of i is required to synchronize the damping force to the velocity (xi being in phase with the velocity). This equation is more often written as:

$$m\ddot{x} + k(1 + i\eta)x = 0$$

where η is the hysteretic damping ratio, that is, the fraction of energy lost in each cycle of the vibration.

Although requiring complex analysis to solve the equation, this model reproduces the real behaviour of many vibrating structures more closely than the viscous model.

A more general model that also requires complex analysis, the fractional model not only includes both the viscous and hysteretic models, but also allows for intermediate cases (useful for some polymers):

$$m\ddot{x} + A\frac{d^r x}{dt^r}i + kx = 0$$

where r is any number, usually between 0 (for hysteretic) and 1 (for viscous), and A is a general damping (h for hysteretic and c for viscous) coefficient.

Chapter 4

Bushing (Isolator) and Balancing of Rotating Masses

Bushing (isolator)

A **bushing** or **rubber bushing** is a type of vibration isolator. It provides an interface between two parts, damping the energy transmitted through the bushing. A common application is in vehicle suspension systems, where a bushing made of rubber (or, more often, synthetic rubber or polyurethane) separates the faces of two metal objects while allowing a certain amount of movement. This movement allows the suspension parts to move freely, for example, when traveling over a large bump, while minimizing transmission of noise and small vibrations through to the chassis of the vehicle. A rubber bushing may also be described as a **flexible mounting** or **antivibration mounting**.

These bushings often take the form of an annular cylinder of flexible material inside a metallic casing or outer tube. They might also feature an internal *crush tube* which protects the bushing from being crushed by the fixings which hold it onto a threaded spigot. Many different types of bushing designs exist. An important difference compared with plain bearings is that the relative motion between the two connected parts is accommodated by strain in the rubber, rather than by shear or friction at the interface. Some rubber bushings, such as the D block for a sway bar, do allow sliding at the interface between one part and the rubber.

Advantages and disadvantages

The main advantage of a bushing, as compared to a solid connection, is less noise and vibration are transmitted. Another advantage is that they require little to no lubrication.

Disadvantages include:

- Rubber bushings can deteriorate quickly in the presence of oils (e.g., motor oil, mineral oil) and extreme heat and cold.

- The flexibility of rubber also introduces an element of play in the suspension system. This may result in camber, caster, or toe changes in the wheels of the vehicle during high-load conditions (cornering and braking), adversely affecting the vehicle's handling. For this reason, a popular aftermarket performance upgrade is the replacement of rubber suspension bushes with bushes made of more rigid materials, such as polyurethane. Polyurethane bushes are also available for many vehicles with the same characteristics as the manufacturers original bushes, but with greatly increased durability. This is useful on vehicles that have a reputation for wearing out standard rubber bushes, but for which harder bushings with increased harshness of ride are not wanted.

Applications

- In vehicles:
    - Sway bar links and mountings
    - Shock absorber mountings
    - Double wishbone suspension assemblies
- Norton Commando motorcycle
- In skateboards, bushings limit the motion of the trucks.
- In fastening, bushings are also used to transfer loads from a fastening to a much larger area in the underlying structure, the object being to reduce the strain on individual fibers within the underlying structure.
- In crankshaft balancing, certain high-speed inline internal combustion engines are prone to torsional vibration of their crankshafts; the straight six and straight eight engines being particularly prone to this problem due to their long crankshaft length. Although straight eight engines faded from the marketplace in the 1950s, many straight six engines have and still do feature crankshaft vibration damping utilizing rubber bushes. The 3,442 cc Jaguar XK 6-cylinder engine of 1948 and most subsequent versions of the ubiquitous Jaguar XK engine used a proprietary Metalastik vibration damper to protect their crankshafts from potentially damaging torsional vibrations. To quote William Heynes, "The Metalastik damper consists of a steel plate to which is bonded, through a thick rubber disk, a malleable iron floating weight. Variations of the weight, rubber volume and mix, give these dampers a very wide field over which they can operate."

History

Charles E. Sorensen credits Walter Chrysler as being a leader in encouraging the adoption of rubber vibration-isolating mounts. In his memoir (1956), he says that, on March 10, 1932, Chrysler called at Ford headquarters to show off a new Plymouth model. "The most radical feature of his car was the novel suspension of its six-cylinder engine so as to cut down vibration. The engine was supported on three points and rested on rubber mounts. Noise and vibration were much less. There was still a lot of movement of the engine when idling, but under a load it settled down. Although it was a great success in the Plymouth, Henry Ford did not like it. For no given reason, he just didn't like it, and that was that. I told Walter that I felt it was a step in the right direction, that it would

smooth out all noises and would adapt itself to axles and springs and steering-gear mounts, which would stop the transfer of road noises into the body. Today rubber mounts are used on all cars. They are also found on electric-motor mounts, in refrigerators, radios, television sets—wherever mechanical noises are apparent, rubber is used to eliminate them. We can thank Walter Chrysler for a quieter way of life. Mr. Ford could have installed this new mount at once in the V-8, but he missed the value of it. Later Edsel and I persuaded him. Rubber mounts are now found also in doors, hinges, windshields, fenders, spring hangers, shackles, and lamps—all with the idea of eliminating squeaks and rattles." Chrysler's novel engine-mounting method was marketed as "Floating Power". Its basic idea soon became the conventional method throughout the automotive industry.

Balancing of rotating masses

The **balancing of rotating bodies** is important to avoid vibration. In heavy industrial machines such as gas turbines and electric generators, vibration can cause catastrophic failure, as well as noise and discomfort. In the case of a simple wheel, balancing simply involves moving the centre of gravity to the centre of rotation. For a system to be in complete balance both force and couple polygons should be closed.

Static balance
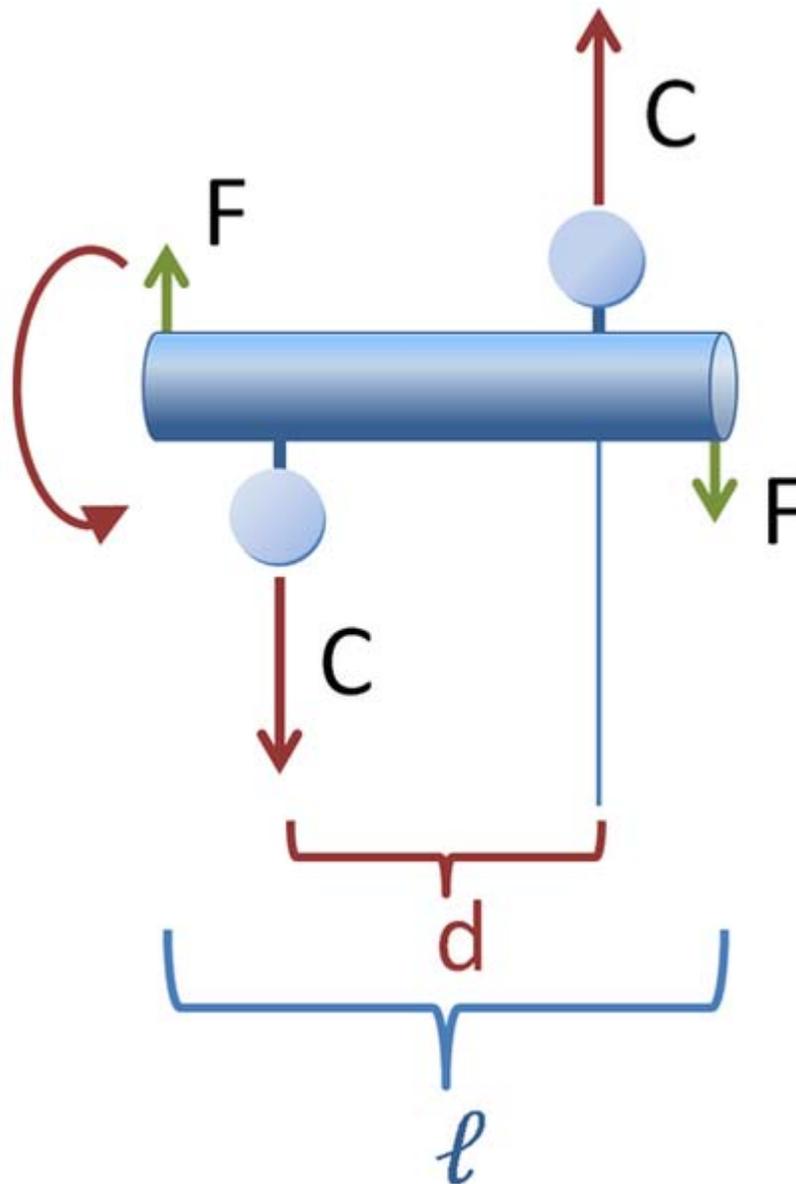
Rotating shaft unbalanced by two identical attached weights, which causes a counterclockwise centrifugal couple Cd that must be resisted by a clockwise couple $F\ell = Cd$ exerted by the bearings. The figure is drawn from the viewpoint of a frame rotating with the shaft, hence the centrifugal forces.

Static balance occurs when there is no resultant centrifugal force (centrifugal couple) and the centre of gravity is on the axis of rotation. To avoid the stress upon the bearings caused by the centrifugal couple, counterbalancing weights must be added. The unbalancing weights are an idealization, of course. For example, in the case of an automobile tire the imbalance is due to imperfections of manufacture that make the tire composition inhomogeneous.

Dynamic balance

This occurs when there is no resulting turning moment along the axis of rotation.

Unbalanced systems

There are many disadvantages to unbalanced systems. When a system is rotating with unbalanced masses unnecessary vibration occurs, further it may generate unwanted noise, excessive stresses in machine elements and reduce the reliability for the all parts associated with the rotating system. If a rotating system is not in balance, due to the unbalanced couple high bearing thrusts are generated. This will cause failure in bearings in short time periods. Shafts with unbalanced masses can be easily bent and fatigue failure may occur in particular number of cycles. When a system is rotating two rotors, although the system is in static balance, always there exists a couple relative to a one rotor. This unbalanced couple will generate a dynamic force couple in the two bearings which is known as rocking couple.

Under conditions where rotating speed is very high even though the mass is low, as in gas turbines or jet engines, or under conditions where rotating speed is low but the mass is high, as in ship propellers, balance of the rotating system should be highly considered, because it may generate large vibrations it will cause failure of the whole system.

Chapter 5

Effective Mass (Spring-Mass System) and Statistical Energy Analysis

Effective mass (spring-mass system)

In a spring-mass system the mass of the spring m, as well as the suspended mass M, have an influence on the motion. However, since not all of the spring moves at the same velocity as the suspended mass, the mass of the spring cannot be simply added to the suspended mass. The *effective mass* of the spring is the mass which must be added to the suspended mass to correctly predict the behavior of the system.

Ideal spring
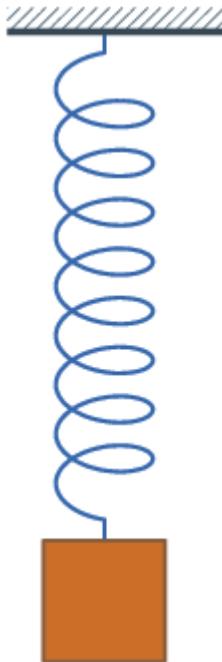
vertical spring-mass system

The effective mass of the spring in a spring-mass system when using an ideal spring is independent whether the direction of the spring-mass system is horizontal, vertical or oblique, is also $\frac{1}{3}$ of the mass of the spring, i.e. $\frac{m}{3}$. This can be shown by integration:

Take an infinitesimally thin segment of the spring which is at a distance y from the fixed end of the spring.

Its length is dy, mass is dm, velocity is u.

$$\therefore dm = \left(\frac{dy}{L}\right) m$$, where L is the length of the spring.

Hence the total kinetic energy of the spring

$$= \int_m \frac{1}{2} u^2 \, dm$$

$$= \int_0^L \frac{1}{2} u^2 \left(\frac{dy}{L}\right) m$$

$$= \frac{1}{2} \frac{m}{L} \int_0^L u^2 \, dy$$

But the velocity of each position of the spring is directly proportional to its length

$$\therefore u : v = y : L$$

$$u = \frac{vy}{L}$$

Hence $\frac{1}{2} \frac{m}{L} \int_0^L u^2 \, dy$

$$= \frac{1}{2} \frac{m}{L} \int_0^L \left(\frac{vy}{L}\right)^2 \, dy$$

$$= \frac{1}{2} \frac{m}{L^3} v^2 \int_0^L y^2 \, dy$$

$$= \frac{1}{2} \frac{m}{L^3} v^2 \left[\frac{y^3}{3}\right]_0^L$$

$$= \frac{1}{2}\frac{m}{3}v^2$$

With comparison of the original kinetic energy formula $\frac{1}{2}mv^2$, we can conclude that effective mass of spring in this case is $\frac{m}{3}$, i.e. $\frac{1}{3}$ of the mass of the spring.

Real spring

However, the above calculations are only suitable for small values of $\frac{M}{m}$. Since Jun-ichi Ueda and Yoshiro Sadamoto have done the experiment. It is found that, as the ratio $\frac{M}{m}$ increases beyond 7, the effective mass of a spring in a vertical spring-mass system becomes smaller than Rayleigh's value $\frac{m}{3}$ and eventually reaches negative values. This unexpected behavior of the effective mass can be explainable in terms of the elastic after-effect.

Statistical energy analysis

Statistical energy analysis (SEA) is a method for predicting resonant sound and vibration transmission in dynamical systems made up of coupled acoustic cavities and structural parts. The vibrational behavior of the system is defined in terms of energy. The energy is here the potential and kinetic energy of the modal resonances (eigenmodes)of the dynamical system.

To solve a sound or vibration problem with SEA, the system is partitioned into components (like plates, shells, beams or cavities) which in turn are represented by one or more modal "subsystems", for whom the vibrational energy is shared among modes with similar dynamic properties. A plate may thus be modeled by three subsystems, one for each type of waves possible (bending,longitudinal and shear). A number of practical prerequisites have to be fulfilled:

1) The subsystems, when the component is isolated, shall have at least a few eigenmodes in each frequency band of interest. 2) The damping of the modes shall not be too high, so that non-resonant transmission becomes significant 3) The coupling strength between subsystems should not be too strong

This means in practice that SEA is useful only at sufficiently high frequencies, e.g. it can by default not predict response of low frequency "global modes".

The core of SEA calculations consists of a relatively small set of energy-balance equations that uses modal densities and internal loss factors (damping) of the subsystems, coupling loss factors between these and the vibrational power injected by the external sources to calculate the vibrational energies of the various subsystems. Engineering response for the subsystems are then expressed as spatial average quantities (sound pressure, vibration velocite etc.), derived from these energies.

SEA is popular in aerospace and automotive industries where various mid- and high frequency sound and vibration transmission properties can be predicted at low cost and for a wide frequency range: random vibration levels of rockets and payloads structures at lift-off, sound transmission into car passenger cabins due to various noise sources (exhaust, engine...). For mechanical engineers, vibrations are a ubiquitous problem which not only cause noise but also reduce life of machinery. Statistical energy analysis is a statistical method to predict multi-modal resonant vibrational levels of components of structures without solving elaborated equations or using finite element software.

SEA was originally stated by Lyon and Maidanik in the early 60s where they found the power exchange relationship between two coupled spring-mass oscillators . Further on, the theory has been extended to the coupling of continuous dynamic systems (beams, plates, shells) . More recently, vibrational entropy has been recognized as being part of SEA with equal importance as energy

Chapter 6

Mechanical Resonance
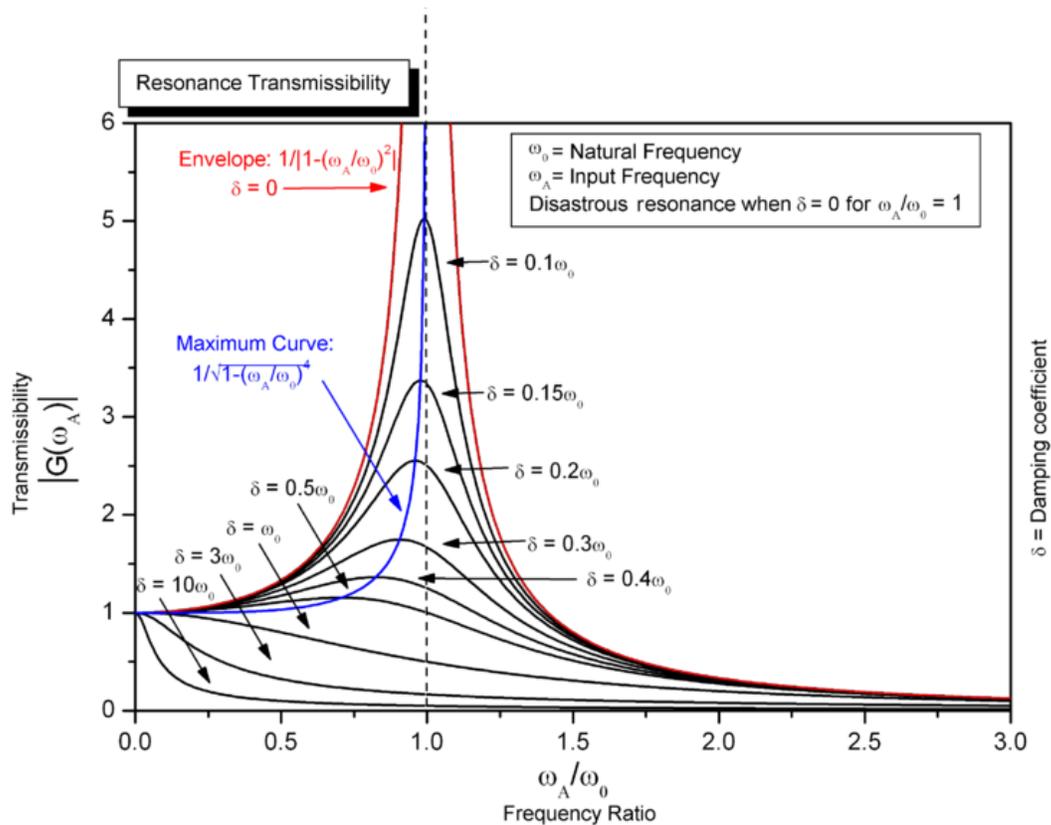
Graph showing mechanical resonance in a mechanical oscillatory system

Mechanical resonance is the tendency of a mechanical system to absorb more energy when the frequency of its oscillations matches the system's natural frequency of vibration (its *resonance frequency* or *resonant frequency*) than it does at other frequencies. It may cause violent swaying motions and even catastrophic failure in improperly constructed

structures including bridges, buildings, and airplanes a phenomenon known as resonance disaster.

Avoiding resonance disasters is a major concern in every building, tower and bridge construction project. As a countermeasure, shock mounts can be installed to absorb resonant frequencies and thus dissipate the absorbed energy. The Taipei 101 building relies on a 660-ton pendulum — a tuned mass damper — to cancel resonance. Furthermore, the structure is designed to resonate at a frequency which does not typically occur. Buildings in seismic zones are often constructed to take into account the oscillating frequencies of expected ground motion. In addition, engineers designing objects having engines must ensure that the mechanical resonant frequencies of the component parts do not match driving vibrational frequencies of the motors or other strongly oscillating parts.

Many resonant objects have more than one resonance frequency, particularly at harmonics (multiples) of the strongest resonance. It will vibrate easily at those frequencies, and less so at other frequencies. Many clocks keep time by mechanical resonance in a balance wheel, pendulum, or quartz crystal.

Description

The natural frequency of a simple mechanical system consisting of a weight suspended by a spring is:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

where *m* is the mass and *k* is the spring constant.

A swing set is a simple example of a resonant system with which most people have practical experience. It is a form of pendulum. If the system is excited (pushed) with a period between pushes equal to the inverse of the pendulum's natural frequency, the swing will swing higher and higher, but if excited at a different frequency, it will be difficult to move. The resonance frequency of a pendulum, the only frequency at which it will vibrate, is given approximately, for small displacements, by the equation:

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

where *g* is the acceleration due to gravity (about 9.8 m/s^2 near the surface of Earth), and *L* is the length from the pivot point to the center of mass.(An elliptic integral yields a description for any displacement). Note that, in this approximation, the frequency does not depend on mass.

Mechanical resonators work by transferring energy repeatedly from kinetic to potential form and back again. In the pendulum, for example, all the energy is stored as gravitational energy (a form of potential energy) when the bob is instantaneously motionless at the top of its swing. This energy is proportional to both the mass of the bob and its height above the lowest point. As the bob descends and picks up speed, its potential energy is gradually converted to kinetic energy (energy of movement), which is proportional to the bob's mass and to the square of its speed. When the bob is at the bottom of its travel, it has maximum kinetic energy and minimum potential energy. The same process then happens in reverse as the bob climbs towards the top of its swing.

Some resonant objects have more than one resonance frequency, particularly at harmonics (multiples) of the strongest resonance. It will vibrate easily at those frequencies, and less so at other frequencies. It will "pick out" its resonance frequency from a complex excitation, such as an impulse or a wideband noise excitation. In effect, it is filtering out all frequencies other than its resonance. In the example above, the swing cannot easily be excited by harmonic frequencies, but can be excited by subharmonics.

Examples

Various examples of mechanical resonance include:

- musical instruments (acoustic resonance).
- Most clocks keep time by mechanical resonance in a balance wheel, pendulum, or quartz crystal.
- tidal resonance of the Bay of Fundy.
- Orbital resonance as in some moons of the solar system's gas giants.
- The resonance of the basilar membrane in the ear.
- Making a child's swing swing higher by pushing it at each swing.
- A wineglass breaking when someone sings a loud note at exactly the right pitch.

Resonance Rings exhibit at California Science Center

Resonance may cause violent swaying motions in improperly constructed structures, such as bridges and buildings. The London Millennium Footbridge (nicknamed the *Wobbly Bridge*) exhibited this problem. A faulty bridge can even be destroyed by its resonance (see "Angers Bridge"; that is why soldiers are trained not to march in lockstep across a bridge, although it is suspected to be a myth, see e.g., MythBusters' 'Breakstep Bridge'. Mechanical systems store potential energy in different forms. For example, a spring/mass system stores energy as tension in the spring, which is ultimately stored as the energy of bonds between atoms.

Resonance disaster

In mechanics and construction a **resonance disaster** describes the destruction of a building or a technical mechanism by induced vibrations at a system's resonance frequency, which causes it to oscillate. Periodic excitation optimally transfers to the system the energy of the vibration and stores it there. Because of this repeated storage and additional energy input the system swings ever more strongly, until its load limit is exceeded. As a countermeasure, shock mounts can be installed to absorb resonant frequencies and thus dissipate the absorbed energy. The Taipei 101 building relies on a 660-ton pendulum — a tuned mass damper — to cancel resonance. Furthermore, the structure is designed to resonate at a frequency which does not typically occur. Buildings

in seismic zones are often constructed to take into account the oscillating frequencies of expected ground motion. Avoiding resonance disasters is a major concern in every building, tower and bridge construction project.

Failure of the original Tacoma Narrows Bridge

The dramatic, rhythmic twisting that resulted in the 1940 collapse of "Galloping Gertie", the original Tacoma Narrows Bridge, is sometimes characterized in physics textbooks as a classic example of resonance; however, this description is misleading. The catastrophic vibrations that destroyed the bridge were not due to simple mechanical resonance, but to a more complicated oscillation caused by interactions between the bridge and the winds passing through its structure — a phenomenon known as aeroelastic flutter. Robert H. Scanlan, father of the field of bridge aerodynamics, wrote an article about this misunderstanding.

Other Examples

- Collapse of Angers Bridge
- Collapse of Königs Wusterhausen Central Tower
- Crash of Partnair Flight 394
- Resonance of the Millenium Bridge

Applications

Various method of inducing mechanical resonance in a medium exist. Mechanical waves can be generated in a medium by subjecting an electromechanical element to an alternating electric field having a frequency which induces mechanical resonance and is below any electrical resonance frequency. Such devices can apply mechanical energy from an external source to an element to mechanically stress the element or apply mechanical energy produced by the element to an external load.

The United States Patent Office classifies devices that tests mechanical resonance under subclass 579, resonance, frequency, or amplitude study, of Class 73, Measuring and testing. This subclass is itself indented under subclass 570, Vibration. Such devices test an article or mechanism by subjecting it to a vibratory force for determining qualities, characteristics, or conditions thereof, or sensing, studying or making analysis of the vibrations otherwise generated in or existing in the mechanism. Devices include methods to cause vibrations at a natural mechanical resonance and measure the frequency and/or amplitude the resonance made. Various devices study the amplitude response over a frequency range is made. This includes nodal points, wave lengths, and standing wave characteristics measured under predetermined vibration conditions.

Earthquake machine

TESLA'S OSCILLATOR AS SHOWN AT THE
COLUMBIAN EXPOSITION.

Tesla's later evolution of his electromechanical oscillator shown at the World's Columbian Exposition.

Nikola Tesla established a laboratory at 46 E Houston Street in New York. There, at one point while experimenting with mechanical oscillators, he allegedly generated a resonance of several buildings causing complaints to the police. As the speed grew he hit the resonance frequency of his own building and belatedly realizing the danger he was forced to apply a sledge hammer to terminate the experiment, just as the astonished police arrived. The Discovery Channel's popular *MythBusters* show examined Tesla's claim that he had created an "Earthquake Machine" in their 60th episode. They tested the physical phenomenon known as mechanical resonance on a traffic bridge, reportedly feeling the bridge vibrating many yards away.

Chapter 7

Noise, Vibration, and Harshness

Noise, vibration, and harshness (NVH), also known as **noise and vibration (N&V)**, is the study and modification of the noise and vibration characteristics of vehicles, particularly cars and trucks. While noise and vibration can be readily measured, **Harshness** is a subjective quality, and is measured either via "jury" evaluations, or with analytical tools that provide results reflecting human subjective impressions. These latter tools belong to the field known as "psychoacoustics."

Interior NVH deals with noise and vibration experienced by the occupants of the cabin, while exterior NVH is largely concerned with the noise radiated by the vehicle, and includes drive-by noise testing.

NVH is mostly engineering, but often objective measurements fail to predict or correlate well with the subjective impression on human observers. This is partly because the human body has its own frequency response, e.g. the ear's response at moderate noise levels is approximated by A-weighting, but this does not mean that two noises with the same A-weighted level are equally disturbing. The field of psychoacoustics is partly concerned with this correlation.

In some cases the NVH engineer is asked to change the sound quality, i.e. adding or subtracting particular harmonics, rather than making the car quieter.

Sources of NVH

The sources of noise in a vehicle are many, including the engine, driveline, tire contact patch and road surface, brakes, and wind. Noise from cooling fans, or the HVAC, alternator, and other engine accessories is also fairly common. Many problems are generated as either vibration or noise, transmitted via a variety of paths, and then radiated acoustically into the cabin. These are classified as "structure-borne" noise. Others are generated acoustically and propagated by airborne paths. Structure-borne noise is

attenuated by isolation, while airborne noise is reduced by absorption or through the use of barrier materials. Vibrations are sensed at the steering wheel, the seat, armrests, or the floor and pedals. Some problems are sensed visually - such as the vibration of the header rail or rear view mirror on open topped cars.

Tonal versus broadband

NVH can be tonal, such as engine noise, or broadband, such as road noise or wind noise, normally. Some resonant systems respond at characteristic frequencies, but in response to random excitation. Therefore, although they look like tonal problems on any one spectrum, their amplitude varies considerably. Other problems are self resonant, such as whistles from antennas.

Tonal noises often have harmonics. Here is the noise spectrum of Michael Schumacher's Ferrari at 16680 rpm, showing the various harmonics. The x axis is given in terms of multiples of engine speed. The y axis is logarithmic, and uncalibrated.
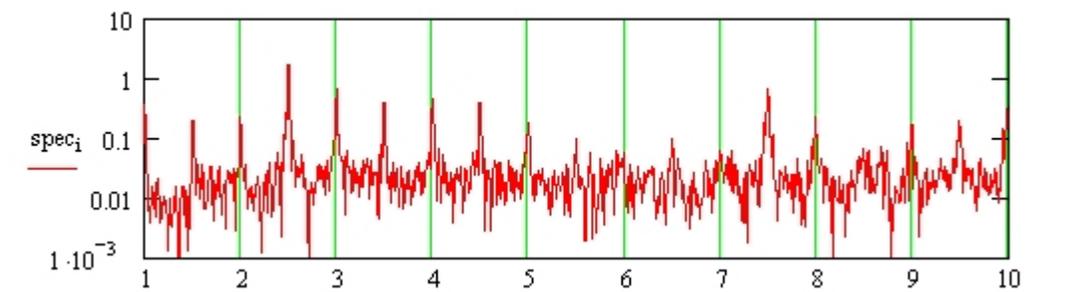
Instrumentation

Typical instrumentation used to measure NVH include microphones, accelerometers and force gauges, or load cells. Many NVH facilities will have semi-anechoic chambers, and rolling road dynamometers. Typically signals are recorded direct to hard disk via an Analog-to-digital converter. In the past magnetic or DAT tape recorders were used. The integrity of the signal chain is very important, typically each of the instruments used are fully calibrated in a lab once per year, and any given setup is calibrated as a whole once per day.

Investigative techniques

Techniques used to help identify NVH include part substitution, modal analysis, rig tests, lead cladding, acoustic intensity, transfer path analysis, and partial coherence. Most NVH work is done in the frequency domain, using fast Fourier transforms to convert the time domain signals into the frequency domain. Wavelet analysis, order analysis, statistical energy analysis, and subjective evaluation of signals modified in real time are also used.

Computer-based modelling

NVH needs good representative prototypes of the production vehicle, for testing. These are needed early in the design process as the solutions often need substantial modification to the design, forcing in engineering changes which are much cheaper when made early. These early prototypes are very expensive, so there has been great interest in computer aided predictive techniques for NVH. Sometimes these work. Back-of-envelope calculations are very useful.

One example is the modelling works for structure borne noise and vibration analysis. When the phenomenon being considered occurs below, say, 25-30 Hz, for example the idle shaking of the powertrain, a multi-body model should be established. In contrast, when the phenomenon being considered occurs at relatively high frequency, for example above 1 kHz, a Statistical Energy Analysis (SEA) model should be established.

Typical solutions

There are three principal means of improving NVH:

1. reducing the source strength, as in making a noise source quieter with a muffler, or improving the balance of a rotating mechanism;
2. interrupting the noise or vibration path, with barriers (for noise) or isolators (for vibration); or
3. absorption of the noise or vibration energy, as for example with foam noise absorbers, or tuned vibration dampers.

Deciding which of these to use in solving a particular problem is the challenge facing the NVH engineer.

Specific methods for improving NVH include the use of Tuned mass dampers, Subframes, balancing, modifying the stiffness or mass of structures, retuning exhausts and Intakes, modifying the characteristics of elastomeric isolators, adding sound deadening or absorbing materials, or using active noise control. In some circumstances, substantial changes in vehicle architecture may be the only way to cure some problems cost effectively.

Chapter 8

Torsional Vibration and Vibrating String

Torsional vibration

Torsional vibration is angular vibration of an object—commonly a shaft along its axis of rotation. Torsional vibration is often a concern in power transmission systems using rotating shafts or couplings where it can cause failures if not controlled.

In ideal power transmission systems using rotating parts the torques applied or reacted are "smooth" leading to constant speeds. In reality this is not the case. The torques generated may not be smooth (e.g., internal combustion engines) or the component being driven may not react to the torque smoothly (e.g., reciprocating compressors). Also, the components transmitting the torque can generate non-smooth or alternating torques (e.g., worn gears, misaligned shafts). Because the components in power transmission systems are not infinitely stiff these alternating torques cause vibration about the axis of rotation.

Crankshaft torsional vibration

Torsional vibration is a concern in the crankshafts of internal combustion engines because of several factors.

- Alternating torques are generated by the slider-crank mechanism of the crankshaft, connecting rod, and piston.
  - The motion of the piston mass and connecting rod mass generate alternating torques often referred to as "inertia" torques
  - The cylinder pressure due to combustion is not constant through the combustion cycle.
  - The slider-crank mechanism does not output a smooth torque even if the pressure is constant (e.g., at top dead centre there is no torque generated)

- Engines with several cylinders can have very flexible crankshafts due to their long length.
- There is inherently little damping in a crankshaft to reduce the vibration

If torsional vibration is not controlled in a crankshaft it can cause failure of the crankshaft or any accessories that are being driven by the crankshaft (typically at the front of the engine; the inertia of the flywheel normally reduces the motion at the rear of the engine).

This potentially damaging vibration is often controlled by a torsional damper that is located at the front nose of the crankshaft (in automobiles it is often integrated into the front pulley). There are two main types of torsional dampers.

- Viscous dampers consist of an inertia ring in a viscous fluid. The torsional vibration of the crankshaft forces the fluid through narrow passages that dissipates the vibration as heat. The viscous torsional damper is analogous to the hydraulic shock absorber in a car's suspension.
- Tuned absorber type of "dampers" often referred to as a harmonic dampers or harmonic balancers (even though it technically does not dampen or balance the crankshaft). This damper uses a spring element (often rubber in automobile engines) and an inertia ring that is typically tuned to the first torsional natural frequency of the crankshaft. This type of damper reduces the vibration at specific engine speeds when an excitation torque excites the first natural frequency of the crankshaft, but not at other speeds. This type of damper is analogous to the tuned mass dampers used in skyscrapers to reduce the building motion during an earthquake.

Measuring torsional vibration on physical systems

The most common way to measure torsional vibration is the approach of using equidistant pulses over one shaft revolution. Dedicated shaft encoders as well as gear tooth pickup transducers (induction, hall-effect, variable reluctance,etc.) can generate these pulses. The resulting encoder pulse train is converted into either a digital rpm reading or a voltage proportional to the rpm.

The use of a dual-beam laser is another technique that is used to measure torsional vibrations. The operation of the dual-beam laser is based on the difference in reflection frequency of two perfectly aligned beams pointing at different points on a shaft. Despite its specific advantages, this method yields a limited frequency range, requires line-of-sight from the part to the laser, and represents multiple lasers in case several measurement points need to be measured in parallel.

Vibrating string
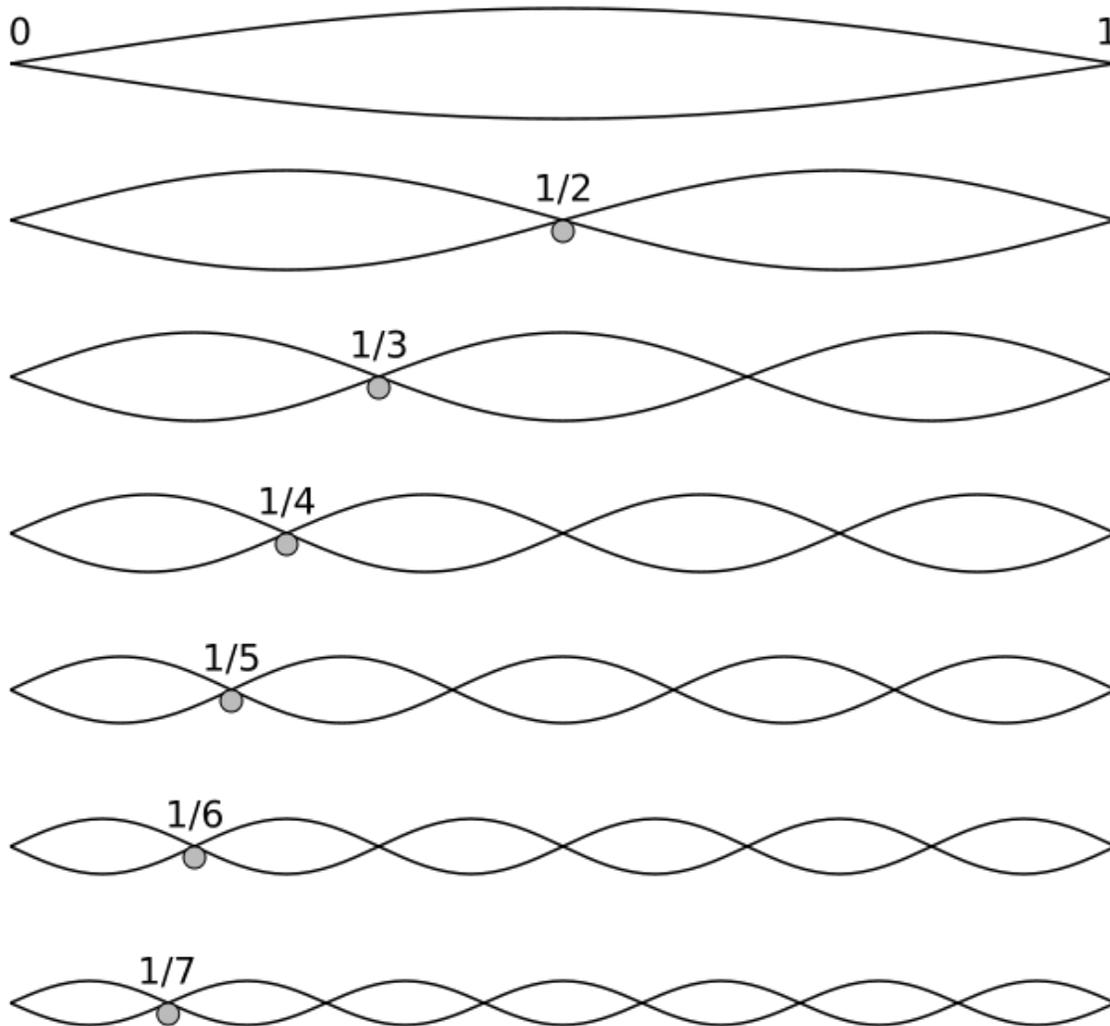
Vibration, standing waves in a string, The fundamental and the first 6 overtones which form a harmonic series

A vibration in a string is a wave. Usually a **vibrating string** produces a sound whose frequency in most cases is constant. Therefore, since frequency characterizes the pitch, the sound produced is a constant note. Vibrating strings are the basis of any string instrument like guitar, cello, or piano.

Wave

The speed of propagation of a wave in a string (v) is proportional to the square root of the tension of the string (T) and inversely proportional to the square root of the linear density (μ) of the string:

$$v = \sqrt{\frac{T}{\mu}}.$$

Derivation
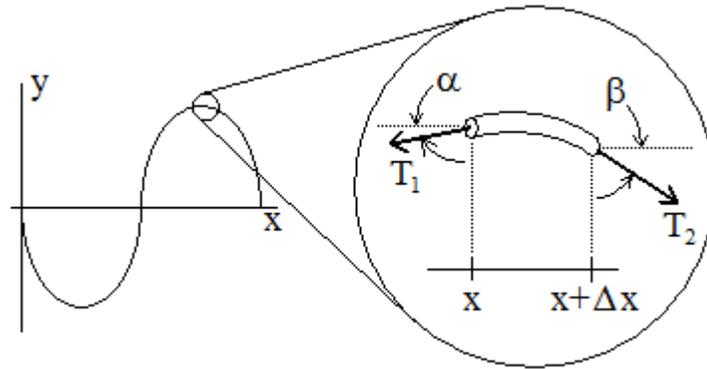
Let Δx be the length of a piece of string, m its mass, and μ its linear mass. If the horizontal component of tension in the string is a constant, T, then the tension acting on each side of the string segment is given by

$$T_{1x} = T_1 \cos(\alpha) \approx T.$$
$$T_{2x} = T_2 \cos(\beta) \approx T.$$

If both angles are small, then the tensions on either side are equal and the net horizontal force is zero. From Newton's second law for the vertical component, the mass of this piece times its acceleration, a, will be equal to the net force on the piece:

$$\Sigma F_y = T_{2y} - T_{1y} = T_2 \sin(\beta) - T_1 \sin(\alpha) = \Delta m a \approx \mu \Delta x \frac{\partial^2 y}{\partial t^2}.$$

Dividing this expression by T and substituting the first and second equations obtains

$$\frac{\mu \Delta x}{T} \frac{\partial^2 y}{\partial t^2} = \frac{T_2 \sin(\beta)}{T_2 \cos(\beta)} - \frac{T_1 \sin(\alpha)}{T_1 \cos(\alpha)} = \tan(\beta) - \tan(\alpha)$$

The tangents of the angles at the ends of the string piece are equal to the slopes at the ends. Using this fact and rearranging provides

$$\frac{1}{\Delta x}\left(\left.\frac{\partial y}{\partial x}\right|^{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|^{x}\right) = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}$$

In the limit that Δx approaches zero, the left hand side is the definition of the second derivative of y:

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 y}{\partial t^2}.$$

This is the wave equation for $y(x,t)$, and the coefficient of the second time derivative term is equal to v^{-2}; thus

$$v = \sqrt{\frac{T}{\mu}},$$

where v is the speed of propagation of the wave in the string. However, this derivation is only valid for vibrations of small amplitude; for those of large amplitude, Δx is not a good approximation for the length of the string piece, the horizontal component of tension is not necessarily constant, and the horizontal tensions are not well approximated by T.

Frequency of the wave

Once the speed of propagation is known, the frequency of the sound produced by the string can be calculated. The speed of propagation of a wave is equal to the wavelength λ divided by the period τ, or multiplied by the frequency f:

$$v = \frac{\lambda}{\tau} = \lambda f.$$

If the length of the string is L, the fundamental harmonic is the one produced by the vibration whose nodes are the two ends of the string, so L is half of the wavelength of the fundamental harmonic. Hence:

$$f = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

where T is the tension, μ is the linear density, and L is the length of the vibrating part of the string. Therefore:

- the shorter the string, the higher the frequency of the fundamental
- the higher the tension, the higher the frequency of the fundamental
- the lighter the string, the higher the frequency of the fundamental

Moreover, if we take the nth harmonic as having a wavelength given by $\lambda_n = 2L\,/\,n$, then we easily get an expression for the frequency of the nth harmonic:

$$f_n = \frac{nv}{2L}$$

And for a string under a tension T with density μ, then

$$f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}$$

Observing string vibrations

One can see the waveforms on a vibrating string if the frequency is low enough and the vibrating string is held in front of a CRT screen such as one of a television or a computer (*not* of an oscilloscope). This effect is called the stroboscopic effect, and the rate at which the string seems to vibrate is the difference between the frequency of the string and the refresh rate of the screen. The same can happen with a fluorescent lamp, at a rate which is the difference between the frequency of the string and the frequency of the alternating current. (If the refresh rate of the screen equals the frequency of the string or an integer multiple thereof, the string will appear still but deformed.) In daylight, this effect does not occur and the string will appear to be still, but thicker and lighter, due to persistence of vision.

A similar but more controllable effect can be obtained using a stroboscope. This device allows the frequency of the xenon flash lamp to be exactly matched to the frequency of vibration of the string; in a darkened room, this clearly shows the waveform. Otherwise, one can use bending or, perhaps more easily, by adjusting the machine heads, to obtain the same frequency, or a multiple of, the AC frequency to achieve the same effect. For example, in the case of a guitar, the bass string pressed to the third fret gives a G at 97.999 Hz; with a slight adjustment, a frequency of 100 Hz can be obtained, exactly one octave above the alternating current frequency in Europe and most countries in Africa and Asia. In most countries of the Americas, where the AC frequency is 60 Hz, one can start from A# at 116.54 Hz, on the fifth string at the first fret, to obtain a frequency of 120 Hz.

Chapter 9

Vibration Isolation

Vibration-isolation is the process of isolating an object, such as a piece of equipment, from the source of vibrations.

Passive isolation

Passive vibration isolation systems consist essentially of a mass, spring and damper (dash-pot).

Negative-Stiffness Vibration Isolator

Negative-Stiffness-Mechanism (NSM) vibration isolation systems offer a unique passive approach for achieving low vibration environments and isolation against sub-Hertz vibrations. "Snap-through" or "over-center" NSM devices are used to reduce the stiffness of elastic suspensions and create compact six-degree-of-freedom systems with low natural frequencies. Practical systems with vertical and horizontal natural frequencies as low as 0.2 to 0.5 Hz provide isolation efficiencies one to two orders of magnitude better than top-performance air tables and pneumatic isolation systems. Electro-mechanical auto-adjust mechanisms compensate for varying weight loads and provide automatic leveling in multiple-isolator systems, similar to the function of leveling valves in pneumatic systems. All-metal systems can be configured which are compatible with high vacuums and other adverse environments such as high temperatures.

These isolation systems enable vibration-sensitive instruments such as scanning probe microscopes, micro-hardness testers and scanning electron microscopes to operate in severe vibration environments sometimes encountered, for example, on upper floors of buildings and in clean rooms. Such operation would not be practical with pneumatic isolation systems. Similarly, they enable vibration-sensitive instruments to produce better images and data than those achievable with pneumatic isolators.

The theory of operation of NSM vibration isolation systems is summarized, some typical systems and applications are described, and data on measured performance is presented. The theory of NSM isolation systems is explained in References 1 and 2. It is summarized briefly for convenience.

Vertical-Motion Isolation

A vertical-motion isolator is shown in Figure 1. It uses a conventional spring connected to an NSM consisting of two bars hinged at the center, supported at their outer ends on pivots, and loaded in compression by forces P. The spring is compressed by weight W to the operating position of the isolator, as shown in Figure 1. The stiffness of the isolator is $K=K_S-K_N$ where K_S is the spring stiffness and K_N is the magnitude of a negative stiffness which is a function of the length of the bars and the load P. The isolator stiffness can be made to approach zero while the spring supports the weight W.
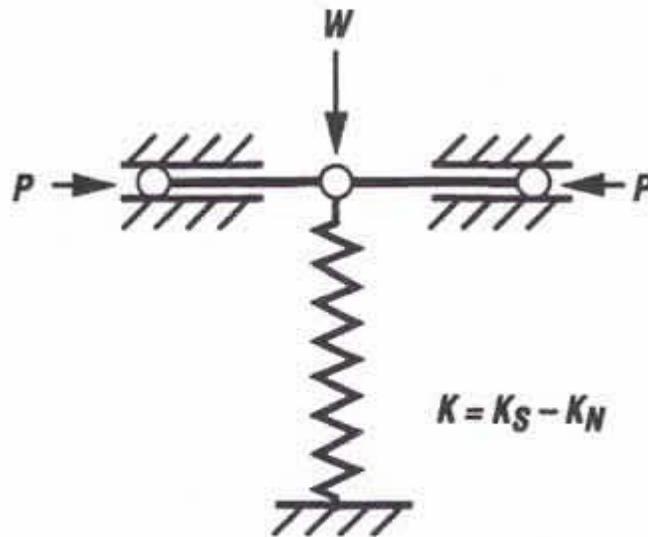
Figure 1 Vertical-Motion Isolator

Horizontal-Motion Isolation

A horizontal-motion isolator consisting of two beam-columns is illustrated in Figure. 2. Each beam-column behaves like two fixed-free beam columns loaded axially by a weight load W. Without the weight load the beam-columns have horizontal stiffness K_S With the weight load the lateral bending stiffness is reduced by the "beam-column" effect. This behavior is equivalent to a horizontal spring combined with an NSM so that the horizontal stiffness is $K=K_S-K_N$, and K_N is the magnitude of the beam-column effect. Horizontal stiffness can be made to approach zero by loading the beam-columns to approach their critical buckling load.
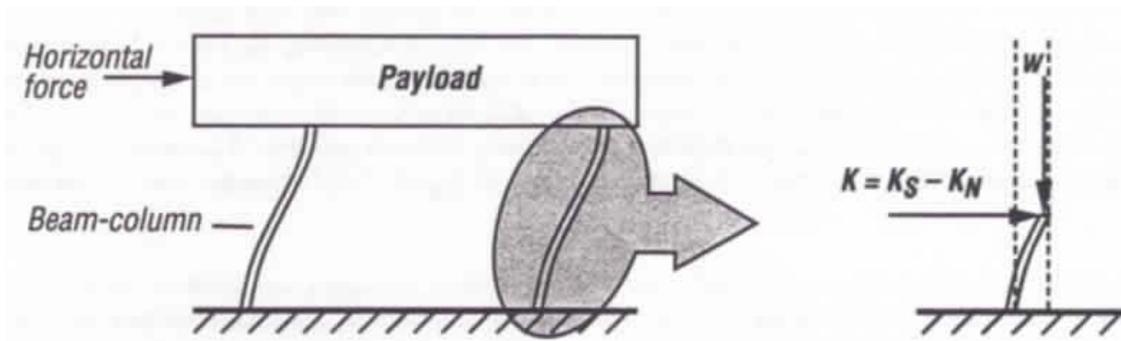
Figure 2 Horizontal-Motion Isolator

Six-Degree-of-Freedom (six-DOF) Isolation

A six-DOF NSM isolator typically uses three isolators stacked in series: a tilt-motion isolator on top of a horizontal-motion isolator on top of a vertical-motion isolator. Figure 3 shows a schematic of a vibration isolation system consisting of a weighted platform supported by a single six-DOF isolator incorporating the isolators of Figures 1 and 2. Flexures are used in place of the hinged bars shown in Figure 1. A tilt flexure serves as the tilt-motion isolator. A vertical-stiffness adjustment screw is used to adjust the compression force on the negative-stiffness flexures thereby changing the vertical stiffness. A vertical load adjustment screw is used to adjust for varying weight loads by raising or lowering the base of the support spring to keep the flexures in their straight,

unbent operating position.
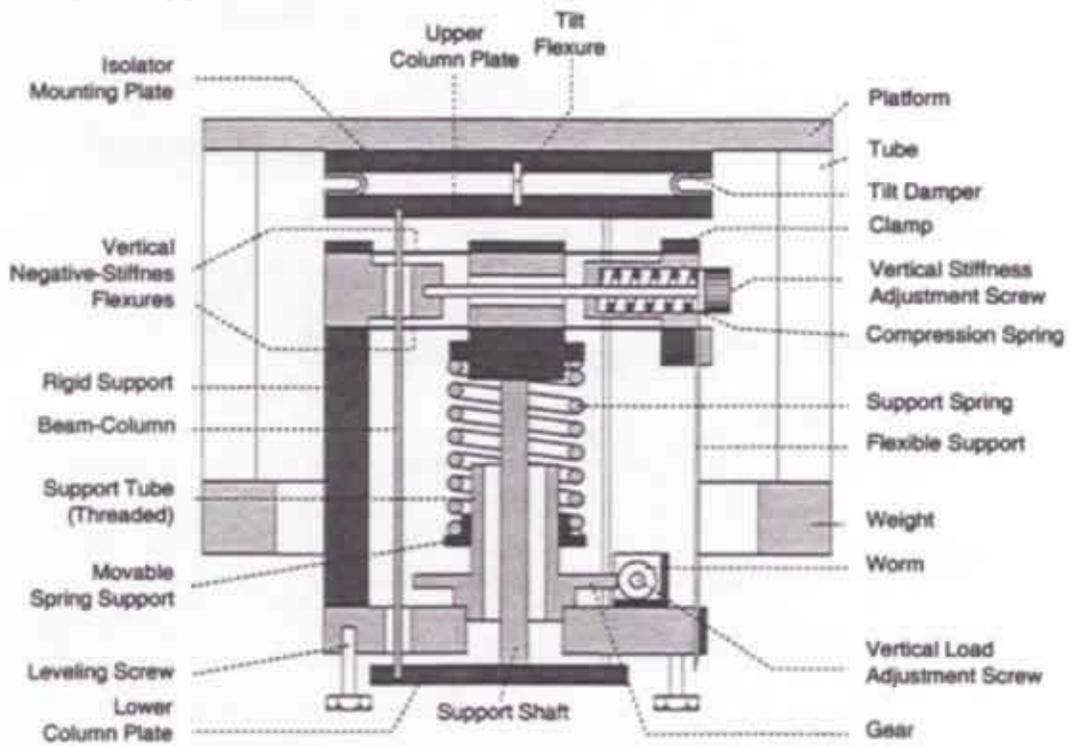
Figure 3 Schematic of Six-DOF Single-Isolator System

Vibration-isolation of supporting joint

○●○●○ — difference of vibration in direction X
□■□■□ — difference of vibration in direction Z
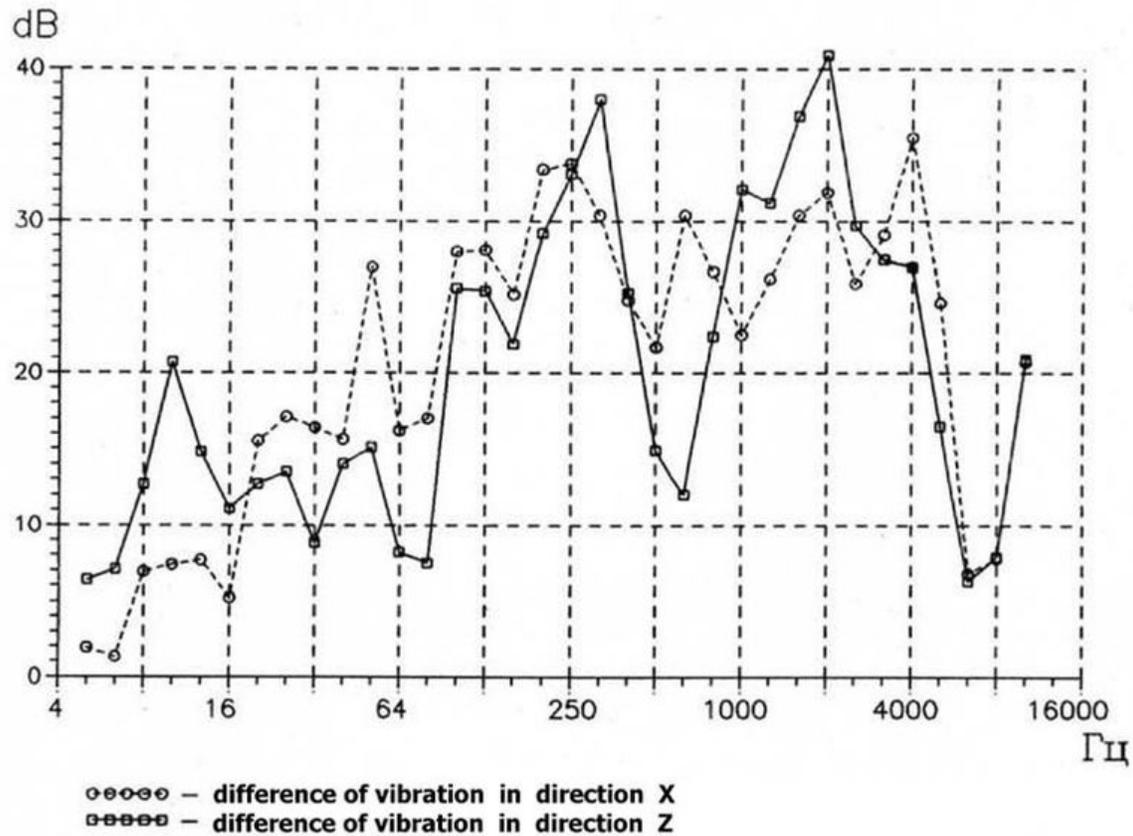
The equipment and gears have joint with surrounding objects (the supporting joint - with the support; the unsupporting joint - the pipe duct or cable). **Vibration-isolation of supporting joint** is realized in the device named vibration-isolator (absorber). On an illustration presented dependence of difference is levels of vibrations which are measured before installation of the functioning gear on vibration-isolator and after installation in a wide range of frequencies.

Vibration-isolator

Vibration-isolator A device that reflects and absorbs waves of oscillatory energy, extending from the working gear or an electrical equipment, with the aid of effect of a vibration insulation. Vibration-isolator is established between a body transferring fluctuations and a body which defend (for example, between the gear and the foundation). On an illustration is presented the image vibration-isolator a series «ВИ» which are applied in shipbuilding of Russia. Shown «ВИ» allow loadings 5, 40 and 300 kg. They differ in the sizes, but have a similar structure . In a structure is used the rubber envelope, which is reinforced by a spring. Rubber and a spring are strongly connected during transformation of crude rubber into rubber envelope by a method of vulcanization. Under action of weight loading of the gear the rubber envelope is of deformation , and a spring are compressed or stretch. Thus, in springs cross section, occurs the twig twist with a material of rubber envelope, causing deformation of shift in rubber envelope. It is known, that the vibration insulation basically cannot be carried out without presence of vibration absorption . The size of deformation of shift in elastic material of isolator- vibration it basis for definition of size of absorption of fluctuations. At action of vibration or shock loadings of deformation increase. Being thus cyclic, it considerably strengthens efficiency of the given device. In the upper part of a design the sleeve, and in the lower part a flange by means of which the vibration-isolator fastens to the gear and the foundation.

Vibration-isolation of unsupporting joint

Vibration-isolation of unsupporting joint is realized in the device named branch pipe a of vibration-isolating.

## A vibration-isolating branch pipe

Среда под давлением
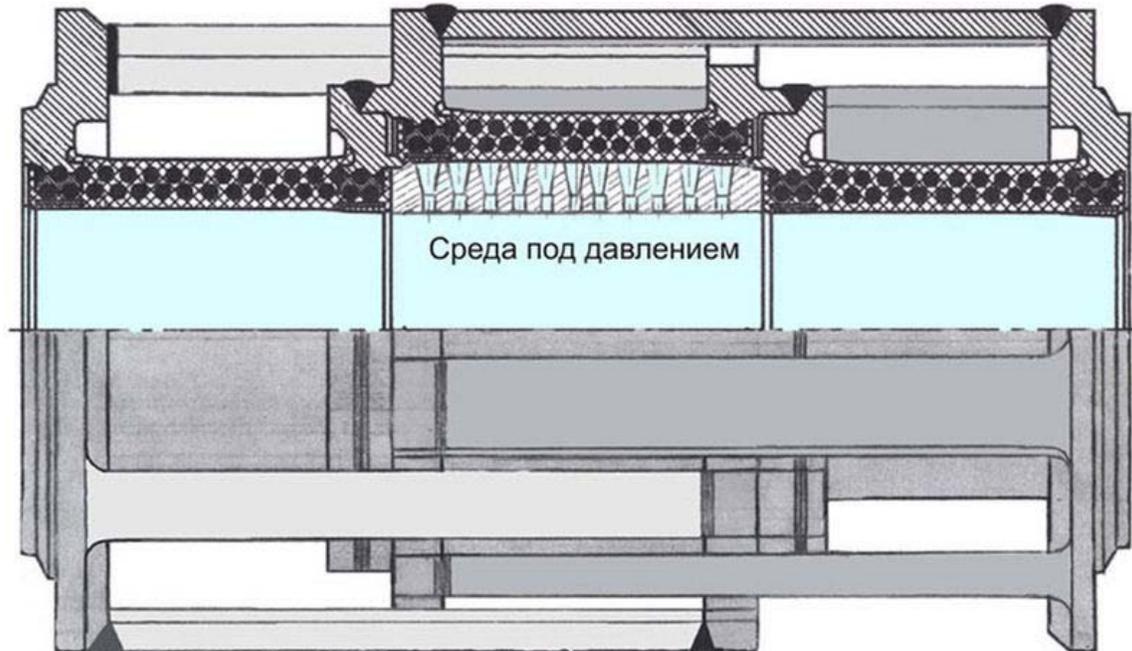
A vibration-isolating branch pipe is a part of a tube with elastic walls for reflection and absorption of waves of the oscillatory energy extending from the working pump over wall of the pipe duct. Is established between the pump and the pipe duct. On an illustration is presented the image a vibration-isolating branch pipe of a series «ВИПБ». In a structure is used the rubber envelope, which is reinforced by a spring. Properties of an envelope are similar envelope to a vibration-isolator. Has the device reducing axial effort from action of internal pressure up to zero.

Active isolation

Active vibration isolation systems contain, along with the spring, a feedback circuit which consists of a piezoelectric accelerometer, a controller, and an electromagnetic transducer. The acceleration (vibration) signal is processed by a control circuit and amplifier. Then it feeds the electromagnetic actuator, which amplifies the signal. As a result of such a feedback system, a considerably stronger suppression of vibrations is achieved compared to ordinary damping.

Subframe isolation
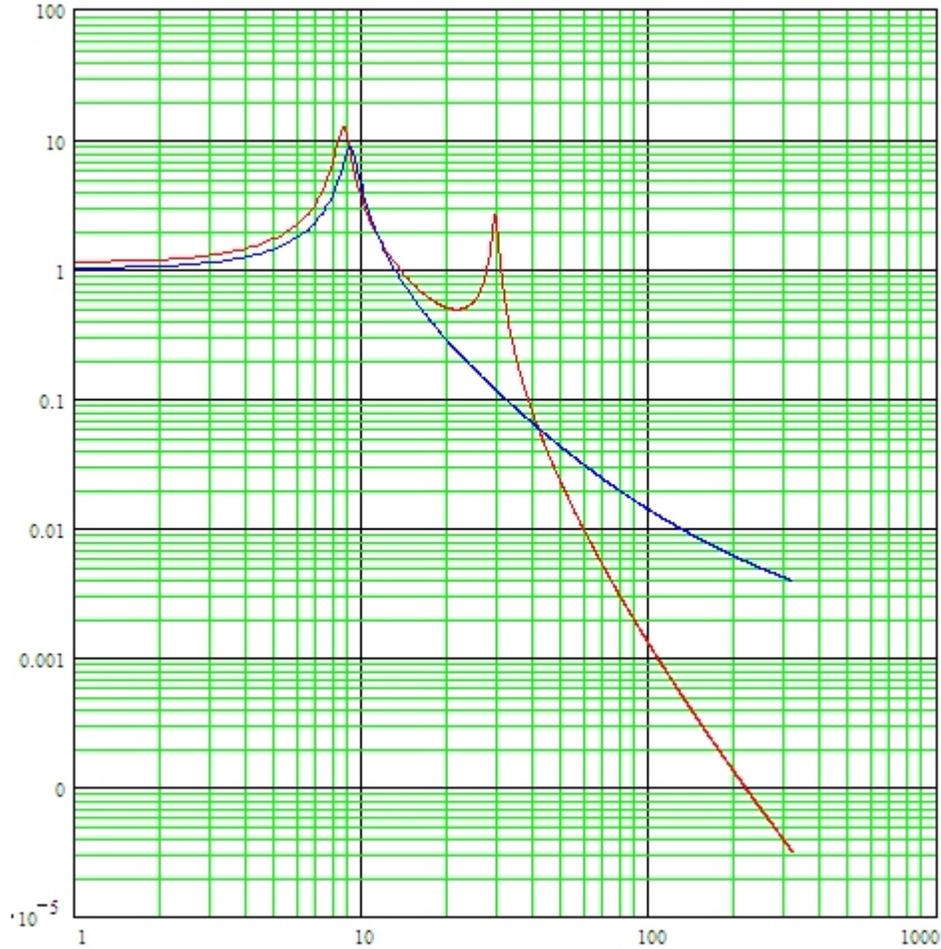
Subframe vibration isolation graph: force transmission on suspended body vs. frequency for rigidly and compliantly mounted subframes.

Another technique used to increase isolation is to use an isolated subframe. This splits the system with an additional mass/spring/damper system. This doubles the high frequency attenuation rolloff, at the cost of introducing additional low frequency modes which may cause the low frequency behaviour to deteriorate. This is commonly used in the rear suspensions of cars with Independent Rear Suspension (IRS), and in the front subframes of some cars. The graph shows the force into the body for a subframe that is rigidly bolted to the body compared with the red curve that shows a compliantly mounted subframe. Above 42 Hz the compliantly mounted subframe is superior, but below that frequency the bolted in subframe is better.

Chapter 10

Vibrations of a Circular Drum
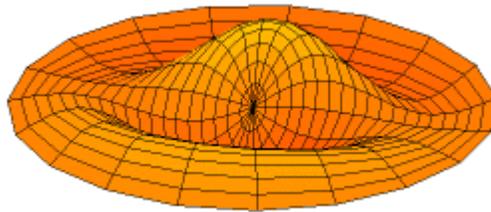
One of the possible modes of vibration of an idealized circular drum (mode u_{12} with the notation below).

The vibrations of an idealized circular drum, essentially an elastic membrane of uniform thickness attached to a rigid circular frame, are solutions of the wave equation with zero boundary conditions.

There exist infinitely many ways in which a drum can vibrate, depending on the shape of the drum at some initial time and the rate of change of the shape of the drum at the initial time. Using separation of variables, it is possible to find a collection of "simple" vibration modes, and it can be proved that any arbitrarily complex vibration of a drum can be decomposed as a series of the simpler vibrations (analogously to the Fourier series).

Motivation

The most obvious relevance of the vibrating drum problem is to the analysis of certain percussion instruments such as drums and timpani. However, there is also a biological application in the working of the eardrum. From an educational point of view the modes of a two-dimensional object are a convenient way to visually demonstrate the meaning of modes, nodes, antinodes and even quantum numbers. These concepts are important to the understanding of the structure of the atom.

The problem

Consider an open disk Ω of radius a centered at the origin, which will represent the "still" drum shape. At any time t, the height of the drum shape at a point (x,y) in Ω measured from the "still" drum shape will be denoted by $u(x,y,t)$, which can take both positive and negative values. Let $\partial\Omega$ denote the boundary of Ω, that is, the circle of radius a centered at the origin, which represents the rigid frame to which the drum is attached.

The mathematical equation that governs the vibration of the drum is the wave equation with zero boundary conditions,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \text{ for } (x,y) \in \Omega$$
$$u = 0 \text{ on } \partial\Omega.$$

Here, c is a positive constant, which gives the "speed" of vibration.

Due to the circular geometry, it will be convenient to use polar coordinates, r and θ. Then, the above equations are written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right) \text{ for } 0 \le r < a, 0 \le \theta \le 2\pi$$
$$u = 0 \text{ for } r = a.$$

The radially symmetric case

We will first study the possible modes of vibration of a circular drum that are radially symmetric. Then, the function u does not depend on the angle θ, and the wave equation simplifies to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right).$$

We will look for solutions in separated variables, $u(r,t) = R(r)T(t)$. Substituting this in the equation above and dividing both sides by $c^2 R(r)T(t)$ yields

$$\frac{T''(t)}{c^2 T(t)} = \frac{1}{R(r)} \left(R''(r) + \frac{1}{r}R'(r) \right).$$

The left-hand side of this equality does not depend on r, and the right-hand side does not depend on t, it follows that both sides must equal to some constant K. We get separate equations for $T(t)$ and $R(r)$:

$$T''(t) = Kc^2 T(t)$$
$$rR''(r) + R'(r) - KrR(r) = 0.$$

The equation for $T(t)$ has solutions which exponentially grow or decay for $K > 0$, are linear or constant for $K = 0$, and are periodic for $K < 0$. Physically it is expected that a solution to the problem of a vibrating drum will be oscillatory in time, and this leaves only the third case, $K < 0$, when $K = -\lambda^2$ (Note that this λ actually plays the role of a wavevector, which is often denoted by k). Then, $T(t)$ is a linear combination of sine and cosine functions,

$$T(t) = A \cos c\lambda t + B \sin c\lambda t.$$

Turning to the equation for $R(r)$, with the observation that $K = -\lambda^2$, all solutions of this second-order differential equation are a linear combination of Bessel functions of order 0,

$$R(r) = c_1 J_0(\lambda r) + c_2 Y_0(\lambda r).$$

The Bessel function Y_0 is unbounded for $r \to 0$, which results in an unphysical solution to the vibrating drum problem, so the constant c_2 must be null. We will also assume $c_1 = 1$, as otherwise this constant can be absorbed later into the constants A and B coming from $T(t)$. It follows that

$$R(r) = J_0(\lambda r).$$

The requirement that height u be zero on the boundary of the drum results in the condition

$$R(a) = J_0(\lambda a) = 0.$$

The Bessel function J_0 has an infinite number of positive roots,

$$0 < \alpha_{01} < \alpha_{02} < \cdots$$

We get that $\lambda a = \alpha_{0n}$, for $n = 1, 2, \ldots$, so

$$R(r) = J_0\left(\frac{\alpha_{0n}}{a} r\right).$$

Therefore, the radially symmetric solutions u of the vibrating drum problem that can be represented in separated variables are

$$u_{0n}(r, t) = (A \cos c\lambda_{0n} t + B \sin c\lambda_{0n} t) J_0(\lambda_{0n} r) \text{ for } n = 1, 2, \ldots,$$

where $\lambda_{0n} = \alpha_{0n} / a$.

The general case

The general case, when u can also depend on the angle θ, is treated similarly. We assume a solution in separated variables,

$$u(r, \theta, t) = R(r)\Theta(\theta)T(t).$$

Substituting this into the wave equation and separating the variables, gives

$$\frac{T''(t)}{c^2 T(t)} = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2 \Theta(\theta)} = K$$

where K is a constant. As before, from the equation for $T(t)$ it follows that $K = -\lambda^2$ with $\lambda > 0$ and

$$T(t) = A \cos c\lambda t + B \sin c\lambda t.$$

From the equation

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Theta''(\theta)}{r^2 \Theta(\theta)} = -\lambda^2$$

we obtain, by multiplying both sides by r^2 and separating variables, that

$$\lambda^2 r^2 + \frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = L$$

and

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = L,$$

for some constant L. Since $\Theta(\theta)$ is periodic, with period 2π, θ being an angular variable, it follows that

$$\Theta(\theta) = C \cos m\theta + D \sin m\theta,$$

where $m = 0, 1, \ldots$ and C and D are some constants. This also implies $L = m^2$.

Going back to the equation for $R(r)$, its solution is a linear combination of Bessel functions J_m and Y_m. With a similar argument as in the previous section, we arrive at

$$R(r) = J_m(\lambda_{mn} r), m = 0, 1, \ldots, n = 1, 2, \ldots,$$

where $\lambda_{mn} = \alpha_{mn} / a$, with α_{mn} the n-th positive root of J_m.
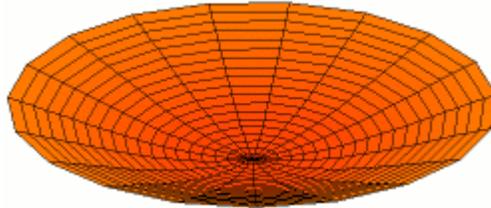
We showed that all solutions in separated variables of the vibrating drum problem are of the form

$$u_{mn}(r, \theta, t) = (A \cos c\lambda_{mn} t + B \sin c\lambda_{mn} t) \, J_m(\lambda_{mn} r) \, (C \cos m\theta + D \sin m\theta)$$
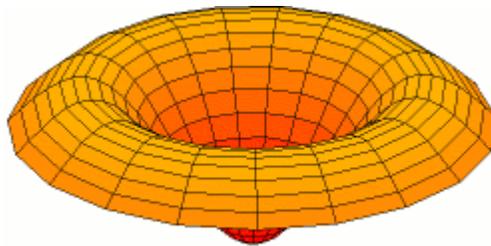
for $m = 0, 1, \ldots, n = 1, 2, \ldots$
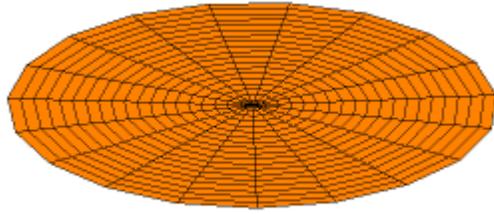
Animations of several vibration modes

A number of modes are shown below together with their quantum numbers. The analogous wave functions of the hydrogen atom are also indicated.

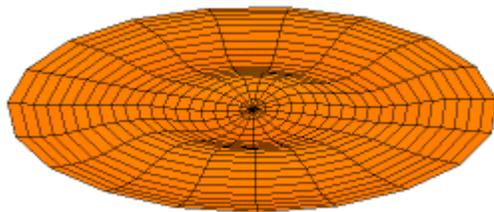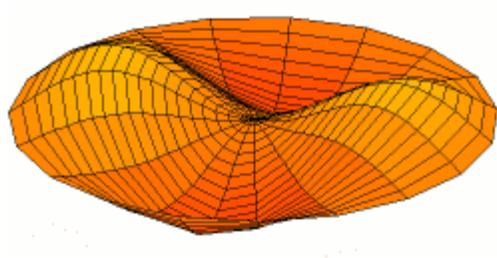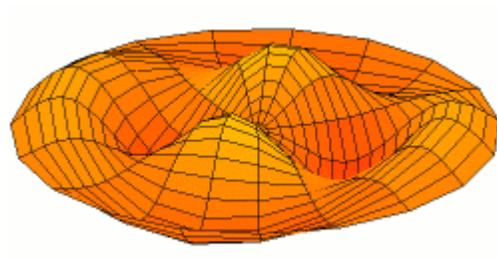Mode u_{01} (1s)
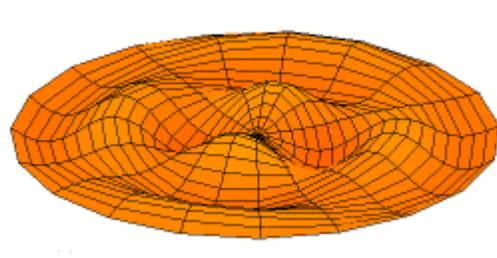

Mode u_{02} (2s)

Mode u_{03} (3s)
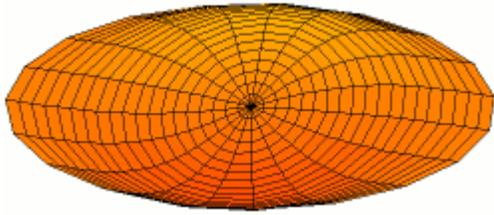
Mode u_{11} (2p)
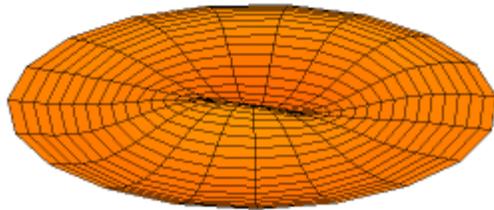
Mode u_{12} (3p)

Mode u_{13} (4p)

Mode u_{21} (3d)

Mode u_{22} (4d)

Mode u_{23} (5d)

Frequency

f = 0.5 Hz
T = 2.0 s

f = 1.0 Hz
T = 1.0 s

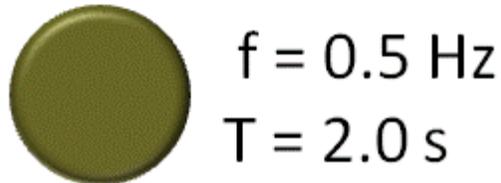f = 2.0 Hz
T = 0.5 s
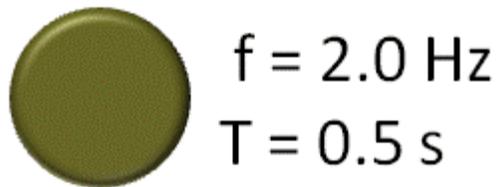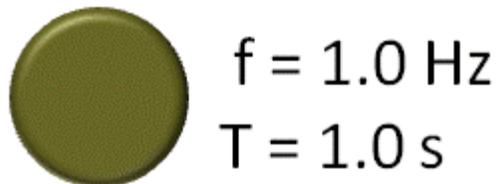
Three flashing lights, from lowest frequency (top) to highest frequency (bottom). f is the frequency in Hertz ("Hz"), meaning the number of flashes per second. T is the period in seconds ("s"), meaning the number of seconds per flash. T and f are reciprocals.

Frequency is the number of occurrences of a repeating event per unit time. It is also referred to as **temporal frequency**. The **period** is the duration of one cycle in a repeating event, so the period is the reciprocal of the frequency. Loosely speaking, 1 year is the

period of the Earth's orbit around the Sun, and the Earth's rotation on its axis has a frequency of 1 rotation per day.

Definitions and units

For cyclical processes, such as rotation, oscillations, or waves, frequency is defined as a number of cycles per unit time. In physics and engineering disciplines, such as optics, acoustics, and radio, frequency is usually denoted by a Latin letter *f* or by a Greek letter *v* (nu).

In SI units, the unit of frequency is the hertz (Hz), named after the German physicist Heinrich Hertz: 1 Hz means that an event repeats once per second. A previous name for this unit was *cycles per second*.
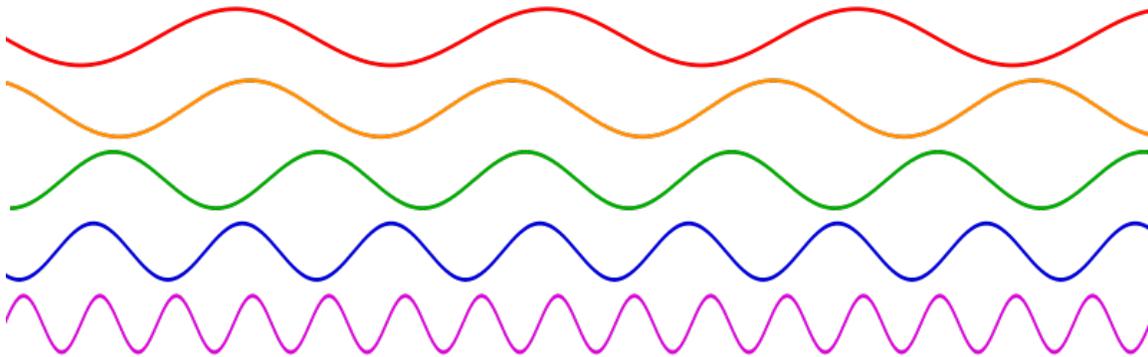
A traditional unit of measure used with rotating mechanical devices is revolutions per minute, abbreviated *RPM*. 60 RPM equals one hertz.

The period, usually denoted by *T,* is the length of time taken by one cycle, and is the reciprocal of the frequency *f*:

$$T = \frac{1}{f}$$

The SI unit for period is the second.

Measurement

Sinusoidal waves of various frequencies; the bottom waves have higher frequencies than those above. The horizontal axis represents time.

By counting

Calculating the frequency of a repeating event is accomplished by counting the number of times that event occurs within a specific time period, then dividing the count by the

length of the time period. For example, if 71 events occur within 15 seconds the frequency is:

$$f = \frac{71}{15\,\text{sec}} = 4.7\,\text{hertz}$$

If the number of counts is not very large, it is more accurate to measure the time interval for a predetermined number of occurrences, rather than the number of occurrences within a specified time. The latter method introduces a random error into the count of between zero and one count, so on average half a count. This is called *gating error* and causes an average error in the calculated frequency of $\Delta f = 1/(2\,T_m)$, or a fractional error of $\Delta f / f = 1/(2\,f\,T_m)$ where T_m is the timing interval and f is the measured frequency. This error decreases with frequency, so it is a problem at low frequencies where the number of counts N is small.

By stroboscope

An older method of measuring the frequency of rotating or vibrating objects is to use a stroboscope. This is an intense repetitively flashing light (strobe light) whose frequency can be adjusted with a calibrated timing circuit. The strobe light is pointed at the rotating object and the frequency adjusted up and down. When the frequency of the strobe equals the frequency of the rotating or vibrating object, the object completes one cycle of oscillation and returns to its original position between the flashes of light, so when illuminated by the strobe the object appears stationary. Then the frequency can be read from the calibrated readout on the stroboscope. A downside of this method is that an object rotating at a multiple of the strobing frequency will also appear stationary.

By frequency counter

Higher frequencies are usually measured with a frequency counter. This is an electronic instrument which measures the frequency of an applied repetitive electronic signal and displays the result in hertz on a digital display. It uses digital logic to count the number of cycles during a time interval established by a precision quartz time base. Cyclic processes that are not electrical in nature, such as the rotation rate of a shaft, mechanical vibrations, or sound waves, can be converted to a repetitive electronic signal by transducers and the signal applied to a frequency counter. Frequency counters can currently cover the range up to about 100 GHz. This represents the limit of direct counting methods; frequencies above this must be measured by indirect methods.

Heterodyne methods

Above the range of frequency counters, frequencies of electromagnetic signals are often measured indirectly by means of heterodyning (frequency conversion). A reference signal of a known frequency near the unknown frequency is mixed with the unknown frequency in a nonlinear mixing device such as a diode. This creates a heterodyne or "beat" signal at the difference between the two frequencies, which is low enough to be measured by a

frequency counter. Of course, this process just measures the unknown frequency by its offset from the reference frequency, which must be determined by some other method. To reach higher frequencies, several stages of heterodyning can be used. Current research is extending this method to infrared and light frequencies (optical heterodyne detection).

Frequency of waves

Frequency has an inverse relationship to the concept of wavelength; simply, frequency is inversely proportional to wavelength λ (lambda). The frequency f is equal to the phase velocity v of the wave divided by the wavelength λ of the wave:

$$f = \frac{v}{\lambda}.$$

In the special case of electromagnetic waves moving through a vacuum, then $v = c$, where c is the speed of light in a vacuum, and this expression becomes:

$$f = \frac{c}{\lambda}.$$

When waves from a monochrome source travel from one medium to another, their frequency remains exactly the same — only their wavelength and speed change.

Examples

Physics of light

Visible light is an electromagnetic wave, consisting of oscillating electric and magnetic fields traveling through space. The frequency of the wave determines its color: 4×10^{14} Hz is red light, 8×10^{14} Hz is violet light, and between these (in the range 4-8×10^{14} Hz) are all the other colors of the rainbow. An electromagnetic wave can have a frequency less than 4×10^{14} Hz, but it will be invisible to the human eye; such waves are called infrared (IR) radiation. At even lower frequency, the wave is called a microwave, and at still lower frequencies it is called a radio wave. Likewise, an electromagnetic wave can have a frequency higher than 8×10^{14} Hz, but it will be invisible to the human eye; such waves are called ultraviolet (UV) radiation. Even higher-frequency waves are called X-rays, and higher still are gamma rays.

All of these waves, from the lowest-frequency radio waves to the highest-frequency gamma rays, are fundamentally the same, and they are all called electromagnetic radiation. They all travel through a vacuum at the speed of light.

Another property of an electromagnetic wave is its wavelength. The wavelength is inversely proportional to the frequency, so an electromagnetic wave with a higher frequency has a shorter wavelength, and vice-versa.
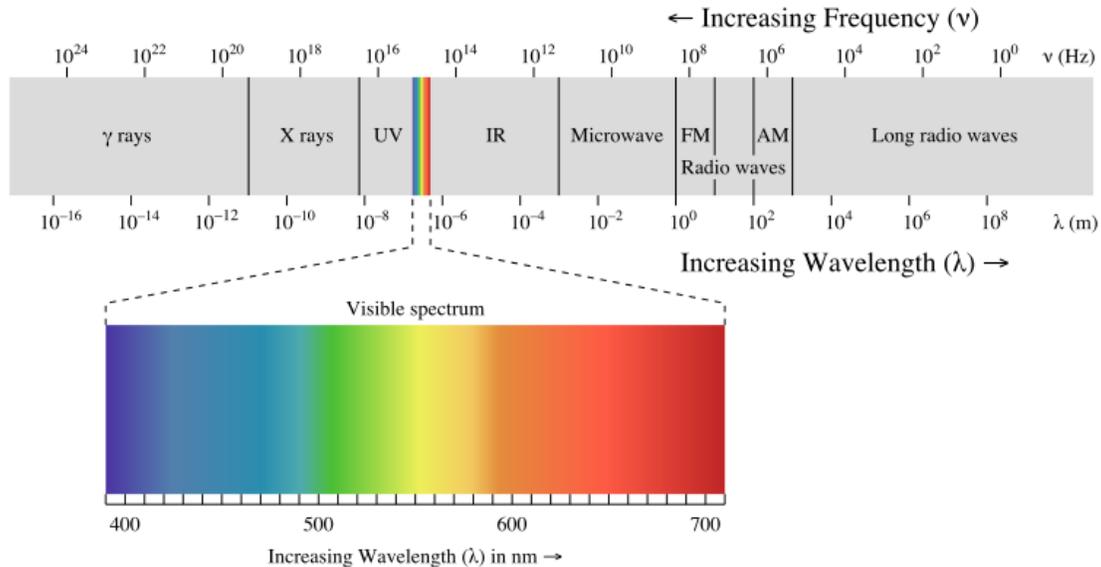
Complete spectrum of electromagnetic radiation with the visible portion highlighted

Physics of sound

Sound is vibration transmitted through a solid, liquid, or gas; particularly, sound means those vibrations composed of frequencies able to be detected by ears. For humans, hearing is limited to frequencies between about 20 Hz and 20,000 Hz (20 kHz), with the upper limit generally decreasing with age. Other species have a different range of hearing. For example, some dog breeds can perceive vibrations up to 60,000 Hz. As a signal perceived by one of the major senses, sound is used by many species for detecting danger, navigation, predation, and communication.

The mechanical vibrations that can be interpreted as sound are able to travel through all forms of matter: gases, liquids, solids, and plasmas. The matter that supports the sound is called the medium. Sound cannot travel through vacuum.

Other examples

In Europe, Africa, Australia, Southern South America, most of Asia, and Russia, the frequency of the alternating current in household electrical outlets is 50 Hz (close to the tone G), whereas in North America and Northern South America, the frequency of the alternating current in household electrical outlets is 60 Hz (between the tones B ♭ and B; that is, a minor third above the European frequency). The frequency of the 'hum' in an

audio recording can show where the recording was made, in countries using a European, or an American, grid frequency.

Period versus frequency

As a matter of convenience, longer and slower waves, such as ocean surface waves, tend to be described by wave period rather than frequency. Short and fast waves, like audio and radio, are usually described by their frequency instead of period. These commonly used conversions are listed below:

Frequency	1 mHz (10^{-3})	1 Hz (10^{0})	1 kHz (10^{3})	1 MHz (10^{6})	1 GHz (10^{9})	1 THz (10^{12})
Period (time)	1 ks (10^{3})	1 s (10^{0})	1 ms (10^{-3})	1 µs (10^{-6})	1 ns (10^{-9})	1 ps (10^{-12})

Other types of frequency
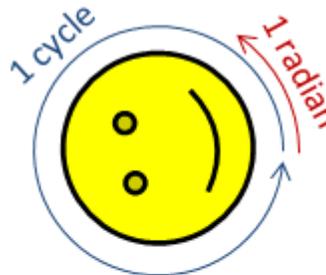
Time (in seconds) = 1.00 s
Rotation (in radians) = 1.75 rad
Rotation (in cycles) = 0.28 cycle

$$\omega = \frac{1.75\ \text{rad}}{1.00\ \text{s}} = 1.75\ \text{rad/s}$$

$$v = \frac{0.28\ \text{cycle}}{1.00\ \text{s}} = 0.28\ \text{Hz}$$

Angular frequency ω (in radians per second), is larger than frequency ν (in cycles per second, also called Hz), by a factor of 2π.

- Angular frequency *ω* is defined as the rate of change of angular displacement (during rotation), or in the phase of a sinusoidal waveform (e.g. in oscillations and waves):

$$\omega = 2\pi f.$$

In other words,

$$\sin(\omega t) = \sin(2\pi f t).$$

Angular frequency is measured in radians per second (rad/s).

- Spatial frequency is analogous to temporal frequency, but the time axis is replaced by one or more spatial displacement axes.
- Wavenumber is the spatial analogue of angular frequency. In case of more than one spatial dimension, wavenumber is a vector quantity.

Frequency ranges

Many systems are characterized by the range of frequencies to which they respond. Musical instruments produce different ranges of notes within the hearing range. The electromagnetic spectrum can be divided into many different ranges such as visible light, infrared or ultraviolet radiation, radio waves, X-rays and so on, and each of these ranges can in turn be divided into smaller ranges. A radio communications signal must occupy a range of frequencies carrying most of its energy, called its bandwidth. Allocation of radio frequency ranges to different uses is a major function of radio spectrum allocation.

The frequency range of a system is the range over which is is considered to provide a useful level of signal with acceptable distortion characteristics. A listing of the upper and lower limits of frequency limits for a system is not useful without a criterion for what the range represents.

Chapter 12

Harmonic Oscillator
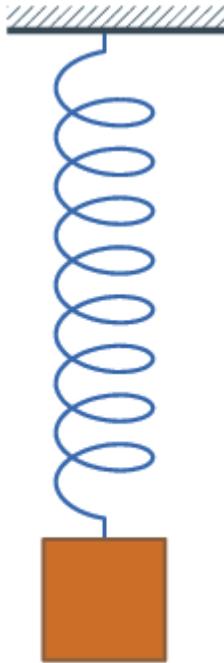
An undamped spring-mass system is a simple harmonic oscillator.

In classical mechanics, a **harmonic oscillator** is a system that, when displaced from its equilibrium position, experiences a restoring force, F, proportional to the displacement, x:

$$F = -kx$$

where k is a positive constant.

If F is the only force acting on the system, the system is called a **simple harmonic oscillator**, and it undergoes simple harmonic motion: sinusoidal oscillations about the

equilibrium point, with a constant amplitude and a constant frequency (which does not depend on the amplitude).

If a frictional force (damping) proportional to the velocity is also present, the harmonic oscillator is described as a **damped oscillator**. Depending on the friction coefficient, the system can:

- Oscillate with a frequency smaller than in the non-damped case, and an amplitude decreasing with time (*underdamped* oscillator).
- Decay exponentially to the equilibrium position, without oscillations (*overdamped* oscillator).

If an external time dependent force is present, the harmonic oscillator is described as a **driven oscillator**.

Mechanical examples include pendula (with small angles of displacement), masses connected to springs, and acoustical systems. Other analogous systems include electrical harmonic oscillators such as RLC circuits. The harmonic oscillator model is very important in physics, because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. Harmonic oscillators occur widely in nature and are exploited in many manmade devices, such as clocks and radio circuits. They are the source of virtually all sinusoidal vibrations and waves.

Simple harmonic oscillator
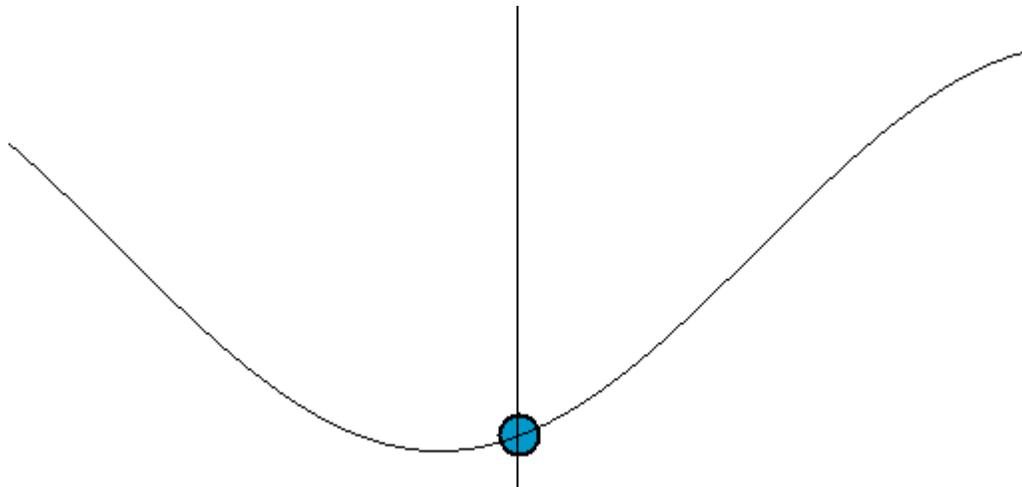
Simple harmonic motion.

A simple harmonic oscillator is an oscillator that is neither driven nor damped. It consists of a mass m, which experiences a single force, F, which pulls the mass in the direction of the point $x=0$ and depends only on the mass's position x and a constant k. Newton's second law for the system is

$$F = ma = m\frac{d^2x}{dt^2} = -kx.$$

Solving this differential equation, we find that the motion is described by the function

$$x(t) = A\cos\left(2\pi f t + \phi\right),$$

where

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{T}.$$

The motion is periodic— repeating itself in a sinusoidal fashion with constant amplitude, A. In addition to its amplitude, the motion of a simple harmonic oscillator is characterized by its period T, the time for a single oscillation or its frequency $f = \frac{1}{T}$, the number of cycles per unit time. The position at a given time t also depends on the phase, φ, which determines the starting point on the sine wave. The period and frequency are determined by the size of the mass m and the force constant k, while the amplitude and phase are determined by the starting position and velocity.

The velocity and acceleration of a simple harmonic oscillator oscillate with the same frequency as the position but with shifted phases. The velocity is maximum for zero displacement, while the acceleration is in the opposite direction as the displacement.

The potential energy stored in a simple harmonic oscillator at position x is

$$U = \frac{1}{2}kx^2.$$

Damped harmonic oscillator
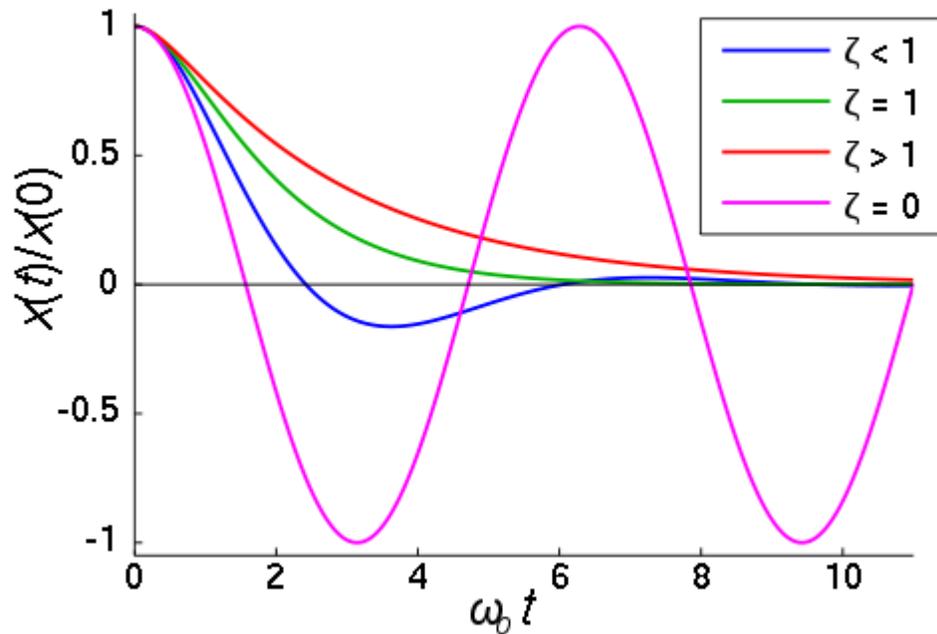
Dependence of the system behavior on the value of the damping ratio ζ.

In real oscillators friction, or **damping**, slows the motion of the system. In many vibrating systems the frictional force F_f can be modeled as being proportional to the velocity v of the object: $F_f = -cv$, where c is called the *viscous damping coefficient*.

Newton's second law for damped harmonic oscillators is then
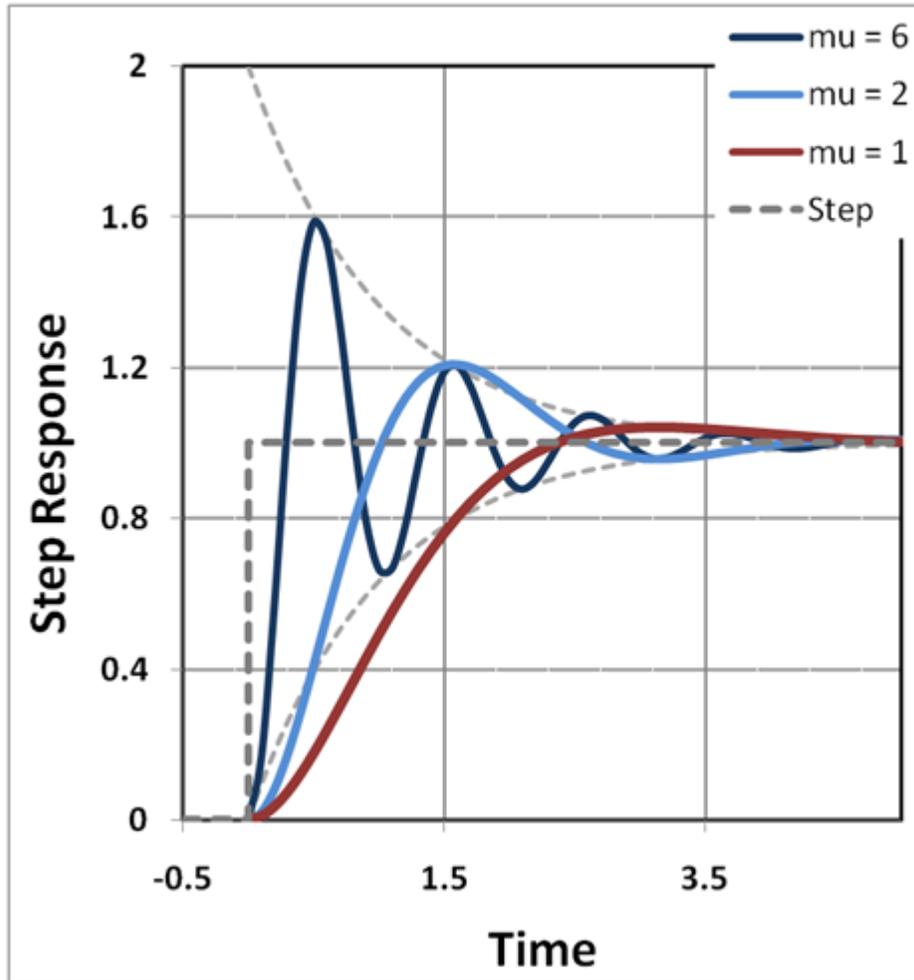
$$F = -kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}.$$

This is rewritten into the form

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = 0,$$

where

$\omega_0 = \sqrt{\dfrac{k}{m}}$ is called the 'undamped angular frequency of the oscillator' and

$\zeta = \dfrac{c}{2m\omega_0}$ is called the 'damping ratio'.

Step-response of a damped harmonic oscillator; curves are plotted for three values of $\mu =$ $\omega_1 = \omega_0\sqrt{1-\zeta^2}$. Time is in units of the decay time $\tau = 1/(\zeta\omega_0)$.

The value of the damping ratio ζ critically determines the behavior of the system. A damped harmonic oscillator can be:

- *Overdamped* ($\zeta > 1$): The system returns (exponentially decays) to equilibrium without oscillating. Larger values of the damping ratio ζ return to equilibrium slower.
- *Critically damped* ($\zeta = 1$): The system returns to equilibrium as quickly as possible without oscillating. This is often desired for the damping of systems such as doors.
- *Underdamped* ($\zeta < 1$): The system oscillates (with a slightly different frequency than the undamped case) with the amplitude gradually decreasing to zero. The angular frequency of the underdamped harmonic oscillator is given by

$$\omega_1 = \omega_0\sqrt{1 - \zeta^2}.$$

The Q factor of a damped oscillator is defined as

$$Q = 2\pi \times \frac{\text{Energy stored}}{\text{Energy lost per cycle}}$$

Q is related to the damping ratio by the equation $Q = \dfrac{1}{2\zeta}$.

Driven harmonic oscillators

Driven harmonic oscillators are damped oscillators further affected by an externally applied force $F(t)$.

Newton's second law takes the form

$$F(t) - kx - c\frac{dx}{dt} = m\frac{d^2x}{dt^2}.$$

It is usually rewritten into the form

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m}.$$

This equation can be solved exactly for any driving force using the solutions $z(t)$ to the unforced equation, which satisfy:

$$\frac{d^2z}{dt^2} + 2\zeta\omega_0\frac{dz}{dt} + \omega_0^2 z = 0,$$

which can be expressed as a damped sinusoidal oscillation:

$$z(t) = Ae^{-\zeta\omega_0 t}\sin\left(\sqrt{1-\zeta^2}\,\omega_0 t + \varphi\right),$$

in the case where $\zeta \leq 1$. The amplitude A and phase φ determine the behavior needed to match the initial conditions.

Step input

In the case $\zeta < 1$ and a unit step input with $x(0) = 0$:

$$F(t) = \begin{cases} \omega_0^2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

the solution is:

$$x(t) = 1 - e^{-\zeta \omega_0 t} \frac{\sin\left(\sqrt{1 - \zeta^2}\,\omega_0 t + \varphi\right)}{\sin(\varphi)} \ ,$$

with phase φ given by

$$\cos\varphi = \zeta.$$

This behavior is found in (for example) feedback amplifiers, where the amplifier design is adjusted to obtain the fastest step response possible without undue overshoot or undershoot and with an adequate settling time.

The time an oscillator needs to adapt to changed external conditions is of the order $\tau = 1/(\zeta\omega_0)$. In physics, the adaptation is called relaxation, and τ is called the relaxation time.

In electrical engineering, a multiple of τ is called the *settling time*, i.e. the time necessary to insure the signal is within a fixed departure from final value, typically within 10%. The term *overshoot* refers to the extent the maximum response exceeds final value, and *undershoot* refers to the extent the response falls below final value for times following the maximum response.

Sinusoidal driving force

In the case of a sinusoidal driving force:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_0\frac{dx}{dt} + \omega_0^2 x = \frac{1}{m}F_0\sin(\omega t)\ ,$$

where F_0 is the driving amplitude and ω is the driving frequency for a sinusoidal driving mechanism. This type of system appears in AC driven RLC circuits (resistor-inductor-capacitor) and driven spring systems having internal mechanical resistance or external air resistance.

The general solution is a sum of a transient solution that depends on initial conditions, and a steady state that is independent of initial conditions and depends only on the driving amplitude F_0, driving frequency, ω, undamped angular frequency ω_0, and the damping ratio ζ.

The steady-state solution is proportional to the driving force with an induced phase change of ϕ:

$$x(t) = \frac{F_0}{mZ_m\omega}\sin(\omega t + \phi)$$

where

$$Z_m = \sqrt{(2\omega_0\zeta)^2 + \frac{1}{\omega^2}(\omega_0^2 - \omega^2)^2}$$

is the absolute value of the impedance or linear response function and

$$\phi = \arctan\left(\frac{2\omega\omega_0\zeta}{\omega^2 - \omega_0^2}\right)$$

is the phase of the oscillation relative to the driving force.

For a particular driving frequency called the resonance frequency $\omega_r = \omega_0\sqrt{1 - 2\zeta^2}$, the amplitude (for a given F_0) is maximum. For underdamped systems the value of the amplitude can become quite large near the resonance frequency.

The transient solutions are the same as the unforced ($F_0 = 0$) damped harmonic oscillator and represent the systems response to other events that occurred previously. The transient solutions typically die out rapidly enough that they can be ignored.

Parametric oscillators

A parametric oscillator is a harmonic oscillator whose parameters oscillate in time. For example, a well known parametric oscillator is a child on a swing where periodically changing the child's center of gravity causes the swing to oscillate. The varying of the parameters drives the system. Examples of parameters that may be varied are its resonance frequency ω and damping β.

Parametric oscillators are used in many applications. The classical varactor parametric oscillator oscillates when the diode's capacitance is varied periodically. The circuit that varies the diode's capacitance is called the "pump" or "driver". In microwave electronics, waveguide/YAG based parametric oscillators operate in the same fashion. The designer varies a parameter periodically to induce oscillations.

Parametric oscillators have been developed as low-noise amplifiers, especially in the radio and microwave frequency range. Thermal noise is minimal, since a reactance (not a resistance) is varied. Another common use is frequency conversion, e.g., conversion from

audio to radio frequencies. For example, the Optical parametric oscillator converts an input laser wave into two output waves of lower frequency (ω_s, ω_i).

Parametric resonance occurs in a mechanical system when a system is parametrically excited and oscillates at one of its resonant frequencies. Parametric excitation differs from forcing, since the action appears as a time varying modification on a system parameter. This effect is different from regular resonance because it exhibits the instability phenomenon.

Universal oscillator equation

The equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\zeta\frac{\mathrm{d}q}{\mathrm{d}\tau} + q = 0$$

is known as the **universal oscillator equation** since all second order linear oscillatory systems can be reduced to this form. This is done through nondimensionalization.

If the forcing function is $f(t) = \cos(\omega t) = \cos(\omega t_c \tau) = \cos(\omega \tau)$, where $\omega = \omega t_c$, the equation becomes

$$\frac{\mathrm{d}^2 q}{\mathrm{d}\tau^2} + 2\zeta\frac{\mathrm{d}q}{\mathrm{d}\tau} + q = \cos(\omega \tau).$$

The solution to this differential equation contains two parts, the "transient" and the "steady state".

Transient solution

The solution based on solving the ordinary differential equation is for arbitrary constants c_1 and c_2

$$q_t(\tau) = \begin{cases} e^{-\zeta\tau}\left(c_1 e^{\tau\sqrt{\zeta^2-1}} + c_2 e^{-\tau\sqrt{\zeta^2-1}}\right) & \zeta > 1 \text{ (overdamping)} \\ e^{-\zeta\tau}(c_1 + c_2\tau) = e^{-\tau}(c_1 + c_2\tau) & \zeta = 1 \text{ (critical damping)} \\ e^{-\zeta\tau}\left[c_1 \cos\left(\sqrt{1-\zeta^2}\tau\right) + c_2 \sin\left(\sqrt{1-\zeta^2}\tau\right)\right] & \zeta < 1 \text{ (underdamping)} \end{cases}$$

The transient solution is independent of the forcing function.

Steady-state solution

Apply the "complex variables method" by solving the auxiliary equation below and then finding the real part of its solution:

$$\frac{d^2q}{d\tau^2} + 2\zeta\frac{dq}{d\tau} + q = \cos(\omega\tau) + i\sin(\omega\tau) = e^{i\omega\tau}.$$

Supposing the solution is of the form

$$q_s(\tau) = Ae^{i(\omega\tau+\phi)}.$$

Its derivatives from zero to 2nd order are

$$q_s = Ae^{i(\omega\tau+\phi)}, \quad \frac{dq_s}{d\tau} = i\omega Ae^{i(\omega\tau+\phi)}, \quad \frac{d^2q_s}{d\tau^2} = -\omega^2 Ae^{i(\omega\tau+\phi)}.$$

Substituting these quantities into the differential equation gives

$$-\omega^2 Ae^{i(\omega\tau+\phi)} + 2\zeta i\omega Ae^{i(\omega\tau+\phi)} + Ae^{i(\omega\tau+\phi)} = (-\omega^2 A + 2\zeta i\omega A + A)e^{i(\omega\tau+\phi)} = e^{i\omega\tau}.$$
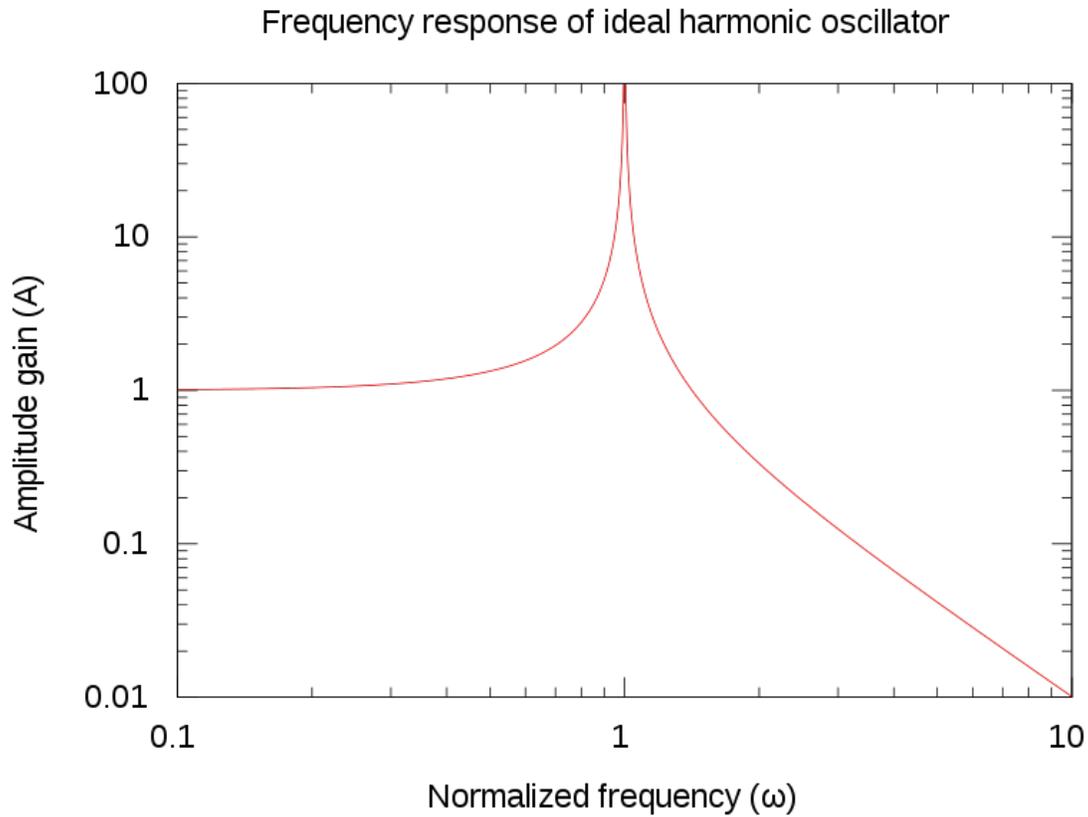
Dividing by the exponential term on the left results in

$$-\omega^2 A + 2\zeta i\omega A + A = e^{-i\phi} = \cos\phi - i\sin\phi.$$

Equating the real and imaginary parts results in two independent equations

$$A(1-\omega^2) = \cos\phi \qquad 2\zeta\omega A = -\sin\phi.$$

Amplitude part

Frequency response of ideal harmonic oscillator

Bode plot of the frequency response of an ideal harmonic oscillator.

Squaring both equations and adding them together gives

$$\left.\begin{array}{l} A^2(1-\omega^2)^2 = \cos^2\phi \\ (2\zeta\omega A)^2 = \sin^2\phi \end{array}\right\} \Rightarrow A^2[(1-\omega^2)^2 + (2\zeta\omega)^2] = 1.$$

By convention the positive root is taken since amplitude is usually considered a positive quantity. Therefore,

$$A = A(\zeta,\omega) = \frac{1}{\sqrt{(1-\omega^2)^2 + (2\zeta\omega)^2}}.$$

Compare this result with the theory section on resonance, as well as the "magnitude part" of the RLC circuit. This amplitude function is particularly important in the analysis and understanding of the frequency response of second-order systems.

Phase part

To solve for φ, divide both equations to get

$$\tan\phi = -\frac{2\zeta\omega}{1-\omega^2} = \frac{2\zeta\omega}{\omega^2-1} \Rightarrow \phi \equiv \phi(\zeta,\omega) = \arctan\left(\frac{2\zeta\omega}{\omega^2-1}\right).$$

This phase function is particularly important in the analysis and understanding of the frequency response of second-order systems.

Full solution

Combining the amplitude and phase portions results in the steady-state solution

$$q_s(\tau) = A(\zeta,\omega)\cos(\omega\tau + \phi(\zeta,\omega)) = A\cos(\omega\tau + \phi).$$

The solution of original universal oscillator equation is a superposition (sum) of the transient and steady-state solutions

$$q(\tau) = q_t(\tau) + q_s(\tau).$$

Equivalent systems

Harmonic oscillators occurring in a number of areas of engineering are equivalent in the sense that their mathematical models are identical. Below is a table showing analogous quantities in four harmonic oscillator systems in mechanics and electronics. If analogous parameters on the same line in the table are given numerically equal values, the behavior of the oscillators—their output waveform, resonant frequency, damping factor, etc.—are the same.

Translational Mechanical	Torsional Mechanical	Series RLC Circuit	Parallel RLC Circuit
Position x	Angle θ	Charge q	Voltage e
Velocity $\dfrac{dx}{dt}$	Angular velocity $\dfrac{d\theta}{dt}$	Current $\dfrac{dq}{dt}$	$\dfrac{de}{dt}$
Mass M	Moment of inertia I	Inductance L	Capacitance C
Spring constant K	Torsion constant μ	Elastance $1/C$	Susceptance $1/L$
Friction γ	Rotational friction Γ	Resistance R	Conductance $1/R$
Drive force $F(t)$	Drive torque $\tau(t)$	e	di/dt

Undamped resonant frequency f_n:

$$\frac{1}{2\pi}\sqrt{\frac{K}{M}} \qquad \frac{1}{2\pi}\sqrt{\frac{\mu}{I}} \qquad \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \qquad \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$$

Differential equation:

$$M\ddot{x} + \gamma\dot{x} + Kx = \qquad I\ddot{\theta} + \Gamma\dot{\theta} + \mu\theta = \qquad L\ddot{q} + R\dot{q} + q/C \qquad C\ddot{e} + \dot{e}/R + \epsilon$$

Application to a conservative force

The problem of the simple harmonic oscillator occurs frequently in physics, because a mass at equilibrium under the influence of any conservative force, in the limit of small motions, behaves as a simple harmonic oscillator.

A conservative force is one that has a potential energy function. The potential energy function of a harmonic oscillator is:

$$V(x) = \frac{1}{2}kx^2$$

Given an arbitrary potential energy function $V(x)$, one can do a Taylor expansion in terms of x around an energy minimum ($x = x_0$) to model the behavior of small perturbations from equilibrium.

$$V(x) = V(x_0) + (x - x_0)V'(x_0) + \frac{1}{2}(x - x_0)^2 V^{(2)}(x_0) + O(x - x_0)^3$$

Because $V(x_0)$ is a minimum, the first derivative evaluated at x_0 must be zero, so the linear term drops out:

$$V(x) = V(x_0) + \frac{1}{2}(x - x_0)^2 V^{(2)}(x_0) + O(x - x_0)^3$$

The constant term $V(x_0)$ is arbitrary and thus may be dropped, and a coordinate transformation allows the form of the simple harmonic oscillator to be retrieved:

$$V(x) \approx \frac{1}{2}x^2 V^{(2)}(0) = \frac{1}{2}kx^2$$

Thus, given an arbitrary potential energy function $V(x)$ with a non-vanishing second derivative, one can use the solution to the simple harmonic oscillator to provide an approximate solution for small perturbations around the equilibrium point.

Examples

Simple pendulum
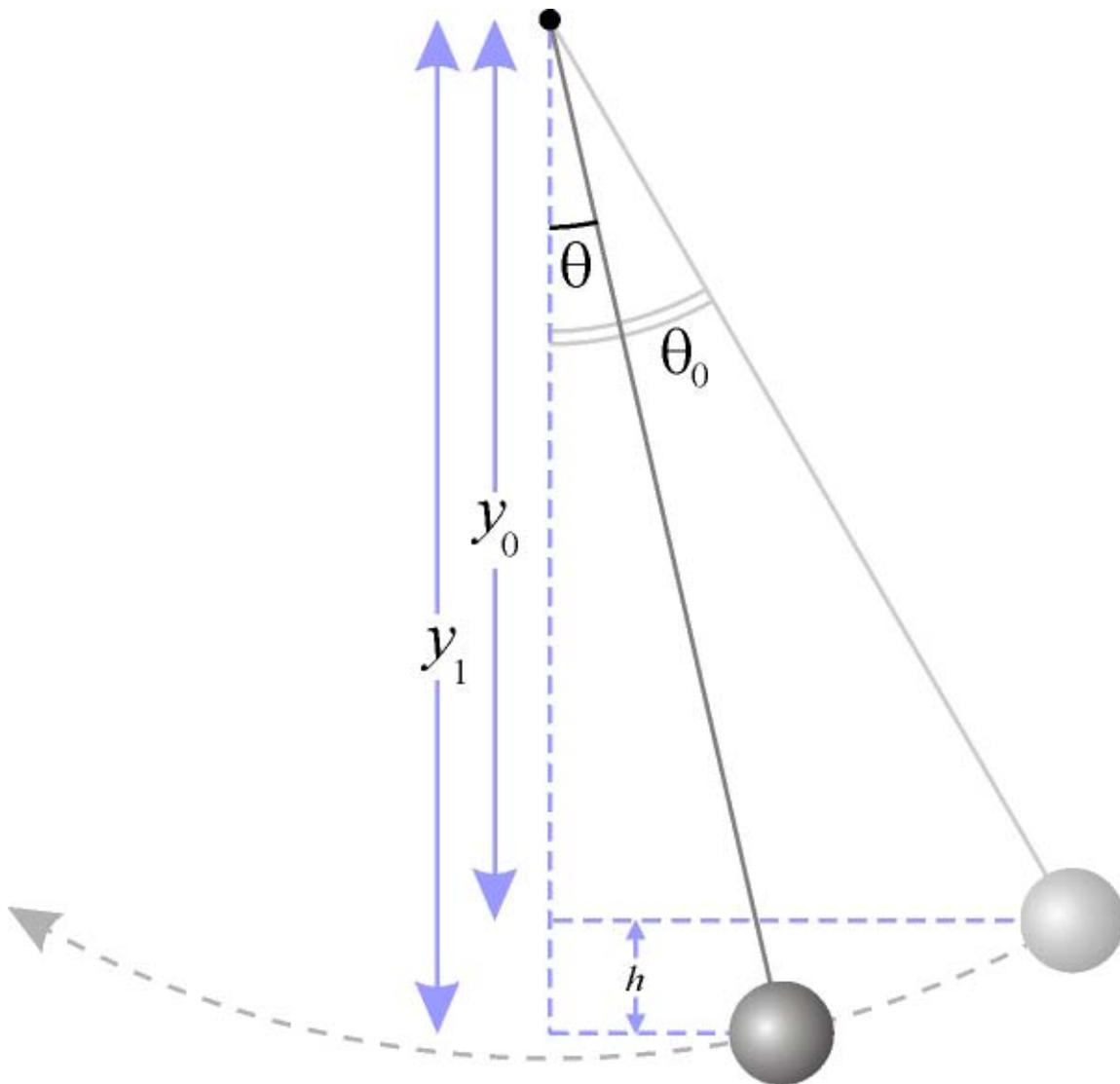
A simple pendulum exhibits simple harmonic motion under the conditions of no damping and small amplitude.

Assuming no damping and small amplitudes, the differential equation governing a simple pendulum is

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \frac{g}{\ell}\theta = 0.$$

The solution to this equation is given by:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}t\right) \qquad |\theta_0| \ll 1$$

where θ_0 is the largest angle attained by the pendulum. The period, the time for one complete oscillation, is given by 2π divided by whatever is multiplying the time in the argument of the cosine ($\sqrt{\frac{g}{\ell}}$ here).

$$T_0 = 2\pi\sqrt{\frac{\ell}{g}} \qquad |\theta_0| \ll 1.$$

Pendulum swinging over turntable

Simple harmonic motion can in some cases be considered to be the one-dimensional projection of two-dimensional circular motion. Consider a long pendulum swinging over the turntable of a record player. On the edge of the turntable there is an object. If the object is viewed from the same level as the turntable, a projection of the motion of the object seems to be moving backwards and forwards on a straight line. When the frequency of rotation of the turntable has perfect synchronization with the motion of the pendulum, the angular speed of the turntable is known as the pulsation of the pendulum. Note that the pulsation of the pendulum is not the angular speed of the pendulum itself, but it is the angular speed of the corresponding circular motion.

In general, the pulsation of straight-line simple harmonic motion is the angular speed of the corresponding circular motion. Mathematically, a motion with period T and frequency $f=1/T$ has pulsation

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Spring–mass system
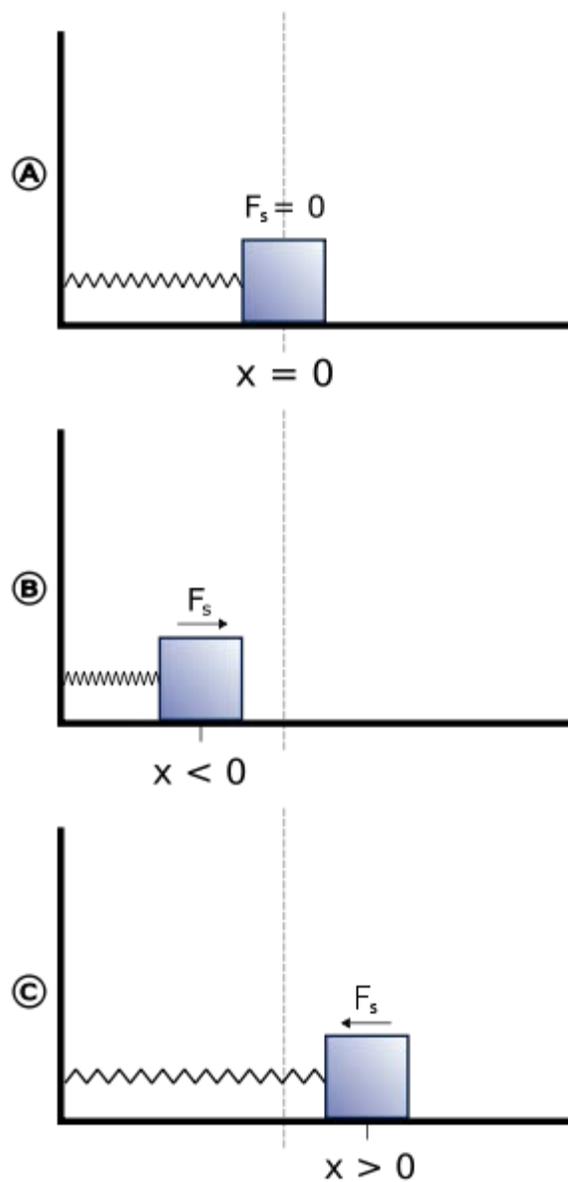
Spring–mass system in equilibrium (A), compressed (B) and stretched (C) states.

When a spring is stretched or compressed by a mass, the spring develops a restoring force. Hooke's law gives the relationship of the force exerted by the spring when the spring is compressed or stretched a certain length:

$$F\left(t\right) = -kx\left(t\right)$$

where F is the force, k is the spring constant, and x is the displacement of the mass with respect to the equilibrium position. This relationship shows that the distance of the spring is always opposite to the force of the spring.

By using either force balance or an energy method, it can be readily shown that the motion of this system is given by the following differential equation:

$$F(t) = -kx(t) = m\frac{d^2}{dt^2}x(t) = ma.$$

...the latter evidently being Newton's second law of motion.

If the initial displacement is A, and there is no initial velocity, the solution of this equation is given by:

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right).$$

Given an ideal massless spring, m is the mass on the end of the spring. If the spring itself has mass, its effective mass must be included in m.

Energy variation in the spring–damping system

In terms of energy, all systems have two types of energy, potential energy and kinetic energy. When a spring is stretched or compressed, it stores elastic potential energy, which then is transferred into kinetic energy. The potential energy within a spring is determined by the equation $U = kx^2 / 2$.

When the spring is stretched or compressed, kinetic energy of the mass gets converted into potential energy of the spring. By conservation of energy, assuming the datum is defined at the equilibrium position, when the spring reaches its maximum potential energy, the kinetic energy of the mass is zero. When the spring is released, it tries to return to equilibrium, and all its potential energy converts to kinetic energy of the mass.