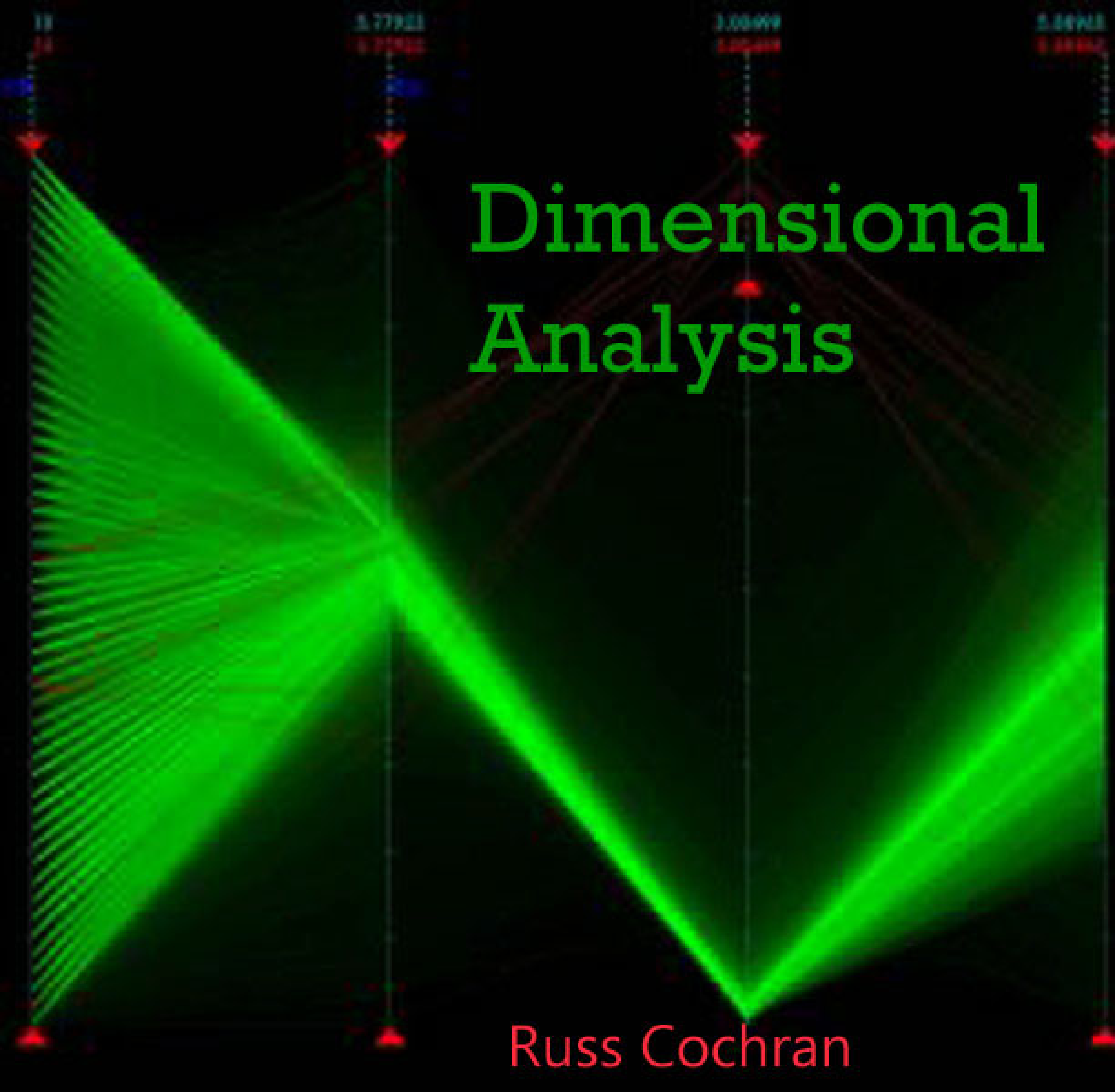


Dimensional Analysis

Russ Cochran

First Edition, 2012

ISBN 978-81-323-4356-1

Published by:
White Word Publications
4735/22 Prakashdeep Bldg,
Ansari Road, Darya Ganj,
Delhi - 110002
Email: info@wtbooks.com

Table of Contents

Chapter 1

Dimensional Analysis

In physics and science, **dimensional analysis** is a tool to find or check relations among physical quantities by using their dimensions. The dimension of a physical quantity is the combination of the basic physical dimensions (usually mass, length, time, electric charge, and temperature) which describe it; for example, speed has the dimension length / time, and may be measured in meters per second, miles per hour, or other units. Dimensional analysis is based on the fact that a physical law must be independent of the units used to measure the physical variables. A straightforward practical consequence is that any meaningful equation (and any inequality and inequation) must have the same dimensions in the left and right sides. Checking this is the basic way of performing dimensional analysis.

Dimensional analysis is routinely used to check the plausibility of derived equations and computations. It is also used to form reasonable hypotheses about complex physical situations that can be tested by experiment or by more developed theories of the phenomena, and to categorize types of physical quantities and units based on their relations to or dependence on other units, or their dimensions if any.

The basic principle of dimensional analysis was known to Isaac Newton (1686) who referred to it as the "*Great Principle of Similitude*". James Clerk Maxwell played a major role in establishing modern use of dimensional analysis by distinguishing mass, length, and time as fundamental units, while referring to other units as derived. The 19th-century French mathematician Joseph Fourier made important contributions based on the idea that physical laws like $F = ma$ should be independent of the units employed to measure the physical variables. This led to the conclusion that meaningful laws must be homogeneous equations in their various units of measurement, a result which was eventually formalized in the Buckingham π theorem. This theorem describes how every physically meaningful equation involving n variables can be equivalently rewritten as an equation of $n - m$ dimensionless parameters, where m is the number of fundamental dimensions used. Furthermore, and most importantly, it provides a method for computing these dimensionless parameters from the given variables.

A dimensional equation can have the dimensions reduced or eliminated through nondimensionalization, which begins with dimensional analysis, and involves scaling quantities by characteristic units of a system or natural units of nature. This gives insight into the fundamental properties of the system, as illustrated in the examples below.

Introduction

Definition

The dimensions of a physical quantity are associated with combinations of mass, length, time, electric charge, and temperature, represented by sans-serif symbols **M**, **L**, **T**, **Q**, and **Θ**, respectively, each raised to rational powers.

The term *dimension* is more abstract than *scale* unit: *mass* is a dimension, while kilograms are a scale unit (choice of standard) in the mass dimension.

As examples, the dimension of the physical quantity speed is *distance/time* ($\mathbf{L}/\mathbf{T}$ or $\mathbf{LT}^{-1}$), and the dimension of the physical quantity force is "mass × acceleration" or "mass×(distance/time)/time" ($\mathbf{ML}/\mathbf{T}^2$ or $\mathbf{MLT}^{-2}$). In principle, other dimensions of physical quantity could be defined as "fundamental" (such as momentum or energy or electric current) in lieu of some of those shown above. Most physicists do not recognize temperature, **Θ**, as a fundamental dimension of physical quantity since it essentially expresses the energy per particle per degree of freedom, which can be expressed in terms of energy (or mass, length, and time). Still others do not recognize electric charge, **Q**, as a separate fundamental dimension of physical quantity, since it has been expressed in terms of mass, length, and time in unit systems such as the cgs system. There are also physicists that have cast doubt on the very existence of incompatible fundamental dimensions of physical quantity.

The unit of a physical quantity and its dimension are related, but not identical concepts. The units of a physical quantity are defined by convention and related to some standard; e.g., length may have units of meters, feet, inches, miles or micrometres; but any length always has a dimension of **L**, independent of what units are arbitrarily chosen to measure it. Two different units of the same physical quantity have conversion factors that relate them. For example: 1 in = 2.54 cm; then (2.54 cm/in) is the conversion factor, and is itself dimensionless and equal to one. Therefore multiplying by that conversion factor does not change a quantity. Dimensional symbols do not have conversion factors.

Mathematical properties

Dimensional symbols, such as **L**, form a group: The identity is defined as $\mathbf{L}^0 = 1$, and the inverse to **L** is $1/\mathbf{L}$ or $\mathbf{L}^{-1}$. **L** raised to any rational power p is a member of the group, having an inverse of $\mathbf{L}^{-p}$ or $1/\mathbf{L}^p$. The operation of the group is multiplication, having the usual rules for handling exponents ($\mathbf{L}^n \times \mathbf{L}^m = \mathbf{L}^{n+m}$).

Dimensional symbols form a vector space over the rational numbers, with for example dimensional symbol $\mathbf{M}^i\mathbf{L}^j\mathbf{T}^k$ corresponding to the vector (i,j,k). When physical measured quantities (be they like-dimensioned or unlike-dimensioned) are multiplied or divided by one other, their dimensional units are likewise multiplied or divided; this corresponds to addition or subtraction in the vector space. When measurable quantities are raised to a rational power, the same is done to the dimensional symbols attached to those quantities; this corresponds to scalar multiplication in the vector space.

A basis for a given vector space of dimensional symbols is called a set of fundamental units or fundamental dimensions, and all other vectors are called derived units. As in any vector space, one may choose different bases, which yields different systems of units (e.g., choosing whether the unit for charge is derived from the unit for current, or vice versa).

Dimensionless quantities correspond to the origin in this vector space.

The set of units of the physical quantities involved in a problem correspond to a set of vectors (or a matrix). The kernel describes some number (e.g., m) of ways in which these vectors can be combined to produce a zero vector. These correspond to producing (from the measurements) a number of dimensionless quantities, $\{\pi_1,...,\pi_m\}$. (In fact these ways completely span the null subspace of another different space, of powers of the measurements.) Every possible way of multiplying (and exponating) together the measured quantities to produce something with the same units as some derived quantity X can be expressed in the general form $X\prod_{i=1}^{m}(\pi_i)^{k_i}$

Consequently, every possible commensurate equation for the physics of the system can be rewritten in the form $f(\pi_1,\pi_2,...,\pi_m) = 0$. Knowing this restriction can be a powerful tool for obtaining new insight into the system.

Mechanics

In mechanics, the dimension of any physical quantity can be expressed in terms of the fundamental dimensions (or *base dimensions*) **M**, **L**, and **T** – these form a 3-dimensional vector space. This is not the only possible choice, but it is the one most commonly used. For example, one might choose force, length and mass as the base dimensions (as some have done), with associated dimensions **F**, **L**, **M**; this corresponds to a different basis, and one may convert between these representations by a change of basis. The choice of the base set of dimensions is, thus, partly a convention, resulting in increased utility and familiarity. It is, however, important to note that the choice of the set of dimensions cannot be chosen arbitrarily – it is not *just* a convention – because the dimensions must form a basis: they must span the space, and be linearly independent.

For example, **F**, **L**, **M** form a set of fundamental dimensions because they form an equivalent basis to *M, L, T:* the former can be expressed as [$\mathbf{F}=\mathbf{ML}/\mathbf{T}^2$],**L**,**M** while the latter can be expressed as **M**,**L**,[$\mathbf{T}=(\mathbf{ML}/\mathbf{F})^{1/2}$].

On the other hand, using length, velocity and time (*L, V, T*) as base dimensions will not work well (they do not form a set of fundamental dimensions), for two reasons:

- There is no way to obtain mass — or anything derived from it, such as force — without introducing another base dimension (thus these do not *span the space*).
- Velocity, being derived from length and time (**V**=**L**/**T**), is redundant (the set is not *linearly independent*).

Other fields of physics and chemistry

Depending on the field of physics, it may be advantageous to choose one or another extended set of dimensional symbols. In electromagnetism, for example, it may be useful to use dimensions of **M**, **L**, **T**, and **Q**, where **Q** represents quantity of electric charge. In thermodynamics, the base set of dimensions is often extended to include a dimension for temperature, **Θ**. In chemistry the number of moles of substance (loosely, but not precisely, related to the number of molecules or atoms) is often involved and a dimension for this is used as well. The choice of the dimensions or even the number of dimensions to be used in different fields of physics is to some extent arbitrary, but consistency in use and ease of communications are very important.

Commensurability

The most basic consequence of dimensional analysis is:

Only *commensurable* quantities (quantities with the same dimensions) may be *compared, equated, added,* or *subtracted.*

However,

One may take *ratios* of *incommensurable* quantities (quantities with different dimensions), and *multiply* or *divide* them.

For example, it makes no sense to ask if 1 hour is more or less than 1 kilometer, as these have different dimensions, nor to add 1 hour to 1 kilometer. On the other hand, if an object travels 100 km in 2 hours, one may divide these and conclude that the object's average speed was 50 km/hour.

As a corollary of this requirement, it follows that in a physically meaningful *expression,* only quantities of the same dimension can be added, subtracted, or compared. For example, if m_{man}, m_{rat} and L_{man} denote, respectively, the mass of some man, the mass of a rat and the length of that man, the expression $m_{\text{man}} + m_{\text{rat}}$ is meaningful, but $m_{\text{man}} + L_{\text{man}}$ is meaningless. However, $m_{\text{man}}/L^2_{\text{man}}$ is fine. Thus, dimensional analysis may be used as a sanity check of physical equations: the two sides of any equation must be commensurable or have the same dimensions, i.e., the equation must be *dimensionally homogeneous.*

Even when two physical quantities have identical dimensions, it may be meaningless to compare or add them. For example, although torque and energy share the dimension ML^2T^{-2}, they are fundamentally different physical quantities.

To compare, add, or subtract quantities with the same dimensions but expressed in different units, the standard procedure is to first convert them all to the same units. For example, to compare 32 metres with 35 yards, use 1 yard = 0.9144 m to convert 35 yards to 32.004 m.

Polynomials and transcendental functions

Scalar arguments to transcendental functions such as exponential, trigonometric and logarithmic functions, or to inhomogeneous polynomials, must be dimensionless quantities. (Note: this requirement is somewhat relaxed in Siano's orientational analysis described below, in which the square of certain dimensioned quantities are dimensionless)

This requirement is clear when one observes the Taylor expansions for these functions (a sum of various powers of the function argument). For example, the logarithm of 3 kg is undefined even though the logarithm of 3 is nearly 0.477. An attempt to compute ln 3 kg would produce, if one naively took ln 3 kg to mean the dimensionally meaningless "ln (1 + 2 kg)",

$$2\,\mathrm{kg} - \frac{4\,\mathrm{kg}^2}{2} + \cdots$$

which is dimensionally incompatible – the sum has no meaningful dimension – requiring the argument of transcendental functions to be dimensionless.

Another way to understand this problem is that the different coefficients *scale* differently under change of units – were one to reconsider this in grams as "ln 3000 g" instead of "ln 3 kg", one could compute ln 3000, but in terms of the Taylor series, the degree 1 term would scale by 1000, the degree-2 term would scale by 1000^2, and so forth – the overall output would not scale as a particular dimension.

While most mathematical identities about dimensionless numbers translate in a straightforward manner to dimensional quantities, care must be taken with logarithms of ratios: the identity log(a/b) = log a - log b, where the logarithm is taken in any base, holds for dimensionless numbers a and b, but it does *not* hold if a and b are dimensional, because in this case the left-hand side is well-defined but the right-hand side is not.

Similarly, while one can evaluate monomials (x^n) of dimensional quantities, one cannot evaluate polynomials of mixed degree with dimensionless coefficients on dimensional quantities: for x^2, the expression $(3\text{ m})^2 = 9\text{ m}^2$ makes sense (as an area), while for $x^2 + x$, the expression $(3\text{ m})^2 + 3\text{ m} = 9\text{ m}^2 + 3\text{ m}$ does not make sense.

However, polynomials of mixed degree can make sense if the coefficients are suitably chosen physical quantities that are not dimensionless. For example,

$$\frac{1}{2}\cdot\left(-32\frac{\text{foot}}{\text{second}^2}\right)\cdot t^2+\left(500\frac{\text{foot}}{\text{second}}\right)\cdot t.$$

This is the height to which an object rises in time *t* if the acceleration of gravity is 32 feet per second per second and the initial upward speed is 500 feet per second. It is not even necessary for *t* to be in *seconds*. For example, suppose *t* = 0.01 minutes. Then the first term would be

$$\frac{1}{2}\cdot\left(-32\frac{\text{foot}}{\text{second}^2}\right)\cdot(0.01\text{ minute})^2$$

$$=\frac{1}{2}\cdot-32\cdot(0.01^2)\left(\frac{\text{minute}}{\text{second}}\right)^2\cdot\text{foot}$$

$$=\frac{1}{2}\cdot-32\cdot(0.01^2)\cdot60^2\cdot\text{foot}.$$

Incorporating units

The value of a dimensional physical quantity *Z* is written as the product of a unit [*Z*] within the dimension and a dimensionless numerical factor, *n*.

$$Z=n\times[Z]=n[Z]$$

In a strict sense, when like-dimensioned quantities are added or subtracted or compared, these dimensioned quantities must be expressed in consistent units so that the numerical values of these quantities may be directly added or subtracted. But, in concept, there is no problem adding quantities of the same dimension expressed in different units. For example, 1 meter added to 1 foot *is* a length, but it would not be correct to add 1 to 1 to get the result. A conversion factor, which is a ratio of like-dimensioned quantities and is equal to the dimensionless unity, is needed:

$1\text{ ft}=0.3048\text{ m}$ is identical to $1=\frac{0.3048\text{ m}}{1\text{ ft}}.$

The factor $0.3048\ \frac{\text{m}}{\text{ft}}$ is identical to the dimensionless 1, so multiplying by this conversion factor changes nothing. Then when adding two quantities of like dimension, but expressed in different units, the appropriate conversion factor, which is essentially the dimensionless 1, is used to convert the quantities to identical units so that their numerical values can be added or subtracted.

$$\begin{aligned} 1\ \mathrm{m} + 1\ \mathrm{ft} &= 1\ \mathrm{m} + 1\ \mathrm{ft} \times 0.3048\ \frac{\mathrm{m}}{\mathrm{ft}} \\ &= 1\ \mathrm{m} + 1\ \cancel{\mathrm{ft}} \times 0.3048\ \frac{\mathrm{m}}{\cancel{\mathrm{ft}}} \\ &= 1\ \mathrm{m} + 0.3048\ \mathrm{m} \\ &= 1.3048\ \mathrm{m} \end{aligned}$$

Only in this manner is it meaningful to speak of adding like-dimensioned quantities of differing units.

Position vs displacement

Some discussions of dimensional analysis implicitly describes all quantities are mathematical vectors. (In mathematics scalars are considered a special case of vectors; the emphasis here is that vectors are closed under addition, subtraction, and scalar multiplication, and permit scalar division.). This assumes an implicit point of reference—an origin. While this is useful and often perfectly adequate, allowing many important errors to be caught, it can fail to model certain aspects of physics. A more rigorous approach requires distinguishing between position and displacement (or moment in time versus duration, or absolute temperature versus temperature change).

Consider points on a line, each with a position with respect to a given origin, and distances among them. Positions and displacements all have units of length, but their meaning is not interchangeable:

- adding two displacements should yield a new displacement (walking ten paces then twenty paces gets you thirty paces forward),
- adding a displacement to a position should yield a new position (walking one block down the street from an intersection gets you to the next intersection),
- subtracting two positions should yield a displacement,
- but one may *not* add two positions.

This illustrates the subtle distinction between *affine* quantities (ones modeled by an affine space, such as position) and *vector* quantities (ones modeled by a vector space, such as displacement).

- Vector quantities may be added to each other, yielding a new vector quantity, and a vector quantity may be added to a suitable affine quantity (a vector space *acts on* an affine space), yielding a new affine quantity.
- Affine quantities cannot be added, but may be subtracted, yielding *relative* quantities which are vectors, and these *relative differences* may then be added to each other or to an affine quantity.

Properly then, positions have dimension of *affine* length, while displacements have dimension of *vector* length. To assign a number to an *affine* unit, one must not only choose a unit of measurement, but also a point of reference, while to assign a number to a *vector* unit only requires a unit of measurement.

Thus some physical quantities are better modeled by vectorial quantities while others tend to require affine representation, and the distinction is reflected in their dimensional analysis.

This distinction is particularly important in the case of temperature for which there is an absolute zero that is different in different measuring systems. That is, for absolute temperatures

$$0\ \text{K} = -273.15\ °\text{C} = -459.67\ °\text{F} = 0\ °\text{R},$$

but for relative temperatures,

$$1\ \text{K} = 1\ °\text{C} \neq 1\ °\text{F} = 1\ °\text{R}$$

Unit conversion for relative temperatures, where no temperature difference is zero in all units, is simply a matter of multiplying by, e.g., 1 °F / 1 K. But because these systems for absolute temperatures have different origins, conversion from one absolute temperature requires accounting for that. As a result, simple dimensional analysis can still lead to errors if it becomes ambiguous if 1 K equals −272.15 °C or 1 °C.

Orientation and frame of reference

Similar to the issue of a point of reference is the issue of orientation: a displacement in 2 or 3 dimensions is not just a length, but is a length together with a *direction.* (This issue does not arise in 1 dimension, or rather is equivalent to the distinction between positive and negative.) Thus, to compare or combine two dimensional quantities in a multi-dimensional space, one also needs an orientation: they need to be compared to a frame of reference.

This leads to the extensions discussed below, namely Huntley's directed dimensions and Siano's orientational analysis.

Other uses

Dimensional analysis is also used to derive relationships between the physical quantities that are involved in a particular phenomenon that one wishes to understand and characterize. It was used for the first time (Pesic, 2005) in this way in 1872 by Lord Rayleigh, who was trying to understand why the sky is blue.

Examples

A simple example: period of a harmonic oscillator

What is the period of oscillation *T* of a mass *m* attached to an ideal linear spring with spring constant *k* suspended in gravity of strength *g*? The four quantities have the following dimensions: *T* [T]; *m* [M]; *k*[M / T^2]; and *g*[L / T^2]. From these we can form only one dimensionless product of powers of our chosen variables, $G_1 = T^2k / m$. The dimensionless product of powers of variables is sometimes referred to as a dimensionless group of variables, but the group, G_1, referred to means "collection" rather than mathematical group. They are often called dimensionless numbers as well.

Note that no other dimensionless product of powers involving *g* with k, m, T, and g alone can be formed, because only g involves L . Dimensional analysis can sometimes yield strong statements about the *irrelevance* of some quantities in a problem, or the need for additional parameters. If we have chosen enough variables to properly describe the problem, then from this argument we can conclude that the period of the mass on the spring is independent of *g*: it is the same on the earth or the moon. The equation demonstrating the existence of a product of powers for our problem can be written in an entirely equivalent way: $T = \kappa\sqrt{m/k}$, for some dimensionless constant κ.

When faced with a case where our analysis rejects a variable (g, here) that we feel sure really belongs in a physical description of the situation, we might also consider the possibility that the rejected variable is in fact relevant, and that some other relevant variable has been omitted, which might combine with the rejected variable to form a dimensionless quantity. That is, however, not the case here.

When dimensional analysis yields a solution of problems where only one dimensionless product of powers is involved, as here, there are no unknown functions, and the solution is said to be "complete."

A more complex example: energy of a vibrating wire

Consider the case of a vibrating wire of length ℓ (*L*) vibrating with an amplitude *A* (*L*). The wire has a linear density *ρ* (*M*/*L*) and is under tension *s* (ML/T^2), and we want to know the energy *E* (ML^2/T^2) in the wire. Let π_1 and π_2 be two dimensionless products of powers of the variables chosen, given by

$$\pi_1 = E/As$$
$$\pi_2 = \ell/A.$$

The linear density of the wire is not involved. The two groups found can be combined into an equivalent form as an equation

$$F(E/As, \ell/A) = 0,$$

where F is some unknown function, or, equivalently as

$$E = Asf(\ell/A),$$

where f is some other unknown function. Here the unknown function implies that our solution is now incomplete, but dimensional analysis has given us something that may not have been obvious: the energy is proportional to the first power of the tension. Barring further analytical analysis, we might proceed to experiments to discover the form for the unknown function f. But our experiments are simpler than in the absence of dimensional analysis. We'd perform none to verify that the energy is proportional to the tension. Or perhaps we might guess that the energy is proportional to ℓ, and so infer that $E = \ell s$. The power of dimensional analysis as an aid to experiment and forming hypotheses becomes evident.

The power of dimensional analysis really becomes apparent when it is applied to situations, unlike those given above, that are more complicated, the set of variables involved are not apparent, and the underlying equations hopelessly complex. Consider, for example, a small pebble sitting on the bed of a river. If the river flows fast enough, it will actually raise the pebble and cause it to flow along with the water. At what critical velocity will this occur? Sorting out the guessed variables is not so easy as before. But dimensional analysis can be a powerful aid in understanding problems like this, and is usually the very first tool to be applied to complex problems where the underlying equations and constraints are poorly understood. In such cases, the answer may depend on a dimensionless number such as the Reynolds number, which may be interpreted by dimensional analysis.

Extensions

Huntley's extension: directed dimensions

Huntley (Huntley, 1967) has pointed out that it is sometimes productive to refine our concept of dimension. Two possible refinements are:

- The magnitude of the components of a vector are to be considered dimensionally distinct. For example, rather than an undifferentiated length unit L, we may have L_x represent length in the x direction, and so forth. This requirement stems ultimately from the requirement that each component of a physically meaningful equation (scalar, vector, or tensor) must be dimensionally consistent.

- Mass as a measure of quantity is to be considered dimensionally distinct from mass as a measure of inertia.

As an example of the usefulness of the first refinement, suppose we wish to calculate the distance a cannon ball travels when fired with a vertical velocity component V_y and a horizontal velocity component V_x, assuming it is fired on a flat surface. Assuming no use of directed lengths, the quantities of interest are then V_x, V_y, both dimensioned as L / T, R, the distance travelled, having dimension L, and g the downward acceleration of gravity, with dimension L / T^2

With these four quantities, we may conclude that the equation for the range R may be written:

$$R \propto V_x^a V_y^b g^c.$$

Or dimensionally

$$L = (L/T)^{a+b} (L/T^2)^c$$

from which we may deduce that $a + b + c = 1$ and $a + b + 2c = 0$, which leaves one exponent undetermined. This is to be expected since we have two fundamental quantities L and T and four parameters, with one equation.

If, however, we use directed length dimensions, then V_x will be dimensioned as L_x / T, V_y as L_y / T, R as L_x and g as L_y / T^2. The dimensional equation becomes:

$$L_x = (L_x/T)^a \, (L_y/T)^b (L_y/T^2)^c$$

and we may solve completely as $a = 1$, $b = 1$ and $c = -1$. The increase in deductive power gained by the use of directed length dimensions is apparent.

In a similar manner, it is sometimes found useful (e.g., in fluid mechanics and thermodynamics) to distinguish between mass as a measure of inertia (inertial mass), and mass as a measure of quantity (substantial mass). For example, consider the derivation of Poiseuille's Law. We wish to find the rate of mass flow of a viscous fluid through a circular pipe. Without drawing distinctions between inertial and substantial mass we may choose as the relevant variables

- $\dot{m}$ the mass flow rate with dimensions M / T
- p_x the pressure gradient along the pipe with dimensions M / L^2T^2
- ρ the density with dimensions M / L^3
- η the dynamic fluid viscosity with dimensions M / LT
- r the radius of the pipe with dimensions L

There are three fundamental variables so the above five equations will yield two dimensionless variables which we may take to be $\pi_1 = \dot{m}/\eta r$ and $\pi_2 = p_x \rho r^5 / \dot{m}^2$ and we may express the dimensional equation as

$$C = \pi_1 \pi_2^a = \left(\frac{\dot{m}}{\eta r}\right)\left(\frac{p_x \rho r^5}{\dot{m}^2}\right)^a$$

where ***C*** and ***a*** are undetermined constants. If we draw a distinction between inertial mass with dimensions M_i and substantial mass with dimensions M_s, then mass flow rate and density will use substantial mass as the mass parameter, while the pressure gradient and coefficient of viscosity will use inertial mass. We now have four fundamental parameters, and one dimensionless constant, so that the dimensional equation may be written:

$$C = \frac{p_x \rho r^4}{\eta \dot{m}}$$

where now only C is an undetermined constant (found to be equal to π / 8 by methods outside of dimensional analysis). This equation may be solved for the mass flow rate to yield Poiseuille's law.

Siano's extension: orientational analysis

Huntley's extension has some serious drawbacks:

- It does not deal well with vector equations involving the *cross product,*
- nor does it handle well the use of *angles* as physical variables.

It also is often quite difficult to assign the L, L_x, L_y, L_z, symbols to the physical variables involved in the problem of interest. He invokes a procedure that involves the "symmetry" of the physical problem. This is often very difficult to apply reliably: It is unclear as to what parts of the problem that the notion of "symmetry" is being invoked. Is it the symmetry of the physical body that forces are acting upon, or to the points, lines or areas at which forces are being applied? What if more than one body is involved with different symmetries? Consider the spherical bubble attached to a cylindrical tube, where one wants the flow rate of air as a function of the pressure difference in the two parts. What are the Huntley extended dimensions of the viscosity of the air contained in the connected parts? What are the extended dimensions of the pressure of the two parts? Are they the same or different? These difficulties are responsible for the limited application of Huntley's addition to real problems.

Angles are, by convention, considered to be dimensionless variables, and so the use of angles as physical variables in dimensional analysis can give less meaningful results. As an example, consider the projectile problem mentioned above. Suppose that, instead of the x- and y-components of the initial velocity, we had chosen the magnitude of the velocity v and the angle θ at which the projectile was fired. The angle is, by convention, considered to be dimensionless, and the magnitude of a vector has no directional quality, so that no dimensionless variable can be composed of the four variables g, v, R, and θ. Conventional analysis will correctly give the powers of g and v, but will give no information concerning the dimensionless angle θ.

Siano (Siano, 1985-I, 1985-II) has suggested that the directed dimensions of Huntley be replaced by using *orientational symbols* 1_x 1_y 1_z to denote vector directions, and an orientationless symbol 1_0. Thus, Huntley's 1_x becomes $L\ 1_x$ with L specifying the dimension of length, and 1_x specifying the orientation. Siano further shows that the orientational symbols have an algebra of their own. Along with the requirement that $1_i^{-1} = 1_i$, the following multiplication table for the orientation symbols results:

	$\mathbf{1_0}$	$\mathbf{1_x}$	$\mathbf{1_y}$	$\mathbf{1_z}$
$\mathbf{1_0}$	1_0	1_x	1_y	1_z
$\mathbf{1_x}$	1_x	1_0	1_z	1_y
$\mathbf{1_y}$	1_y	1_z	1_0	1_x
$\mathbf{1_z}$	1_z	1_y	1_x	1_0

Note that the orientational symbols form a group (the Klein four-group or "Viergruppe"). In this system, scalars always have the same orientation as the identity element, independent of the "symmetry of the problem." Physical quantities that are vectors have the orientation expected: a force or a velocity in the z-direction has the orientation of 1_z. For angles, consider an angle θ that lies in the z-plane. Form a right triangle in the z plane with θ being one of the acute angles. The side of the right triangle adjacent to the angle then has an orientation 1_x and the side opposite has an orientation 1_y. Then, since $\tan(\theta) = 1_y/1_x = \theta + \ldots$ we conclude that an angle in the xy plane must have an orientation $1_y/1_x = 1_z$, which is not unreasonable. Analogous reasoning forces the conclusion that $\sin(\theta)$ has orientation 1_z while $\cos(\theta)$ has orientation 1_0. These are different, so one concludes (correctly), for example, that there are no solutions of physical equations that are of the form $a \cos(\theta)+b \sin(\theta)$, where a and b are real scalars. Note that an expression such as $\sin(\theta + \pi / 2) = \cos(\theta)$ is not dimensionally inconsistent since it is a special case of the sum of angles formula and should properly be written:

$$\sin(a\,1_z + b\,1_z) = \sin(a\,1_z)\cos(b\,1_z) + \sin(b\,1_z)\cos(a\,1_z)$$

which for $a = \theta$ and $b = \pi / 2$ yields $\sin(\theta\,1_z + (\pi/2)\,1_z) = 1_z\cos(\theta\,1_z)$. Physical quantities may be expressed as complex numbers (e.g. $e^{i\theta}$) which imply that the complex quantity i has an orientation equal to that of the angle it is associated with (1_z in the above example).

The assignment of orientational symbols to physical quantities and the requirement that physical equations be orientationally homogeneous can actually be used in a way that is similar to dimensional analysis to derive a little more information about acceptable solutions of physical problems. In this approach one sets up the dimensional equation and solves it as far as one can. If the lowest power of a physical variable is fractional, both sides of the solution is raised to a power such that all powers are integral. This puts it into "normal form". The orientational equation is then solved to give a more restrictive condition on the unknown powers of the orientational symbols, arriving at a solution that is more complete than the one that dimensional analysis alone gives. Often the added information is that one of the powers of a certain variable is even or odd.

As an example, for the projectile problem, using orientational symbols, θ, being in the *xy*-plane will thus have dimension 1_z and the range of the projectile R will be of the form:

$$R = g^a\, v^b\, \theta^c \text{ which means } L\, 1_x \sim \left(\frac{L\, 1_y}{T^2}\right)^a \left(\frac{L}{T}\right)^b 1_z^c.$$

Dimensional homogeneity will now correctly yield $a = -1$ and $b = 2$, and orientational homogeneity requires that c be an odd integer. In fact the required function of theta will be $\sin(\theta)\cos(\theta)$ which is a series of odd powers of θ.

It is seen that the Taylor series of $\sin(\theta)$ and $\cos(\theta)$ are orientationally homogeneous using the above multiplication table, while expressions like $\cos(\theta) + \sin(\theta)$ and $\exp(\theta)$ are not, and are (correctly) deemed unphysical.

It should be clear that the multiplication rule used for the orientational symbols is not the same as that for the cross product of two vectors. The cross product of two identical vectors is zero, while the product of two identical orientational symbols is the identity element.

Percentages and derivatives

Percentages are dimensionless quantities, since they are ratios of two quantities with the same dimensions.

Derivatives with respect to a quantity add the dimensions of the variable one is differentiating with respect to on the denominator. Thus:

- position (x) has units of L (Length);
- derivative of position with respect to time (dx/dt, velocity) has units of L/T – Length from position, Time from the derivative;
- the second derivative (d^2x/dt^2, acceleration) has units of L/T^2.

In economics, one distinguishes between stocks and flows: a stock has units of "units" (say, widgets or dollars), while a flow is a derivative of a stock, and has units of "units/time" (say, dollars/year).

Beware that in some contexts, dimensional quantities are expressed as dimensionless quantities or percentages by omitting some dimensions. This may or may not be misleading. For example, Debt to GDP ratios are generally expressed as percentages: total debt outstanding (dimension of Currency) divided by annual GDP (dimension of Currency) – but one may argue that in comparing a stock to a flow, annual GDP should have dimensions of Currency/Time (Dollars/Year, for instance), and thus Debt to GDP should have units of years.

Dimensionless concepts

Constants

The dimensionless constants that arise in the results obtained, such as the C in the Poiseuille's Law problem and the κ in the spring problems discussed above come from a more detailed analysis of the underlying physics, and often arises from integrating some differential equation. Dimensional analysis itself has little to say about these constants, but it is useful to know that they very often have a magnitude of order unity. This observation can allow one to sometimes make "back of the envelope" calculations about the phenomenon of interest, and therefore be able to more efficiently design experiments to measure it, or to judge whether it is important, etc.

Formalisms

Paradoxically, dimensional analysis can be a useful tool even if all the parameters in the underlying theory are dimensionless, e.g., lattice models such as the Ising model can be used to study phase transitions and critical phenomena. Such models can be formulated in a purely dimensionless way. As we approach the critical point closer and closer, the distance over which the variables in the lattice model are correlated (the so-called correlation length, ξ) becomes larger and larger. Now, the correlation length is the relevant length scale related to critical phenomena, so one can, e.g., surmize on "dimensional grounds" that the non-analytical part of the free energy per lattice site should be ~$1 / \xi^d$ where *d* is the dimension of the lattice.

It has been argued by some physicists, e.g., Michael Duff, that the laws of physics are inherently dimensionless. The fact that we have assigned incompatible dimensions to Length, Time and Mass is, according to this point of view, just a matter of convention, borne out of the fact that before the advent of modern physics, there was no way to relate mass, length, and time to each other. The three independent dimensionful constants: c, $\hbar$, and G, in the fundamental equations of physics must then be seen as mere conversion factors to convert Mass, Time and Length into each other.

Just as in the case of critical properties of lattice models, one can recover the results of dimensional analysis in the appropriate scaling limit; e.g., dimensional analysis in mechanics can be derived by reinserting the constants $\hbar$, c, and G (but we can now consider them to be dimensionless) and demanding that a nonsingular relation between quantities exists in the limit $c \rightarrow \infty, \hbar \rightarrow 0$and $G \rightarrow 0$. In problems involving a gravitational field the latter limit should be taken such that the field stays finite.

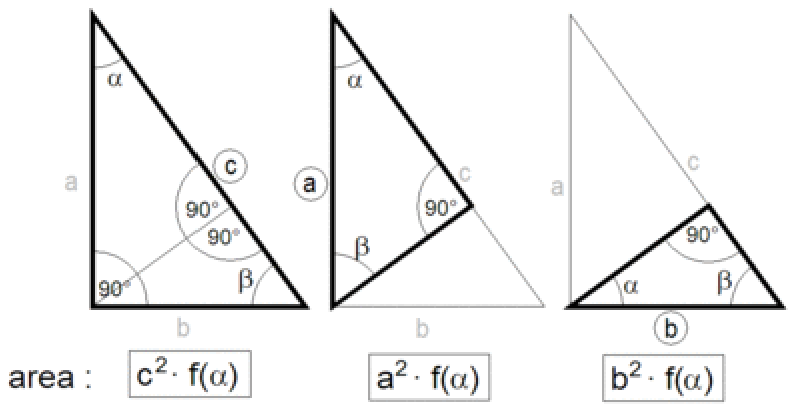
Applications

Dimensional analysis is most often used in physics and chemistry- and in the mathematics thereof- but finds some applications outside of those fields as well.

Mathematics

A simple application of dimensional analysis to mathematics is in computing the form of the volume of an n-ball (the solid ball in n-dimensions), or the area of its surface, the n-sphere: being an n-dimensional figure, the volume scales as x^n, while the surface area, being $(n - 1)$-dimensional, scales as x^{n-1}. Thus the volume of the n-ball in terms of the radius is $C_n r^n$, for some constant C_n. Determining the constant takes more involved mathematics, but the form can be deduced and checked by dimensional analysis alone.

Proof of the Pythagorean theorem



Dimensional proof of the Pythagorean theorem

A very simple proof of the Pythagorean theorem can be obtained by just dimensional reasoning.

The area of any triangle depends on its size and shape, which can be unambiguously identified by the length of one of its edges (for example, the largest) and by any two of its angles (the third being determined by the fact that the sum of all three is π). Thus, *recalling that an area has the dimensions of a length squared*, we can write:

$$area = largest\_edge^2 \cdot f(angle\_1, angle\_2),$$

where f is an adimensional function of the angles.

Now, referring to the figure at right, if we divide a right triangle in two smaller ones by tracing the segment perpendicular to its hypotenuse and passing by the opposite vertex, and express the obvious fact that the total area is the sum of the two smaller ones, by applying the previous equation we have:

$$c^2 \cdot f(\alpha, \pi/2) = a^2 \cdot f(\alpha, \pi/2) + b^2 \cdot f(\alpha, \pi/2).$$

And, eliminating f:

$$c^2 = a^2 + b^2 \text{ , Q.E.D.}$$

Note that the result is obtained without specifying the form of the adimensional function *f*.

Finance, economics, and accounting

In finance, economics, and accounting, dimensional analysis is most commonly used in interpreting various financial ratios, economics ratios, and accounting ratios.

- For example, the P/E ratio has dimensions of time (units of years), and can be interpreted as "years of earnings to earn the price paid."
- In economics, debt-to-GDP ratio also has units of years (debt has units of currency, GDP has units of currency/year).
- More surprisingly, bond duration also has units of years, which can be shown by dimensional analysis, but takes some financial intuition to understand.
- Velocity of money has units of 1/Years (GDP/Money supply has units of Currency/Year over Currency): how often a unit of currency circulates per year.

Dimensional analysis is rarely used in (mainstream/neoclassical) economic modeling, and economic models are often dimensionally inconsistent. The equation of exchange is the most notable example of a dimensional equation in economic modeling, while the widely-used Cobb–Douglas model does not use dimensions in a meaningful way. This lack of dimensional consistency is criticized by heterodox economics, notably Austrian economics, while dimensional consistency is not considered necessary or desirable by mainstream economists.

Chapter 2

Dimensionless Quantity

In dimensional analysis, a **dimensionless quantity** is a quantity without an associated physical dimension. It is thus a "pure" number, and as such always has a dimension of 1. Dimensionless quantities are widely used in mathematics, physics, engineering, economics, and in everyday life (such as in counting). Numerous well-known quantities, such as π, e, and φ, are dimensionless.

Dimensionless quantities are often defined as products or ratios of quantities that are not dimensionless, but whose dimensions cancel out when their powers are multiplied. This is the case, for instance, with the engineering strain, a measure of deformation. It is defined as change in length over initial length but, since these quantities both have dimensions L (length), the result is a dimensionless quantity.

Properties

- Even though a dimensionless quantity has no physical dimension associated with it, it can still have dimensionless units. It is sometimes helpful to use the same units in both the numerator and denominator, such as kg/kg, to show the quantity being measured (for example, to distinguish a mass ratio from a volume ratio). The quantity may also be given as a ratio of two different units that have the same dimension (for instance, light years over meters). This may be the case when calculating slopes in graphs, or when making unit conversions. Such notation does not indicate the presence of physical dimensions, and is purely a notational convention. Other common dimensionless units are % (= 0.01), ‰ (= 0.001), ppm (= 10^{-6}), ppb (= 10^{-9}), ppt (= 10^{-12}) and angle units (radians, grad, degrees). Units of amount such as the dozen and the gross are also dimensionless.
- The – dimensionless – ratio of two quantities with the same dimensions has the same value regardless of the units used to calculate them. For instance, if body **A** exerts a force of magnitude F on body **B**, and **B** exerts a force of magnitude f on **A**, then the ratio F/f will always be equal to 1, regardless of the actual units used to measure F and f. This is a fundamental property of dimensionless proportions and follows from the assumption that the laws of physics are independent of the system of units used in their expression. In this case, if the ratio F/f was not

always equal to 1, but changed if we switched from SI to CGS, for instance, that would mean that Newton's Third Law's truth or falsity would depend on the system of units used, which would contradict this fundamental hypothesis. The assumption that the laws of physics are not contingent upon a specific unit system is also closely related to the Buckingham π theorem. A formulation of this theorem is that any physical law can be expressed as an identity (always true equation) involving only dimensionless combinations (ratios or products) of the variables linked by the law (e. g., pressure and volume are linked by Boyle's Law – they are inversely proportional). If the dimensionless combinations' values changed with the systems of units, then the equation would not be an identity, and Buckingham's theorem would not hold.

Buckingham π theorem

Another consequence of the Buckingham π theorem of dimensional analysis is that the functional dependence between a certain number (say, n) of variables can be reduced by the number (say, k) of independent dimensions occurring in those variables to give a set of $p = n - k$ independent, dimensionless quantities. For the purposes of the experimenter, different systems which share the same description by dimensionless quantity are equivalent.

Example

The power consumption of a stirrer with a given shape is a function of the density and the viscosity of the fluid to be stirred, the size of the stirrer given by its diameter, and the speed of the stirrer. Therefore, we have $n = 5$ variables representing our example.

Those $n = 5$ variables are built up from $k = 3$ dimensions which are:

- Length: L (m)
- Time: T (s)
- Mass: M (kg).

According to the π-theorem, the $n = 5$ variables can be reduced by the $k = 3$ dimensions to form $p = n - k = 5 - 3 = 2$ independent dimensionless numbers which are, in case of the stirrer:

- Reynolds number (a dimensionless number describing the fluid flow regime)
- Power number (describing the stirrer and also involves the density of the fluid)

Standards efforts

The CIPM Consultative Committee for Units contemplated defining the unit of 1 as the 'uno', but the idea was dropped.

Examples

Consider this example: Sarah says, *"Out of every 10 apples I gather, 1 is rotten."*. The rotten-to-gathered ratio is (1 apple) / (10 apples) = 0.1 = 10%, which is a dimensionless quantity. Another more typical example in physics and engineering is the measure of plane angles. An angle is measured as the ratio of the length of a circle's arc subtended by an angle whose vertex is the centre of the circle to some other length. The ratio, length divided by length, is dimensionless. When using radians as the unit, the length that is compared is the length of the radius of the circle. When using degree as the units, the arc's length is compared to 1/360 of the circumference of the circle.

List of dimensionless quantities

All numbers are dimensionless quantities. Certain dimensionless quantities of some importance are given below:

Name	Standard symbol	Definition	Field of application
Abbe number	*V*		optics (dispersion in optical materials)
Activity coefficient	*γ*		chemistry (Proportion of "active" molecules or atoms)
Albedo	α		climatology, astronomy (reflectivity of surfaces or bodies)
Archimedes number	*Ar*		motion of fluids due to density differences
Arrhenius number	α		Ratio of activation energy to thermal energy
Atomic weight	*M*		chemistry
Bagnold number	*Ba*		flow of bulk solids such as grain and sand.
Blowdown circulation number	*BC*		deviation from isothermal flow in blowdown (rapid depressurization) of a pressure vessel
Bejan number (thermodynamics)	*Be*		the ratio of heat transfer irreversibility to total irreversibility due to heat transfer and fluid friction
Bejan number (fluid mechanics)	*Be*		dimensionless pressure drop along a channel
Bingham number	*Bm*	$Bm = \frac{\tau_y L}{\mu V}$	Ratio of yield stress to viscous stress
Biot number	*Bi*		surface vs. volume conductivity of

			solids
Blake number	*Bl* or *B*		relative importance of inertia compared to viscous forces in fluid flow through porous media
Bodenstein number			residence-time distribution
Bond number	*Bo*		capillary action driven by buoyancy
Brinkman number	*Br*		heat transfer by conduction from the wall to a viscous fluid
Brownell-Katz number			combination of capillary number and Bond number
Capillary number	*Ca*		fluid flow influenced by surface tension
Coefficient of static friction	μ_s		friction of solid bodies at rest
Coefficient of kinetic friction	μ_k		friction of solid bodies in translational motion
Colburn j factor			dimensionless heat transfer coefficient
Courant-Friedrich-Levy number	ν		numerical solutions of hyperbolic PDEs
Damkohler number	*Da*		reaction time scales vs. transport phenomena
Damping ratio	ζ	$\zeta = \frac{c}{2\sqrt{km}}$	the level of damping in a system
Darcy friction factor	C_f or *f*		fluid flow
Dean number	*D*		vortices in curved ducts
Deborah number	*De*		rheology of viscoelastic fluids
Decibel	*dB*		ratio of two intensities, often sound
Drag coefficient	C_d		flow resistance
Dukhin number	*Du*		ratio of electric surface conductivity to the electric bulk conductivity in heterogeneous systems
Euler's number	*e*		mathematics
Eckert number	*Ec*		convective heat transfer
Ekman number	*Ek*		geophysics (frictional (viscous)

			forces)
Elasticity (economics)	*E*		widely used to measure how demand or supply responds to price changes
Eötvös number	*Eo*		determination of bubble/drop shape
Ericksen number	*Er*		liquid crystal flow behavior
Euler number	*Eu*		hydrodynamics (pressure forces vs. inertia forces)
Fanning friction factor	*f*		fluid flow in pipes
Feigenbaum constants	α,δ		chaos theory (period doubling)
Fine structure constant	α	$\alpha = \frac{e^2}{2\varepsilon_0 hc}$	quantum electrodynamics (QED)
f-number	*f*		optics, photography
Foppl–von Karman number			thin-shell buckling
Fourier number	*Fo*		heat transfer
Fresnel number	*F*		slit diffraction
Froude number	*Fr*		wave and surface behaviour
Gain			electronics (signal output to signal input)
Galilei number	*Ga*		gravity-driven viscous flow
Golden ratio	φ		mathematics and aesthetics
Graetz number	*Gz*		heat flow
Grashof number	*Gr*		free convection
Gravitational coupling constant	α_G	$\alpha_G = \frac{Gm_e^2}{\hbar c}$	Gravitation
Hatta number	*Ha*		adsorption enhancement due to chemical reaction
Hagen number	*Hg*		forced convection
Hydraulic gradient	*i*		groundwater flow
Karlovitz number			turbulent combustion turbulent combustion
Keulegan–Carpenter number	K_C		ratio of drag force to inertia for a bluff object in oscillatory fluid flow

Knudsen number	Kn	ratio of the molecular mean free path length to a representative physical length scale
Kt/V		medicine
Kutateladze number	K	counter-current two-phase flow
Laplace number	La	free convection within immiscible fluids
Lewis number	Le	ratio of mass diffusivity and thermal diffusivity
Lift coefficient	C_L	lift available from an airfoil at a given angle of attack
Lockhart-Martinelli parameter	χ	flow of wet gases
Lundquist number	S	ratio of a resistive time to an Alfvén wave crossing time in a plasma
Mach number	M	gas dynamics
Magnetic Reynolds number	R_m	magnetohydrodynamics
Manning roughness coefficient	n	open channel flow (flow driven by gravity)
Marangoni number	Mg	Marangoni flow due to thermal surface tension deviations
Morton number	Mo	determination of bubble/drop shape
Mpemba number	K_M	thermal conduction and diffusion in freezing of a solution
Nusselt number	Nu	heat transfer with forced convection
Ohnesorge number	Oh	atomization of liquids, Marangoni flow
Péclet number	Pe	advection–diffusion problems
Peel number		adhesion of microstructures with substrate
Pi	π	mathematics (ratio of a circle's circumference to its diameter)
Poisson's ratio	ν	elasticity (load in transverse and longitudinal direction)

Porosity	φ		geology
Power factor			electronics (real power to apparent power)
Power number	N_p		power consumption by agitators
Prandtl number	Pr	$\mathrm{Pr} = \frac{\nu}{\alpha} = \frac{c_p \mu}{k}$	convection heat transfer (thickness of thermal and momentum boundary layers)
Pressure coefficient	C_P		pressure experienced at a point on an airfoil
Q factor	Q		describes how under-damped an oscillator or resonator is
Radian	rad		measurement of angles
Rayleigh number	Ra		buoyancy and viscous forces in free convection
Refractive index	n		electromagnetism, optics
Reynolds number	Re	$Re = \frac{vL\rho}{\mu}$	Ratio of fluid inertial and viscous forces
Relative density	RD		hydrometers, material comparisons
Richardson number	Ri		effect of buoyancy on flow stability
Rockwell scale			mechanical hardness
Rolling resistance coefficient	C_{rr}	$C_{rr} = \frac{N_f}{F}$	Vehicle dynamics
Rossby number	R_o		inertial forces in geophysics
Rouse number	Z or P		sediment transport
Schmidt number	Sc		fluid dynamics (mass transfer and diffusion)
Shape factor	H		ratio of displacement thickness to momentum thickness in boundary layer flow
Sherwood number	Sh		mass transfer with forced convection
Shields parameter	τ_* or θ		threshold of sediment movement due to fluid motion
Sommerfeld number			boundary lubrication
Stanton number	St		heat transfer in forced convection
Stefan number	Ste		heat transfer during phase change
Stokes number	Stk		particle dynamics
Strain	ε		materials science, elasticity

Strouhal number	*St* or *Sr*		nondimensional frequency, continuous and pulsating flow
Taylor number	*Ta*		rotating fluid flows
Ursell number	*U*		nonlinearity of surface gravity waves on a shallow fluid layer
Vadasz number	*Va*	$Va = \frac{\phi Pr}{Da}$	governs the effects of porosity φ, the Prandtl number and the Darcy number on flow in a porous medium
van 't Hoff factor	*i*		quantitative analysis (K_f and K_b)
Wallis parameter	J^*		nondimensional superficial velocity in multiphase flows
Weaver flame speed number			laminar burning velocity relative to hydrogen gas
Weber number	*We*		multiphase flow with strongly curved surfaces
Weissenberg number	*Wi*		viscoelastic flows
Womersley number	α		continuous and pulsating flows

Dimensionless physical constants

Certain fundamental physical constants, such as the speed of light in a vacuum, the universal gravitational constant, and the constants of Planck and Boltzmann, are normalized to 1 if the units for time, length, mass, charge, and temperature are chosen appropriately. The resulting system of units is known as natural. However, not all physical constants cannot be eliminated in **any** system of units; the values of the remaining ones must be determined experimentally. Resulting constants include:

- α, the fine structure constant, the coupling constant for the electromagnetic interaction;
- μ or β, the proton-to-electron mass ratio, the rest mass of the proton divided by that of the electron. More generally, the rest masses of all elementary particles relative to that of the electron;
- α_s, the coupling constant for the strong force;
- α_G, the gravitational coupling constant.

Chapter 3

Natural Units

In physics, **natural units** are physical units of measurement based only on universal physical constants. For example the elementary charge *e* is a natural unit of electric charge, or the speed of light *c* is a natural unit of speed. A purely natural system of units is defined in such a way that some set of selected universal physical constants are normalized to unity; that is, their numerical values in terms of these units become exactly 1.

Introduction

Natural units are intended to elegantly simplify particular algebraic expressions appearing in physical law or to normalize some chosen physical quantities that are properties of universal elementary particles and that may be reasonably believed to be constant. However, what may be believed and forced to be constant in one system of natural units can very well be allowed or even assumed to vary in another natural unit system.

Natural units are natural because the origin of their definition comes only from properties of nature and not from any human construct. Planck units are often, without qualification, called "natural units", when in fact they constitute only one of several systems of natural units, albeit the best known such system. Planck units might be considered unique in that the set of units are not based on properties of any prototype, object, or particle but are solely derived from the properties of free space.

As with other systems of units, the base units of a set of natural units will include definitions and values for length, mass, time, temperature, and electric charge (in lieu of electric current). Some physicists do not recognize temperature as a fundamental physical quantity, since it simply expresses the energy per degree of freedom of a particle, which can be expressed in terms of energy (or mass, length, and time). Virtually every system of natural units normalizes Boltzmann's constant k_B to 1, which can be thought of as simply a way of defining the unit temperature.

In the SI unit system, electric charge is a separate fundamental dimension of physical quantity, but in natural unit systems charge is expressed in terms of the mechanical units of mass, length, and time, similarly to cgs. There are two common ways to relate charge to mass, length, and time: In "rationalized" (or "Heaviside-Lorentz") natural unit systems, Coulomb's law is $F=q_1q_2/(4\pi r^2)$, and in "non-rationalized" (or "Gaussian"), Coulomb's law is $F=q_1q_2/r^2$. Both possibilities are incorporated into different natural unit systems.

Notation and use

Natural units are most commonly used by *setting the units to one*. For example, many natural unit systems include the equation $c = 1$ in the unit-system definition, where c is the speed of light. If a velocity v is half the speed of light, then from the equations $v = \frac{1}{2}c$ and $c = 1$, the consequence is $v = \frac{1}{2}$. The equation $v = \frac{1}{2}$ means "the velocity v has the value one-half when measured in Planck units", or "the velocity v is one-half the Planck unit of velocity".

The equation $c = 1$ can be plugged in anywhere else. For example, Einstein's equation $E = mc^2$ can be rewritten in Planck units as $E = m$. This equation means "The rest-energy of a particle, measured in Planck units of energy, equals the rest-mass of a particle, measured in Planck units of mass."

Advantages and disadvantages

Compared to SI or other unit systems, natural units have both advantages and disadvantages:

- **Simplified equations:** By setting constants to 1, equations containing those constants appear more compact and in some cases may be simpler to understand. For example, the special relativity equation $E^2 = p^2c^2 + m^2c^4$ appears somewhat complicated, but the natural units version, $E^2 = p^2 + m^2$, appears simpler.

- **Physical interpretation:** Natural unit systems automatically incorporate dimensional analysis. For example, in Planck units, the units are defined by properties of quantum mechanics and gravity. Not coincidentally, the Planck unit of length is approximately the length where quantum gravity effects become important. Likewise, atomic units are based on the mass and charge of an electron, and not coincidentally the atomic unit of length is the Bohr radius describing the orbit of the electron in a hydrogen atom.

- **No prototypes:** A prototype is a physical object that defines a unit, such as the International Prototype Kilogram, a certain cylinder whose mass is by definition exactly one kilogram. A prototype definition always has imperfect reproducibility between different places and between different times, and it is an advantage of natural unit systems that they use no prototypes. (They share this advantage with other *non-natural* unit systems, such as conventional electrical units.)

- **Less precise measurements:** SI units are designed to be used in precision measurements. For example, the second is defined by an atomic transition frequency in cesium atoms, because this transition frequency can be precisely reproduced with atomic clock technology. Natural unit systems are *not* generally based on quantities that can be precisely reproduced in a lab. Therefore, a quantity measured in natural units can have fewer digits of precision than the same quantity measured in SI. For example, Planck units use the gravitational constant G, which is measurable in a laboratory only to four significant digits.

- **Greater potential for ambiguity:** Consider the equation $a = 10^{10}$ in Planck units. If a represents a length, then the equation means $a = 1.6\times10^{-25}$ m. If a represents a mass, then the equation means $a = 220$ kg. Therefore, if the variable a was not clearly defined, then the equation $a = 10^{10}$ might be misinterpreted. By contrast, in SI units, the equation would be $a = 220$ kg, and it would be automatically clear that a represents a mass, not a length or anything else. In fact, natural units are especially useful when this ambiguity is *deliberate*: For example, in special relativity space and time are so closely related that it can be useful to not specify whether a variable represents a distance or a time.

Candidate physical constants used in natural unit systems

The candidate physical constants to be normalized are chosen from those in the following table. Note that only a smaller subset of the following can be normalized in any one system of units without contradiction in definition (*e.g.*, m_e and m_p cannot both be defined as the unit mass in a single system).

Constant	Symbol	Dimension
Speed of light in vacuum	$c = \frac{1}{\sqrt{\mu_0\epsilon_0}}$	L T^{-1}
Magnetic constant	$\mu_0 = \frac{1}{\epsilon_0 c^2}$	Q^{-2} M L
Electric constant	$\epsilon_0 = \frac{1}{\mu_0 c^2}$	Q^{2} M^{-1} L^{-3} T^{2}
Coulomb constant	$k_e = \frac{1}{4\pi\epsilon_0} = \frac{\mu_0 c^2}{4\pi}$	Q^{-2} M L^{3} T^{-2}

Characteristic impedance of free space	$Z_0 = \mu_0 c = \frac{1}{\epsilon_0 c}$	$Q^{-2} M L^2 T^{-1}$
Gravitational constant	G	$M^{-1} L^3 T^{-2}$
Planck constant (reduced)	$\hbar = \frac{h}{2\pi}$	$M L^2 T^{-1}$
Boltzmann constant	k_B	$M L^2 T^{-2} \Theta^{-1}$
Elementary charge	e	Q
Electron mass	m_e	M
Proton mass	m_p	M

Judiciously choosing units can only normalize physical constants that have dimension. Dimensionless physical constants cannot take on a different numerical value no matter what system of units is used. One such constant important to physics is the fine-structure constant, α:

$$\alpha \equiv \frac{e^2}{\hbar c(4\pi\epsilon_0)} = \frac{1}{137.035999679} = 7.2973525376 \cdot 10^{-3}$$

Since α is a fixed dimensionless number not equal to 1, it is not possible to define a system of natural units that will normalize *all* of the physical constants that comprise α. Any three of the four constants: c, $\hbar$, e, or $4\pi\varepsilon_0$, can be normalized (leaving the remaining physical constant to take on a value that is a simple function of α, attesting to the fundamental nature of the fine-structure constant) but not all four.

Systems of natural units

Planck units

Quantity	Expression	Metric value
Length (L)	$l_P = \sqrt{\frac{\hbar G}{c^3}}$	1.616252×10^{-35} m

Mass (M)	$m_P = \sqrt{\frac{\hbar c}{G}}$	$2.17644(11) \times 10^{-8}$ kg
Time (T)	$t_P = \sqrt{\frac{\hbar G}{c^5}}$	5.39124×10^{-44} s
Electric charge (Q)	$q_P = \sqrt{\hbar c (4\pi\epsilon_0)}$	$1.87554573 \times 10^{-18}$ C
Temperature (Θ)	$T_P = \sqrt{\frac{\hbar c^5}{G {k_B}^2}}$	1.416785×10^{32} K

$$c = G = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$e = \sqrt{\alpha} \approx 0.08542$$

Planck units are unique among systems of natural units, because they are not defined in terms of properties of any prototype, physical object, or even elementary particle. They only refer to the basic structure of the laws of physics: c and G are part of the structure of spacetime in general relativity, and $\hbar$ captures the relationship between energy and frequency which is at the foundation of quantum mechanics. This makes Planck units particularly useful and common in theories of quantum gravity, including string theory.

Some may consider Planck units to be "more natural" even than other natural unit systems discussed below. For example, some other systems use the mass of an electron as a parameter to be normalized. But the electron is just one of 15 known massive elementary particles, all with different masses, and there is no compelling reason, within fundamental physics, to emphasize the electron mass over some other elementary particle's mass.

"Natural units" (particle physics)

Unit	Metric value	Derivation
1 eV^{-1} of length	1.97×10^{-7} m	$= (1\text{eV}^{-1})\hbar c$
1 eV of mass	1.78×10^{-36} kg	$= (1\text{eV}) / c^2$
1 eV^{-1} of time	6.58×10^{-16} s	$= (1\text{eV}^{-1})\hbar$
1 unit of electric charge (for rationalized)	5.29×10^{-19} C	$= \sqrt{\hbar c \epsilon_0}$
1 eV of temperature	1.16×10^{4} K	$= 1\text{eV} / k_B$

In particle physics, the phrase "natural units" generally means:

$$\hbar = c = k_B = 1.$$

This is yet not enough information to define a unit system. Next, this equation is almost always supplemented with a definition of charge, from one of two possibilities:

- **Rationalized (Heaviside-Lorentz units)**:

  $\varepsilon_0 = \mu_0 = Z_0 = 1$

- **Not rationalized (Gaussian units)**:

$$\frac{1}{4\pi\epsilon_0} = \mu_0 = 1$$

In Heaviside-Lorentz units, there are factors of 4π in Coulomb's law and the Biot-Savart law but not in Maxwell's equations; In Gaussian units, it is the reverse. Both systems are used, although Heaviside-Lorentz is more common. The value of the elementary charge in the two systems is:

$e = \sqrt{\alpha} \approx 0.08542$ (Gaussian).
$e = \sqrt{4\pi\alpha} \approx 0.3028$ (Lorentz-Heaviside).

Finally, one more unit is needed. Most commonly, electron-volt (eV) is used, despite the fact that this is not a "natural" unit in the sense discussed above. (The SI prefixed multiples of eV are used as well: keV, MeV, GeV, etc.)

With the addition of eV (or any other auxiliary unit), any quantity can be expressed. For example, a distance of 1 cm can be expressed in terms of eV, in natural units, as:

$$1\,\mathrm{cm} = \frac{1\,\mathrm{cm}}{\hbar c} \approx 51000\,\mathrm{eV}^{-1}$$

Stoney units

Quantity	Expression	Metric Value
Length (L)	$l_S = \sqrt{\frac{Ge^2}{c^4(4\pi\epsilon_0)}}$	1.38068×10^{-36} m
Mass (M)	$m_S = \sqrt{\frac{e^2}{G(4\pi\epsilon_0)}}$	1.85921×10^{-9} kg

Time (T)	$t_S = \sqrt{\frac{Ge^2}{c^6(4\pi\epsilon_0)}}$	4.60544×10^{-45} s
Electric charge (Q)	$q_S = e$	1.60218×10^{-19} C
Temperature (Θ)	$T_S = \sqrt{\frac{c^4e^2}{G(4\pi\epsilon_0){k_B}^2}}$	1.21028×10^{31} K

Stoney units are defined by:

$$c = G = e = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$\hbar = \frac{1}{\alpha}$$

where α is the fine-structure constant.

George Johnstone Stoney was the first physicist to introduce the concept of natural units. He presented the idea in a lecture entitled "On the Physical Units of Nature" delivered to the British Association in 1874. Stoney units differ from Planck units by fixing the elementary charge at 1, instead of Planck's constant (only discovered after Stoney's proposal).

Stoney units are rarely used in modern physics for calculations, but they are of historical interest.

Atomic units

Quantity	Expression (Hartree atomic units)	Metric Value (Hartree atomic units)
Length (L)	$l_A = \frac{\hbar^2(4\pi\epsilon_0)}{m_e e^2}$	5.29177×10^{-11} m
Mass (M)	$m_A = m_e$	9.10938×10^{-31} kg
Time (T)	$t_A = \frac{\hbar^3(4\pi\epsilon_0)^2}{m_e e^4}$	2.41889×10^{-17} s
Electric charge (Q)	$q_A = e$	1.60218×10^{-19} C
Temperature (Θ)	$T_A = \frac{m_e e^4}{\hbar^2(4\pi\epsilon_0)^2 k_B}$	3.15774×10^{5} K

There are two types of atomic units, closely related: **Hartree atomic units**:

$$e = m_e = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$c = \frac{1}{\alpha}$$

Rydberg atomic units:

$$\frac{e}{\sqrt{2}} = 2m_e = \hbar = \frac{1}{4\pi\epsilon_0} = k_B = 1$$
$$c = \frac{2}{\alpha}$$

These units are designed to simplify atomic and molecular physics and chemistry, especially the hydrogen atom, and are widely used in these fields. The Hartree units were first proposed by Douglas Hartree, and are more common than the Rydberg units.

The units are designed especially to characterize the behavior of an electron in the ground state of a hydrogen atom. For example, using the Hartree convention, in the Bohr model of the hydrogen atom, an electron in the ground state has orbital velocity = 1, orbital radius = 1, angular momentum = 1, ionization energy = ½, etc.

The unit of energy is called the Hartree energy in the Hartree system and the Rydberg energy in the Rydberg system. They differ by a factor of 2. The speed of light is relatively large in atomic units (137 in Hartree or 274 in Rydberg), which comes from the fact that an electron in hydrogen tends to move much slower than the speed of light. The gravitational constant is extremely small in atomic units (around 10^{-45}), which comes from the fact that the gravitational force between two electrons is far weaker than the Coulomb force. The unit length, m_A, is the Bohr radius, a_0.

Quantum chromodynamics (QCD) system of units

Quantity	Expression
Length (L)	$l_{\mathrm{QCD}} = \frac{\hbar}{m_p c}$
Mass (M)	$m_{\mathrm{QCD}} = m_p$
Time (T)	$t_{\mathrm{QCD}} = \frac{\hbar}{m_p c^2}$
Electric charge (Q)	$q_{\mathrm{QCD}} = e$
Temperature (Θ)	$T_{\mathrm{QCD}} = \frac{m_p c^2}{k_B}$

$$c = e = m_p = \hbar = k_B = 1$$

$$\frac{1}{4\pi\epsilon_0} = \alpha$$

The electron mass is replaced with that of the proton and the permittivity of free space is not fixed by definition. *Strong units* are convenient for work in QCD and nuclear physics, where quantum mechanics and relativity are omnipresent and the proton is an object of central interest.

Geometrized units

$$c = G = 1$$

The geometrized unit system is not a completely defined or unique system. In this system, the base physical units are chosen so that the speed of light and the gravitational constant are set equal to unity, leaving latitude to also set some other constant such as the Boltzmann constant and Coulomb force constant equal to unity:

$$k_B = 1$$
$$\frac{1}{4\pi\epsilon_0} = 1$$

If the reduced Planck constant is also set equal to unity,

$$\hbar = 1$$

then geometrized units are identical to Planck units.

Summary table

Quantity / Symbol	Planck	Stoney	Atomic (Hartree)	"Natural Units" (Rationalized)	QCD
Speed of light in vacuum c	1	1	$\frac{1}{\alpha}$	1	1
Electric constant (vacuum permittivity) ϵ_0	$\frac{1}{4\pi}$	$\frac{1}{4\pi}$	$\frac{1}{4\pi}$	1	$\frac{1}{4\pi\alpha}$

Magnetic constant (vacuum permeability) $\mu_0 = \frac{1}{\epsilon_0 c^2}$	4π	4π	$4\pi\alpha^2$	1	$4\pi\alpha$
Characteristic impedance of vacuum $Z_0 = \frac{1}{\epsilon_0 c}$	4π	4π	$4\pi\alpha$	1	$4\pi\alpha$
Planck's constant (reduced) $\hbar = \frac{h}{2\pi}$	1	$\frac{1}{\alpha}$	1	1	1
Elementary charge e	$\sqrt{\alpha}$	1	1	$\sqrt{4\pi\alpha}$	1
Josephson constant $K_J = \frac{e}{\pi\hbar}$	$\frac{\sqrt{\alpha}}{\pi}$	$\frac{\alpha}{\pi}$	$\frac{1}{\pi}$	$\sqrt{\frac{4\alpha}{\pi}}$	$\frac{1}{\pi}$
von Klitzing constant $R_K = \frac{2\pi\hbar}{e^2}$	$\frac{2\pi}{\alpha}$	$\frac{2\pi}{\alpha}$	2π	$\frac{1}{2\alpha}$	2π
Gravitational constant G	1	1	$\alpha_G = \left(\frac{m_e}{m_P}\right)^2 G c^{-5} \hbar^{-1} \approx 10^{-56}\,\mathrm{eV}^{-2}$		$\mu^2 \alpha_G$

Electron mass m_e	$\sqrt{\alpha_G}\ \sqrt{\frac{\alpha_G}{\alpha}}\,1$	$m_e c^2 \approx 511\,\mathrm{keV}$	$\frac{1}{\mu}$

where:

- α is the fine-structure constant, $7.2973525376 \times 10^{-3} = (137.035999679)^{-1}$,
- α_G is the gravitational coupling constant, $(m_e/m_{Planck})^2 \approx 1.7518 \times 10^{-45}$,
- μ is the proton-to-electron mass ratio, about 1836.15267247.

Chapter 4

Buckingham π Theorem

The **Buckingham π theorem** is a key theorem in dimensional analysis. It is a formalization of Rayleigh's method of dimensional analysis. The theorem loosely states that if we have a physically meaningful equation involving a certain number, n, of physical variables, and these variables are expressible in terms of k independent fundamental physical quantities, then the original expression is equivalent to an equation involving a set of $p = n - k$ dimensionless parameters constructed from the original variables: it is a scheme for nondimensionalization. This provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Statement

More formally, the number of dimensionless terms that can be formed, p, is equal to the nullity of the dimensional matrix, and k is the rank. For the purposes of the experimenter, different systems which share the same description in terms of these dimensionless numbers are equivalent.

In mathematical terms, if we have a physically meaningful equation such as

$$f(q_1, q_2, \ldots, q_n) = 0$$

where the q_i are the n physical variables, and they are expressed in terms of k independent physical units, then the above equation can be restated as

$$F(\pi_1, \pi_2, \ldots, \pi_p) = 0$$

where the π_i are dimensionless parameters constructed from the q_i by $p = n - k$ equations of the form

$$\pi_i = q_1^{a_1}\, q_2^{a_2} \cdots q_n^{a_n}$$

where the exponents a_i are rational numbers (they can always be taken to be integers: just raise it to a power to clear denominators).

The use of the π_i as the dimensionless parameters was introduced by Edgar Buckingham in his original 1914 paper on the subject from which the theorem draws its name.

Significance

The Vaschy-Buckingham π theorem provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown. However, the choice of dimensionless parameters is not unique: Buckingham's theorem only provides a way of generating sets of dimensionless parameters, and will not choose the most 'physically meaningful'.

Two systems for which these parameters coincide are called *similar* (as with similar triangles, they differ only in scale); they are equivalent for the purposes of the equation, and the experimentalist who wants to determine the form of the equation can choose the most convenient one.

Proof

Outline

It will be assumed that the space of fundamental and derived physical units forms a vector space over the rational numbers, with the fundamental units as basis vectors, and with multiplication of physical units as the "vector addition" operation, and raising to powers as the "scalar multiplication" operation: represent a dimensional variable as the set of exponents needed for the fundamental units (with a power of zero if the particular fundamental unit is not present). For instance, the gravitational constant *g* has units of $\ell/t^2 = \ell^1 t^{-2}$ (distance over time squared), so it is represented as the vector $(1, -2)$ with respect to the basis of fundamental units (distance,time).

Making the physical units match across sets of physical equations can then be regarded as imposing linear constraints in the physical unit vector space.

Formal proof

Given a system of *n* dimensional variables (physical variables), in *k* (physical) dimensions, write the *dimensional matrix M*, whose rows are the dimensions and whose columns are the variables: the *(i,j)*th entry is the power of the *i*th unit in the *j*th variable. The matrix can be interpreted as taking in a combination of the dimensional quantities and giving out the dimensions of this product. So

$$M \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

is the units of

$$q_1^{a_1}\, q_2^{a_2} \cdots q_n^{a_n}.$$

A dimensionless variable is a combination whose units are all zero (hence, dimensionless), which is equivalent to the kernel of this matrix; a dimensionless variable is a linear relation between units of dimensional variables.

By the rank-nullity theorem, a system of n vectors in k dimensions (where all dimensions are necessary) satisfies a *(p=n-k)*-dimensional space of relations. Any choice of basis will have p elements, which are the dimensionless variables.

The dimensionless variables can always be taken to be integer combinations of the dimensional variables (by clearing denominators). There is mathematically no natural choice of dimensionless variables; some choices of dimensionless variables are more physically meaningful, and these are what are ideally used.

Examples

Speed

This example is elementary, but demonstrates the general procedure: Suppose a car is driving at 100 km/hour; how long does it take it to go 200 km?

This question has 2 fundamental physical units: time t and length ℓ, and 3 dimensional variables: distance D, time taken T, and velocity V. Thus there is 3-2=1 dimensionless quantity. The units of the dimensional quantities are:

$$D \sim \ell,\ T \sim t,\ V \sim \ell/t.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

The rows correspond to the dimensions ℓ, and t, and the columns to the dimensional variables D, T, V. For instance, the 3rd column, (1, −1), states that the V (velocity) variable has units of $\ell^1 t^{-1} = \ell/t$.

For a dimensionless constant $\pi = D^{a_1}T^{a_2}V^{a_3}$ we are looking for a vector $a = [a_1,a_2,a_3]$ such that the matrix product of M on a yields the zero vector [0,0]. In linear algebra, this vector is known as the kernel of the dimensional matrix, and it spans the nullspace of the dimensional matrix, which in this particular case is one dimensional. The dimensional matrix as written above is in reduced row echelon form, so one can read off that a kernel vector may be written (to within a multiplicative constant) by:

$$a = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

If the dimensional matrix were not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix in order to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\begin{aligned} \pi &= D^{-1}T^1V^1 \\ &= TV/D \end{aligned}$$

or, in dimensional terms:

$$\pi \sim (\ell)^{-1}(t)^1(\ell/t)^1 \sim 1$$

Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant.

Dimensional analysis has thus provided a general equation relating the three physical variables:

$$f(\pi) = 0$$

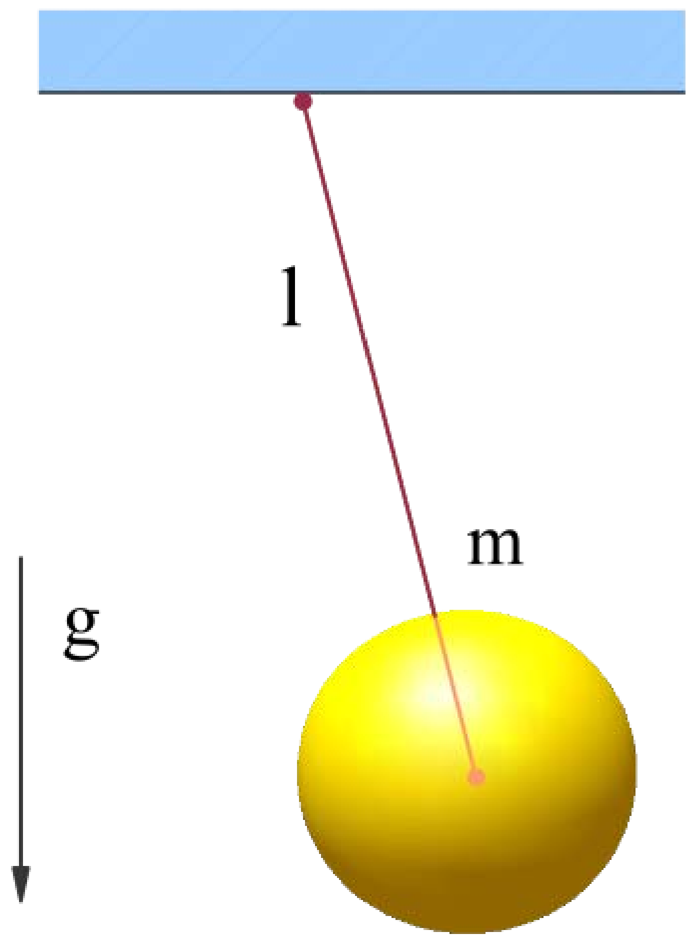
which may be written:

$$T = \frac{CD}{V}$$

where C is one of a set of constants, such that $C = f^{-1}(0)$. The actual relationship between the three variables is simply $D = VT$ so that the actual dimensionless equation ($f(\pi) = 0$) is written:

$$f(\pi) = \pi - 1 = VT / D - 1 = 0$$

In other words, there is only one value of C and it is unity. The fact that there is only a single value of C and that it is equal to unity is a level of detail not provided by the technique of dimensional analysis.



The simple pendulum

We wish to determine the period T of small oscillations in a simple pendulum. It will be assumed that it is a function of the length L , the mass M , and the acceleration due to gravity on the surface of the Earth g, which has units of length divided by time squared. The model is of the form

$$f(T, M, L, g) = 0.$$

(Note that it is written as a relation, not as a function: T isn't here written as a function of M, L, and g.)

There are 3 fundamental physical units in this equation: time t, mass m, and length l, and 4 dimensional variables, T, M, L, and g. Thus we need only 4−3=1 dimensionless parameter, denoted π, and the model can be re-expressed as

$$f(\pi) = 0$$

where π is given by

$$\pi = T^{a1} M^{a2} L^{a3} g^{a4}$$

for some values of a_1, ..., a_4.

The units of the dimensional quantities are:

$$T = t, M = m, L = \ell, g = \ell/t^2.$$

The dimensional matrix is:

$$M = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

(The rows correspond to the dimensions t, m, and l, and the columns to the dimensional variables T, M, L and g. For instance, the 4th column, (−2, 0, 1), states that the g variable has units of $t^{-2} m^0 \ell^1$.)

We are looking for a kernel vector a = [a1,a2,a3,a4] such that the matrix product of M on a yields the zero vector [0,0,0]. The dimensional matrix as written above is in reduced row echelon form, so one can read off that a kernel vector may be written (to within a multiplicative constant) by:

$$a = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Were it not already reduced, one could perform Gauss-Jordan elimination on the dimensional matrix in order to more easily determine the kernel. It follows that the dimensionless constant may be written:

$$\pi = T^2 M^0 L^{-1} g^1$$
$$= gT^2/L$$

In dimensional terms:

$$\pi = (t)^2 (m)^0 (\ell)^{-1} (\ell/t^2)^1 = 1$$

which is dimensionless. Since the kernel is only defined to within a multiplicative constant, if the above dimensionless constant is raised to any arbitrary power, it will yield another equivalent dimensionless constant.

This example is easy because 3 of the dimensional quantities are fundamental units, so the last (*g*) is a combination of the previous. Note that if a_2 were non-zero there would be no way to cancel the *M* value—therefore a_2 *must* be zero. Dimensional analysis has allowed us to conclude that the period of the pendulum is not a function of its mass. (In the 3D space of powers of mass, time, and distance, we can say that the vector for mass is linearly independent from the vectors for the three other variables. Up to a scaling factor, $\vec{g} - 2\vec{T} - \vec{L}$ is the only nontrivial way to construct a vector of a dimensionless parameter.)

The model can now be expressed as:

$$f(gT^2/L) = 0.$$

Assuming the zeroes of f are discrete, we can say $gT^2/L = C_n$ where C_n is the nth zero. If there is only one zero, then $gT^2/L = C$. It requires more physical insight or an experiment to show that there is indeed only one zero and that the constant is in fact given by $C = 4\pi^2$.

For large oscillations of a pendulum, the analysis is complicated by an additional dimensionless parameter, the maximum swing angle. The above analysis is a good approximation as the angle approaches zero.

The atomic bomb

Sir Geoffrey I. Taylor used dimensional analysis to estimate the energy released in an atomic bomb explosion (Taylor, 1950a,b). The first atomic bomb was detonated near Alamogordo, New Mexico on July 16, 1945. In 1947, movies of the explosion were declassified, allowing Sir Geoffrey to complete the analysis and estimate the energy released in the explosion, even though the energy release was still classified. The actual energy released (about 20ktons of TNT) was later declassified and its value was remarkably close to Taylor's estimate (16.8ktons).

Taylor supposed that the process was adequately described by five physical quantities: the time t since the detonation, the energy E which is released at a single point in space at detonation, the radius R of the shock wave at time T, the atmospheric pressure p and the ambient density ρ. There are only three fundamental physical units in this equation: mass, time, and length. Thus we need only 5 − 3 = 2 dimensionless parameters. The dimensional matrix for rows corresponding to L, t and m and columns corresponding to R, T, p, ρ and E is

$$M = \begin{bmatrix} 1 & 0 & -1 & -3 & 2 \\ 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

The nullspace is two dimensional. Two kernel vectors which span this space are:

$$A_0 = [5, -2, 0, 1, -1] \qquad \text{and} \qquad A_1 = [0, 6, 5, -3, -2]$$

which yield the dimensionless constants:

$$\pi_0 = R\left(\frac{\rho}{Et^2}\right)^{1/5} \qquad \text{and} \qquad \pi_1 = p\left(\frac{t^6}{E^2\rho^3}\right)^{1/5}$$

Note that there are an infinite number of dimensionless constant pairs that could be defined by taking linear combinations of the above two particular kernel vectors. The process can now be described by an equation of the form

$$f(\pi_0, \pi_1) = 0,$$

Assuming that inverting the equation yields a single possible value for R, this may be written:

$$R = \left(\frac{Et^2}{\rho}\right)^{1/5} g(\pi_1),$$

where $g(\pi_1)$ is some function of π_1. The energy in the explosion is expected to be huge, so that for times of the order of a second after the explosion, we can estimate π_1 to be approximately zero, and experiments using light explosives can be conducted to determine that $g(0)$ is on the order of unity so that

$$R \approx \left(\frac{Et^2}{\rho}\right)^{1/5}.$$

This is Taylor's equation which, once he knew the radius of the explosion as a function of the time, allowed him to calculate the energy of the explosion. (Wan, 1989)

Chapter 5

Homogeneity (Physics) & Fermi Problem

Homogeneity (Physics)

In general, **homogeneity** is defined as the quality or state of being *homogeneous* (of the same or similar nature, from Greek ὁμός meaning 'same'. It also means having a uniform structure throughout). For instance, a uniform electric field (which has the same strength and the same direction at each point) would be compatible with homogeneity (all points experience the same physics). A material constructed with different constituents can be described as effectively homogeneous in the electromagnetic domain, when interacting with a directed radiation field (light, microwave frequencies, etc.) In physics, homogeneous usually means describing a material or system that has the same properties at every point of the space; in other words, uniform without irregularities. In physics, it also describes a substance or an object whose properties do not vary with position. For example, an object of uniform density is sometimes described as homogeneous.

Another related definition is simply a substance that is uniform in composition.

Mathematically, homogeneity has the connotation of invariance, as all components of the equation have the same degree of value whether or not each of these components are scaled to different values, for example, by multiplication or addition. Cumulative distribution fits this description. "The state of having identical cumulative distribution function or values".

Context

The definition of homogeneous strongly depends on the context used. For example, a composite material is made up of different individual materials, known as "*constituents*" of the material, but may be defined as a homogeneous material when assigned a function. For example, asphalt paves our roads, but is a composite material consisting of asphalt binder and mineral aggregate, and then laid down in layers and compacted.

In another context, any material is actually made up of atoms and molecules. At this level, then, any material is not homogeneous. However, in our normal everyday world, a block of wood, a pane of glass, or a sheet of metal is described as wood, glass, or stainless steel. In other words, these are each described as a homogeneous material.

A few other instances of context are: *Dimensional homogeneity* is the quality of an equation having quantities of same units on both sides; *Homogeneity (in space)* implies conservation of momentum; and *homogeneity in time* implies conservation of energy.

Homogeneous alloy

In the context of composite metals is an alloy. A blend of a metal with one or more metallic or nonmetallic materials is an alloy. The components of an alloy do not combine chemically but, rather, are very finely mixed. An alloy might be homogeneous or might contain small particles of components that can be viewed with a microscope. Brass is an example of an alloy, being a homogeneous mixture of copper and zinc. Another example is steel, which is an alloy of iron with carbon and possibly other metals. The purpose of alloying is to produce desired properties in a metal that naturally lacks them. Brass, for example, is harder than copper and has a more gold-like color. Steel is harder than iron and can even be made rust proof (stainless steel).

Homogeneous cosmology

Homogeneity, in another context plays a role in cosmology. From the perspective of 19th-century cosmology (and before), the universe was infinite, unchanging, homogeneous, and therefore filled with stars. In 1826, this being true according to the German astronomer Heinrich Olbers, then the entire night sky should be filled with light and as bright as daytime, but it is dark. He presented a technical paper in 1826 that attempted to answer this conundrum. The faulty premise, unknown in Olbers' time, was that the universe is not infinite, static, and homogeneous. The Big Bang cosmology replaced this model (expanding, finite, and inhomogeneous universe). However, modern astronomers supply reasonable explanations to answer this question. One of at least several explanations is that distant stars and galaxies are red shifted, which weakens their apparent light and makes the night sky dark.

Translation invariance

By translation invariance, one means independence of (absolute) position, especially when referring to a law of physics, or to the evolution of a physical system.

Fundamental laws of physics should not (explicitly) depend on position in space. That would make them quite useless. In some sense, this is also linked to the requirement that experiments should be reproducible. This principle is true for all laws of mechanics (Newton's laws, etc.), electrodynamics, quantum mechanics, etc.

In practice, this principle is usually violated, since one studies only a small subsystem of the universe, which of course "feels" the influence of rest of the universe. This situation gives rise to "external fields" (electric, magnetic, gravitational, etc.) which make the description of the evolution of the system depending on the position (potential wells, etc.). This only stems from the fact that the objects creating these external fields are not considered as (a "dynamical") part of the system.

Translational invariance as described above is equivalent to shift invariance in system analysis, although here it is most commonly used in linear systems, whereas in physics the distinction is not usually made.

The notion of isotropy, for properties independent of direction, is not a consequence of homogeneity. For example, a uniform electric field (i.e., which has the same strength and the same direction at each point) would be compatible with homogeneity (at each point physics will be the same), but not with isotropy, since the field singles out one "preferred" direction.

Consequences

In Lagrangian formalism, homogeneity (in space) implies conservation of momentum, and homogeneity in time implies conservation of energy. This is shown, using variational calculus, in standard textbooks like the classical reference [Landau & Lifshitz] cited below. This is a particular application of Noether's theorem.

Dimensional homogeneity

As said in the introduction, *dimensional homogeneity* is the quality of an equation having quantities of same units on both sides. A valid equation in physics must be homogeneous, since equality cannot apply between quantities of different nature. This can be used to spot errors in formula or calculations. For example, if one is calculating a speed, units must always combine to [length]/[time]; if one is calculating an energy, units must always combine to [mass]•[length]^2/[time]^2, etc. For example, the following formulae could be valid expressions for some energy:

$$E_k = \frac{1}{2}mv^2; \quad E = mc^2; \quad E = pv; \quad E = hc/\lambda$$

if m is a mass, v and c are velocities, p is a momentum, h is Planck's constant, λ a length. On the other hand, if the units of the right hand side do not combine to [mass]•[length]^2/[time]^2, it cannot be a valid expression for some energy.

Being homogeneous does not necessarily mean the equation will be true, since it does not take into account numerical factors. For example, $E = m \cdot v^2$ could be or could not be the correct formula for the energy of a particle of mass m traveling at speed v, and one cannot know if $h \cdot c/\lambda$ should be divided or multiplied by 2π.

Nevertheless, this is a very powerful tool in finding characteristic units of a given problem.

Theoretical physicists tend to express everything in natural units given by constants of nature, for example by taking $c = \hbar = k = 1$; once this is done, one partly loses the possibility of the above checking. The atomic number was also known as a proton number.

Fermi Problem

In science, particularly in physics or engineering education, a **Fermi problem**, **Fermi question**, or **Fermi estimate** is an estimation problem designed to teach dimensional analysis, approximation, and the importance of clearly identifying one's assumptions. Named after physicist Enrico Fermi, such problems typically involve making justified guesses about quantities that seem impossible to compute given limited available information.

Fermi was known for his ability to make good approximate calculations with little or no actual data, hence the name. One example is his estimate of the strength of the atomic bomb detonated at the Trinity test, based on the distance travelled by pieces of paper dropped from his hand during the blast. Fermi's estimate of 10 kilotons of TNT was remarkably close to the now-accepted value of around 20 kilotons, a difference of less than one order of magnitude.

Examples of Fermi problems

The classic Fermi problem, generally attributed to Fermi, is "*How many piano tuners are there in Chicago?*" A typical solution to this problem would involve multiplying together a series of estimates that would yield the correct answer if the estimates were correct. For example, we might make the following assumptions:

1. There are approximately 5,000,000 people living in Chicago.
2. On average, there are two persons in each household in Chicago.
3. Roughly one household in twenty has a piano that is tuned regularly.
4. Pianos that are tuned regularly are tuned on average about once per year.
5. It takes a piano tuner about two hours to tune a piano, including travel time.
6. Each piano tuner works eight hours in a day, five days in a week, and 50 weeks in a year.

From these assumptions we can compute that the number of piano tunings in a single year in Chicago is

(5,000,000 persons in Chicago) / (2 persons/household) × (1 piano/20 households) × (1 piano tuning per piano per year) = 125,000 piano tunings per year in Chicago.

We can similarly calculate that the average piano tuner performs

(50 weeks/year)×(5 days/week)×(8 hours/day)/(1 piano tuning per 2 hours per piano tuner) = 1000 piano tunings per year per piano tuner.

Dividing gives

(125,000 piano tunings per year in Chicago) / (1000 piano tunings per year per piano tuner) = 125 piano tuners in Chicago.

A famous example of a Fermi-problem-like estimate is the Drake equation, which seeks to estimate the number of intelligent civilizations in the galaxy. The basic question of why, if there is a significant number of such civilizations, ours has never encountered any others is called the Fermi paradox.

Advantages and scope

Scientists often look for Fermi estimates of the answer to a problem before turning to more sophisticated methods to calculate a precise answer. This provides a useful check on the results: where the complexity of a precise calculation might obscure a large error, the simplicity of Fermi calculations makes them far less susceptible to such mistakes. (Performing the Fermi calculation first is preferable because the intermediate estimates might otherwise be biased by knowledge of the calculated answer.)

Fermi estimates are also useful in approaching problems where the optimal choice of calculation method depends on the expected size of the answer. For instance, a Fermi estimate might indicate whether the internal stresses of a structure are low enough that it can be accurately described by linear elasticity; or if the estimate already bears significant relationship in scale relative to some other value, for example, if a structure will be over-engineered to withstand loads several times greater than the estimate.

Although Fermi calculations are often not accurate, as there may be many problems with their assumptions, this sort of analysis does tell us what to look for to get a better answer. For the above example, we might try to find a better estimate of the number of pianos tuned by a piano tuner in a typical day, or look up an accurate number for the population of Chicago. It also gives us a rough estimate that may be good enough for some purposes: if we want to start a store in Chicago that sells piano tuning equipment, and we calculate that we need 10,000 potential customers to stay in business, we can reasonably assume that the above estimate is far enough below 10,000 that we should consider a different business plan (and, with a little more work, we could compute a rough upper bound on the number of piano tuners by considering the most extreme *reasonable* values that could appear in each of our assumptions).

A Fermi calculation that involves the multiplication of several estimated factors (e.g. the number of piano tuners in Chicago) will *probably* be more accurate than might be first supposed (assuming that there is no consistent bias in the estimated factors). This is because if there is no consistent bias, then there will probably (with a binomial distribution) be some factors that are estimated too high and other factors that are estimated too low, and such errors will partially cancel each other out.

Chapter 6

Nondimensionalization

Nondimensionalization is the partial or full removal of units from an equation involving physical quantities by a suitable substitution of variables. This technique can simplify and parameterize problems where measured units are involved. It is closely related to dimensional analysis. In some physical systems, the term **scaling** is used interchangeably with *nondimensionalization*, in order to suggest that certain quantities are better measured relative to some appropriate unit. These units refer to quantities intrinsic to the system, rather than units such as SI units. Nondimensionalization is not the same as converting extensive quantities in an equation to intensive quantities, since the latter procedure results in variables that still carry units.

Nondimensionalization can also recover characteristic properties of a system. For example, if a system has an intrinsic resonance frequency, length, or time constant, nondimensionalization can recover these values. The technique is especially useful for systems that can be described by differential equations. One important use is in the analysis of control systems. One of the simplest characteristic units is the doubling time of a system experiencing exponential growth, or conversely the half-life of a system experiencing exponential decay; a more natural pair of characteristic units is mean age/mean lifetime, which correspond to base *e* rather than base 2.

Many illustrative examples of nondimensionalization originate from simplifying differential equations. This is because a large body of physical problems can be formulated in terms of differential equations. Consider the following:

- List of dynamical systems and differential equations topics
- List of partial differential equation topics
- Differential equations of mathematical physics

Although nondimensionalization is well adapted for these problems, it is not restricted to them. An example of a non-differential-equation application is dimensional analysis, while another is normalization in statistics.

Measuring devices are practical examples of nondimensionalization occurring in everyday life. Measuring devices are calibrated relative to some known unit. Subsequent measurements are made relative to this standard. Then, the absolute value of the measurement is recovered by scaling with respect to the standard.

Rationale

Suppose a pendulum is swinging with a particular period *T*. For such a system, it is advantageous to perform calculations relating to the swinging relative to *T*. In some sense, this is normalizing the measurement with respect to the period.

Measurements made relative to an intrinsic property of a system will apply to other systems which also have the same intrinsic property. It also allows one to compare a common property of different implementations of the same system. Nondimensionalization determines in a systematic manner the **characteristic units** of a system to use, without relying heavily on prior knowledge of the system's intrinsic properties (one should not confuse characteristic units of a *system* with natural units of *nature*). In fact, nondimensionalization can suggest the parameters which should be used for analyzing a system. However, it is necessary to start with an equation that describes the system appropriately.

Nondimensionalization steps

To nondimensionalize a system of equations, one must do the following:

1. Identify all the independent and dependent variables;
2. Replace each of them with a quantity scaled relative to a characteristic unit of measure to be determined;
3. Divide through by the coefficient of the highest order polynomial or derivative term;
4. Choose judiciously the definition of the characteristic unit for each variable so that the coefficients of as many terms as possible become 1;
5. Rewrite the system of equations in terms of their new dimensionless quantities.

The last three steps are usually specific to the problem where nondimensionalization is applied. However, almost all systems require the first two steps to be performed.

As an illustrative example, consider a first order differential equation with constant coefficients:

$$a\frac{dx}{dt} + bx = Af(t).$$

1. In this equation the independent variable here is t, and the dependent variable is x.
2. Set $x = \chi x_c, \; t = \tau t_c$. This results in the equation

$$a\frac{x_c}{t_c}\frac{d\chi}{d\tau} + bx_c\chi = Af(\tau t_c) \stackrel{\text{def}}{=} AF(\tau).$$

3. The coefficient of the highest ordered term is in front of the first derivative term. Dividing by this gives

$$\frac{d\chi}{d\tau} + \frac{bt_c}{a}\chi = \frac{At_c}{ax_c}F(\tau).$$

4. The coefficient in front of χ only contains one characteristic variable t_c, hence it is easiest to choose to set this to unity first:

$\frac{bt_c}{a} = 1 \Rightarrow t_c = \frac{a}{b}$. Subsequently, $\frac{At_c}{ax_c} = \frac{A}{bx_c} = 1 \Rightarrow x_c = \frac{A}{b}$.

5. The final dimensionless equation in this case becomes completely independent of any parameters with units:

$$\frac{d\chi}{d\tau} + \chi = F(\tau).$$

Substitutions

Suppose for simplicity that a certain system is characterized by two variables - a dependent variable *x* and an independent variable *t*, where *x* is a function of *t*. Both *x* and *t* represent quantities with units. To scale these two variables, assume there are two intrinsic units of measurement x_c and t_c with the same units as *x* and *t* respectively, such that these conditions hold:

$$\kappa = \frac{t}{t_c} \Rightarrow t = \kappa t_c$$
$$\chi = \frac{x}{x_c} \Rightarrow x = \chi x_c.$$

These equations are used to replace *x* and *t* when nondimensionalizing. If differential operators are needed to describe the original system, their scaled counterparts become dimensionless differential operators.

Conventions

There are no restrictions on the variable names used to replace "*x*" and "*t*". However, they are generally chosen so that it is convenient and intuitive to use for the problem at hand. For example, if "*x*" represented mass, the letter "*m*" might be an appropriate symbol to represent the dimensionless mass quantity.

Here, the following conventions have been used:

- t - represents the independent variable - usually a time quantity. Its nondimensionalized counterpart is τ.
- x - represents the dependent variable - can be mass, voltage, or any measurable quantity. Its nondimensionalized counterpart is χ.

A subscripted c added to a quantity's variable-name is used to denote the characteristic unit used to scale that quantity. For example, if x is a quantity, then x_c is the characteristic unit used to scale it.

Differential operators

Consider the relationship

$$t = \tau t_c \Rightarrow dt = t_c d\tau \Rightarrow \frac{d\tau}{dt} = \frac{1}{t_c}.$$

The dimensionless differential operators with respect to the independent variable becomes

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{t_c}\frac{d}{d\tau} \Rightarrow \frac{d^n}{dt^n} = \left(\frac{d}{dt}\right)^n = \left(\frac{1}{t_c}\frac{d}{d\tau}\right)^n = \frac{1}{t_c^n}\frac{d^n}{d\tau^n}.$$

Forcing function

If a system has a forcing function $f(t)$, then

$$f(t) = f(\tau t_c) = f(t(\tau)) = F(\tau).$$

Hence, the new forcing function F is made to be dependent on the dimensionless quantity τ.

Linear differential equations with constant coefficients

First order system

Let us consider the differential equation for a first order system:

$$a\frac{dx}{dt} + bx = Af(t).$$

The derivation of the characteristic units for this system gives

$$t_c = \frac{a}{b},\ x_c = \frac{A}{b}.$$

Second order system

A second order system has the form

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = Af(t).$$

Substitution step

Replace the variables *x* and *t* with their scaled quantities. The equation becomes

$$a\frac{x_c}{t_c^2}\frac{d^2\chi}{d\tau^2} + b\frac{x_c}{t_c}\frac{d\chi}{d\tau} + cx_c\chi = Af(\tau t_c) = AF(\tau).$$

This new equation is not dimensionless, although all the variables with units are isolated in the coefficients. Dividing by the coefficient of the highest ordered term, the equation becomes

$$\frac{d^2\chi}{d\tau^2} + t_c\frac{b}{a}\frac{d\chi}{d\tau} + t_c^2\frac{c}{a}\chi = \frac{At_c^2}{ax_c}F(\tau).$$

Now it is necessary to determine the quantities of x_c and t_c so that the coefficients become normalized. Since there are two free parameters, at most only two coefficients can be made to equal unity.

Determination of characteristic units

Consider the variable t_c:

1. If $t_c = \frac{a}{b}$ the first order term is normalized.
2. If $t_c = \sqrt{\frac{a}{c}}$ the zeroth order term is normalized.

Both substitutions are valid. However, for pedagogical reasons, the latter substitution is used for second order systems. Choosing this substitution allows x_c to be determined by normalizing the coefficient of the forcing function:

$$1 = \frac{At_c^2}{ax_c} = \frac{A}{cx_c} \Rightarrow x_c = \frac{A}{c}.$$

The differential equation becomes

$$\frac{d^2\chi}{d\tau^2} + \frac{b}{\sqrt{ac}}\frac{d\chi}{d\tau} + \chi = F(\tau).$$

The coefficient of the first order term is unitless. Define

$$2\zeta \stackrel{\text{def}}{=} \frac{b}{\sqrt{ac}}.$$

The factor 2 is present so that the solutions can be parameterized in terms of ζ. In the context of mechanical or electrical systems, ζ is known as the damping ratio, and is an important parameter required in the analysis of control systems. 2ζ is also known as the linewidth of the system. The result of the definition is the universal oscillator equation.

$$\frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = F(\tau).$$

Higher order systems

The general n-th order linear differential equation with constant coefficients has the form:

$$a_n\frac{d^n x(t)}{dt^n} + a_{n-1}\frac{d^{n-1}x(t)}{dt^{n-1}} + \ldots + a_1\frac{dx(t)}{dt} + a_0 x(t) = \sum_{k=0}^{n} a_k\frac{d^k x(t)}{dt^k} = Af(t).$$

The function $f(t)$ is known as the forcing function.

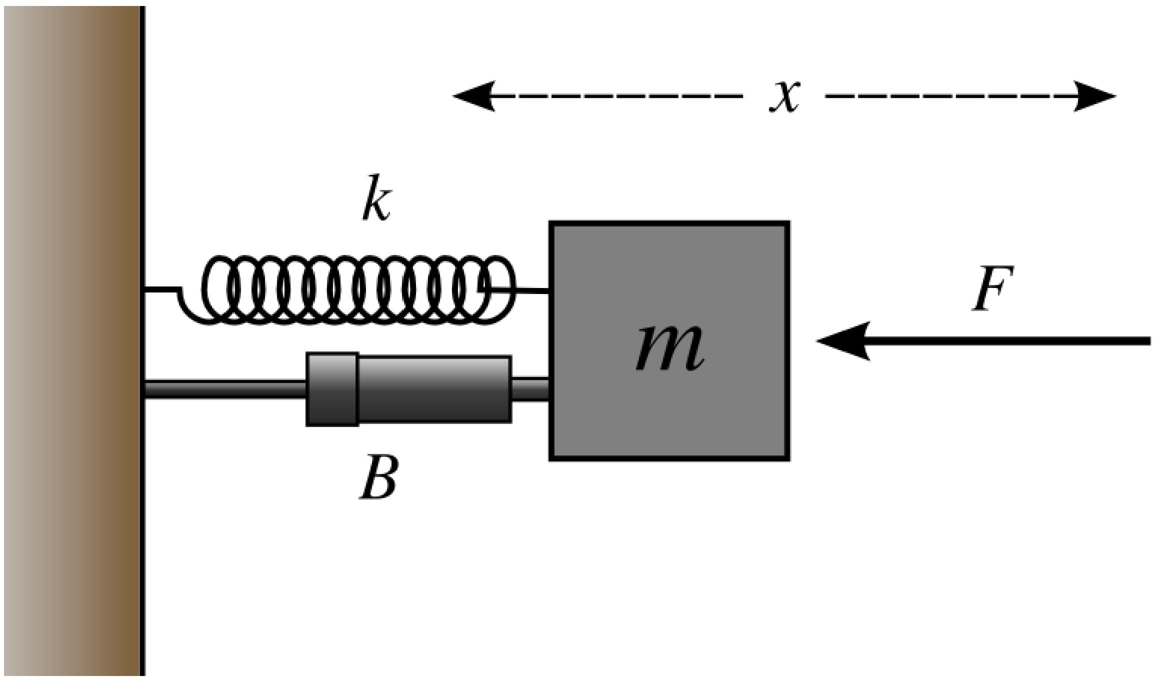
If the differential equation only contains real (not complex) coefficients, then the properties of such a system behaves as a mixture of first and second order systems only. This is because the roots of its characteristic polynomial are either real, or complex conjugate pairs. Therefore, understanding how nondimensionalization applies to first and second ordered systems allows the properties of higher order systems to be determined through superposition.

The number of free parameters in a nondimensionalized form of a system increases with its order. For this reason, nondimensionalization is rarely used for higher order differential equations. The need for this procedure has also been reduced with the advent of symbolic computation.

Examples of recovering characteristic units

A variety of systems can be approximated as either first or second order systems. These include mechanical, electrical, fluidic, caloric, and torsional systems. This is because the fundamental physical quantities involved within each of these examples are related through first and second order derivatives.

Mechanical oscillations



A mass attached to a spring and a dashpot.

Suppose we have a mass attached to a spring and a damper, which in turn are attached to a wall, and a force acting on the mass along the same line.

Define

x = displacement from equilibrium [m]
t = time [s]
f = external force or "disturbance" applied to system [kg m s^{-2}]
m = mass of the block [kg]
B = damping constant of dashpot [kg s^{-1}]
k = force constant of spring [kg s^{-2}]

Suppose the applied force is a sinusoid $F = F_0 \cos(\omega t)$, the differential equation that describes the motion of the block is

$$m\frac{d^2x}{dt^2} + B\frac{dx}{dt} + kx = F_0 \cos(\omega t)$$

Nondimensionalizing this equation the same way as described under second order system yields several characteristics of the system.

The intrinsic unit x_c corresponds to the distance the block moves per unit force

$$x_c = \frac{F_0}{k}.$$

The characteristic variable t_c is equal to the period of the oscillations

$$t_c = \sqrt{\frac{m}{k}}$$

and the dimensionless variable 2ζ corresponds to the linewidth of the system. ζ itself is the damping ratio.

$$2\zeta = \frac{B}{\sqrt{mk}}$$

Electrical oscillations

First-order series RC circuit

For a series RC attached to a voltage source

$$R\frac{dQ}{dt} + \frac{Q}{C} = V(t) \Rightarrow \frac{d\chi}{d\tau} + \chi = F(\tau)$$

with substitutions

$$Q = \chi x_c,\ t = \tau t_c,\ x_c = CV_0,\ t_c = RC,\ F = V.$$

The first characteristic unit corresponds to the total charge in the circuit. The second characteristic unit corresponds to the time constant for the system.

Second-order series RLC circuit

For a series configuration of R,C,L components where Q is the charge in the system

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0\cos(\omega t) \Rightarrow \frac{d^2\chi}{d\tau^2} + 2\zeta\frac{d\chi}{d\tau} + \chi = \cos(\Omega\tau)$$

with the substitutions

$$Q = \chi x_c,\ t = \tau t_c,\ \ x_c = CV_0,\ t_c = \sqrt{LC},\ 2\zeta = R\sqrt{\frac{C}{L}},\ \Omega = t_c\omega.$$

The first variable corresponds to the maximum charge stored in the circuit. The resonance frequency is given by the reciprocal of the characteristic time. The last expression is the linewidth of the system. The Ω can be considered as a normalized forcing function frequency.

Nonlinear differential equation example

Since there are no general methods of solving nonlinear differential equations, each case has to be considered on an individual basis when nondimensionalizing.

Quantum harmonic oscillator

The Schrödinger equation for the one dimensional time independent quantum harmonic oscillator is

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+\frac{1}{2}m\omega^2x^2\right)\psi(x)=E\psi(x).$$

The wavefunction ψ itself represents probability, which is in a sense already dimensionless and normalized. Therefore, there is no need to nondimensionalize the wavefunction. However, it should be rewritten as a function of a dimensionless variable. Furthermore, the variable x has units of length. Hence substitute

$$\chi=\frac{x}{x_c},\ \psi(x)=\psi(x(\chi))=\psi(\chi).$$

The differential equation becomes

$$\left(-\frac{\hbar^2}{2mx_c^2}\frac{d^2}{d\chi^2}+\frac{m\omega^2x_c^2}{2}\chi^2\right)\psi(\chi)=E\psi(\chi)\Rightarrow\left(-\frac{d^2}{d\chi^2}+\frac{m^2\omega^2x_c^4}{\hbar^2}\chi^2\right)\psi(\chi)=\frac{2Emx_c^2}{\hbar^2}\psi(\chi).$$

To make the term in front of χ^2 unitless, set

$$\frac{m^2\omega^2x_c^4}{\hbar^2}=1\Rightarrow x_c=\sqrt{\frac{\hbar}{m\omega}}.$$

Hence, the fully nondimensionalized equation is

$$\left(-\frac{d^2}{d\chi^2}+\chi^2\right)\psi(\chi)=\frac{2E}{\hbar\omega}\psi(\chi)\stackrel{\text{def}}{=}\mathcal{E}\psi(\chi).$$

The nondimensionalization factor for the energy is the same as the ground state of the harmonic oscillator. Usually, the energy term is not made dimensionless because a primary emphasis of quantum mechanics is determining the energies of the states of a system. Rearranging the first equation, the familiar equation for the harmonic oscillator is

$$\frac{\hbar\omega}{2}\left(-\frac{d^2}{d\chi^2}+\chi^2\right)\psi(\chi)=E\psi(\chi).$$

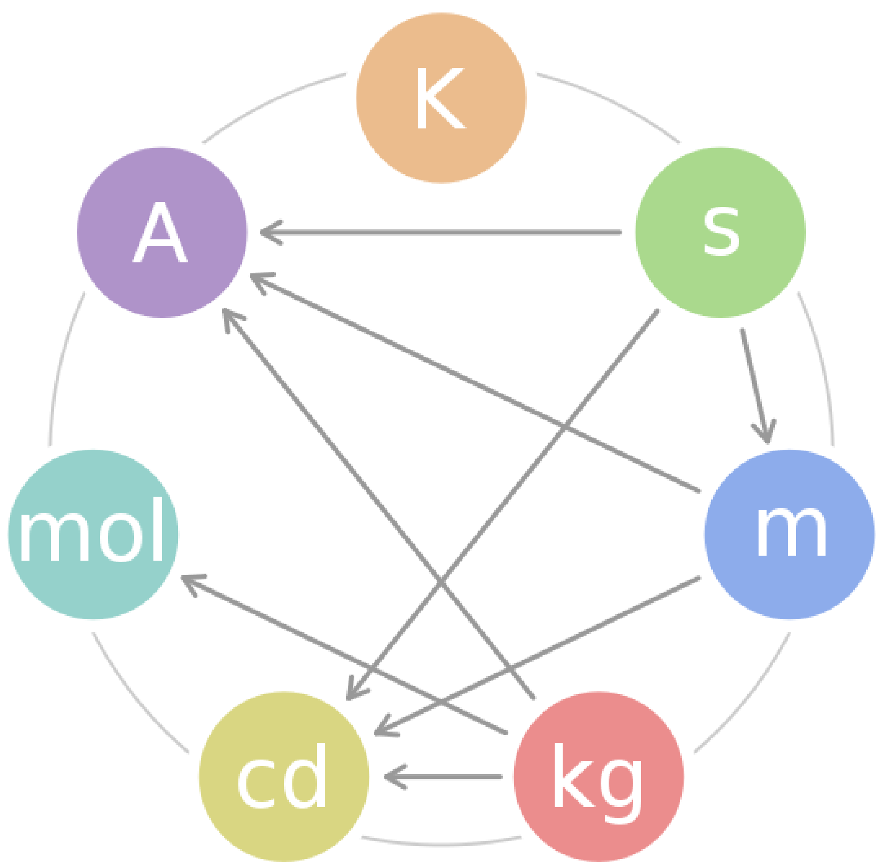
Statistical analogs

In statistics, the analogous process is usually dividing a difference (a distance) by a scale factor (a measure of statistical dispersion), which yields a dimensionless number, which is called *normalization.* Most often, this is dividing errors or residuals by the standard deviation or sample standard deviation, respectively, yielding standard scores and studentized residuals.

Chapter 7

SI Base Unit & Similitude (Model)

SI Base Unit



The seven SI base units and the interdependency of their definitions

The International System of Units (**SI**) defines seven units of measure as a basic set from which all other SI units are derived. These **SI base units** and their physical quantities are:

- metre for length
- kilogram for mass
- second for time
- ampere for electric current
- kelvin for temperature
- candela for luminous intensity
- mole for the amount of substance.

The SI base quantities form a set of mutually independent dimensions as required by dimensional analysis commonly employed in science and technology. However, in a given realization of these units they may well be interdependent, i.e. defined in terms of each other.

The names of all SI units are written in lowercase characters (e.g., the *metre* has the symbol m), except that the symbols of units named after persons are written with an initial capital letter (e.g., the *ampere* has the uppercase symbol A).

Many other units, such as the litre, are formally not part of the SI, but are accepted for use with SI.

SI base units

Name	Symbol	Measure	Definition	Historical Origin / Justification
metre	**m**	length	"The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second." *17th CGPM (1983, Resolution 1, CR, 97)*	$^1⁄_{10,000,000}$ of the distance from the Earth's equator to the North Pole measured on the circumference through Paris.
kilogram	**kg**	mass	"The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram." *3rd CGPM (1901, CR, 70)*	The mass of one litre of water. A litre is one thousandth of a cubic metre.
second	**s**	time	"The second is the duration of 9 192 631	The day is divided in 24 hours, each hour

			770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom." *13th CGPM (1967/68, Resolution 1; CR, 103)* "This definition refers to a caesium atom at rest at a temperature of 0 K." *(Added by CIPM in 1997)*	divided in 60 minutes, each minute divided in 60 seconds. A second is $^{1}/_{(24 \times 60 \times 60)}$ of the day
ampere	**A**	electric current	"The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length." *9th CGPM (1948)*	The original "International Ampere" was defined electrochemically as the current required to deposit 1.118 milligrams of silver per second from a solution of silver nitrate. Compared to the SI ampere, the difference is 0.015%.
kelvin	**K**	thermodynamic temperature	"The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water." *13th CGPM (1967/68, Resolution 4; CR, 104)* "This definition refers to water having the isotopic composition defined exactly by the following amount of substance ratios: 0.000 155 76 mole of ^{2}H per	The Celsius scale: the Kelvin scale uses the degree Celsius for its unit increment, but is a thermodynamic scale (0 K is absolute zero).

			mole of ^{1}H, 0.000 379 9 mole of ^{17}O per mole of ^{16}O, and 0.002 005 2 mole of ^{18}O per mole of ^{16}O." *(Added by CIPM in 2005)*	
mole	**mol**	amount of substance	"1. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is "mol." 2. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles." *14th CGPM (1971, Resolution 3; CR, 78)* "In this definition, it is understood that unbound atoms of carbon 12, at rest and in their ground state, are referred to." *(Added by CIPM in 1980)*	Atomic weight or molecular weight divided by the molar mass constant, 1 g/mol.

candela	**cd**	luminous intensity	"The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian." *16th CGPM (1979, Resolution 3; CR, 100)*	The candlepower, which is based on the light emitted from a burning candle of standard properties.

Future redefinitions

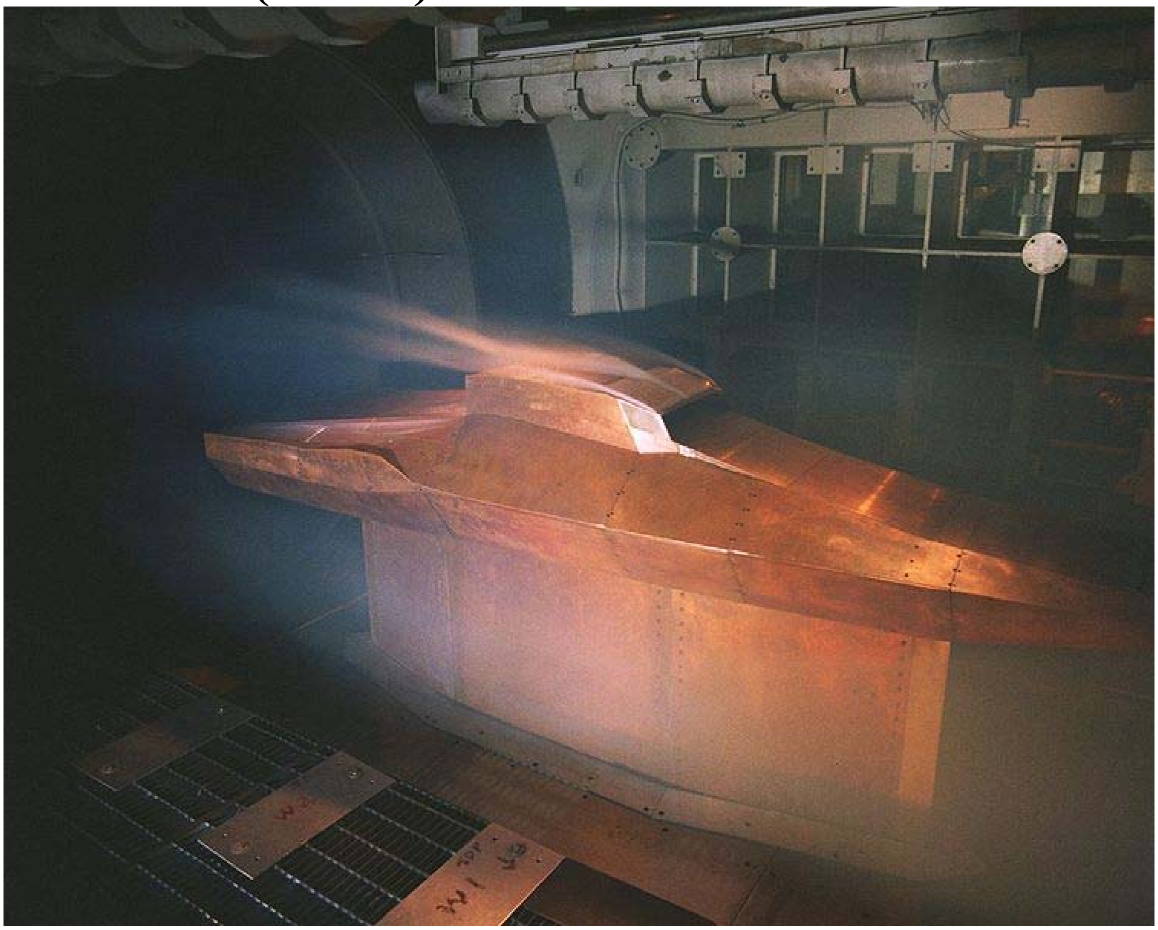
There have been several modifications to the definitions of the base units, and additions of base units, since the Metre Convention in 1875. Since the redefinition of the metre in 1960, the kilogram is the only unit which is directly defined in terms of a physical artifact rather than a property of nature. However, the mole, the ampere and the candela are also linked through their definitions to the mass of this platinum–iridium cylinder stored in a vault near Paris. It has long been an objective of metrology to find a way to define the kilogram in terms of a fundamental constant, in the same way that the metre is now defined in terms of the speed of light.

The 21st General Conference on Weights and Measures (CGPM, 1999) placed these efforts on an official footing, and recommended "that national laboratories continue their efforts to refine experiments that link the unit of mass to fundamental or atomic constants with a view to a future redefinition of the kilogram." Two main possibilities have attracted attention: the Planck constant and the Avogadro constant.

In 2005, the International Committee for Weights and Measures (CIPM) approved the preparation of new definitions for the kilogram, the ampere, and the kelvin and it noted the possibility of a new definition for the mole based on the Avogadro constant. The 23rd CGPM (2007) decided to postpone any legal change until the next General Conference in 2011.

In a note to the CIPM in October 2009, Ian Mills, the President of the CIPM *Consultative Committee - Units (CCU)* cataloged the uncertainties of the fundamental constants of physics according to the current definitions and their values under the proposed new definition. He urged the CIPM to accept the proposed changes in the definition of the *kilogram*, *ampere*, *kelvin* and *mole* so that they are referenced to the values of the fundamental constants, namely Planck's constant (h), the electron charge (e), Boltzmann's constant (k), and Avogadro's constant (N_A).

Similitude (Model)



A full scale X-43 Wind tunnel test. The test is designed to have **dynamic similitude** with the real application to ensure valid results.

Similitude is a concept used in the testing of engineering models. A model is said to have similitude with the real application if the two share geometric similarity, kinematic similarity and dynamic similarity. *Similarity* and *similitude* are interchangeable in this context.

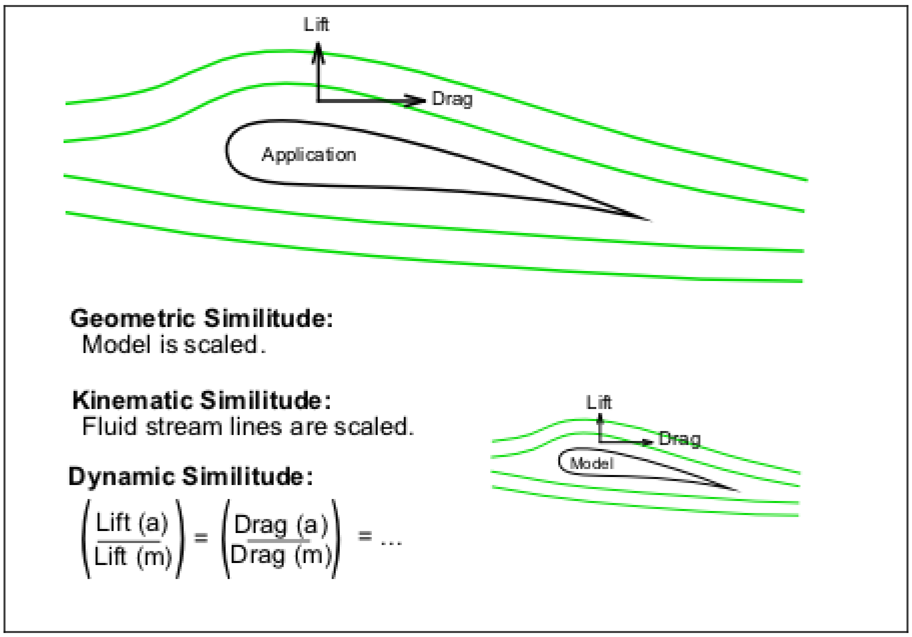
The term **dynamic similitude** is often used as a catch-all because it implies that geometric and kinematic similitude have already been met.

Similitude's main application is in hydraulic and aerospace engineering to test fluid flow conditions with scaled models. It is also the primary theory behind many textbook formulas in fluid mechanics.

Overview

Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations aren't reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process.

Construction of a scale model, however, must be accompanied by an analysis to determine what conditions it is tested under. While the geometry may be simply scaled, other parameters, such as pressure, temperature or the velocity and type of fluid may need to be altered. Similitude is achieved when testing conditions are created such that the test results are applicable to the real design.



The three conditions required for a model to have similitude with an application.

The following criteria are required to achieve similitude;

- **Geometric similarity** - The model is the same shape as the application, usually scaled.
- **Kinematic similarity** - Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)
- **Dynamic similarity** - Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

To satisfy the above conditions the application is analyzed;

1. All parameters required to describe the system are identified using principles from continuum mechanics.

2. Dimensional analysis is used to express the system with as few independent variables and as many dimensionless parameters as possible.
3. The values of the dimensionless parameters are held to be the same for both the scale model and application. This can be done because they are *dimensionless* and will ensure dynamic similitude between the model and the application. The resulting equations are used to derive *scaling laws* which dictate model testing conditions.

It is often impossible to achieve strict similitude during a model test. The greater the departure from the application's operating conditions, the more difficult achieving similitude is. In these cases some aspects of similitude may be neglected, focusing on only the most important parameters.

The design of marine vessels remains more of an art than a science in large part because dynamic similitude is especially difficult to attain for a vessel that is partially submerged: a ship is affected by wind forces in the air above it, by hydrodynamic forces within the water under it, and especially by wave motions at the interface between the water and the air. The scaling requirements for each of these phenomena differ, so models cannot replicate what happens to a full sized vessel nearly so well as can be done for an aircraft or submarine—each of which operates entirely within one medium.

Similitude is a term used widely in fracture mechanics relating to the strain life approach. Under given loading conditions the fatigue damage in an un-notched specimen is comparable to that of a notched specimen. Similitude suggests that the component fatigue life of the two objects will also be similar.

An example

Consider a submarine modeled at 1/40th scale. The application operates in sea water at 0.5 °C, moving at 5 m/s. The model will be tested in fresh water at 20 °C. Find the power required for the submarine to operate at the stated speed.

A free body diagram is constructed and the relevant relationships of force and velocity are formulated using techniques from continuum mechanics. The variables which describe the system are:

Variable	Application	Scaled mode	Units
L (diameter of submarine)	1	1/40	(m)
V (speed)	5	*calculate*	(m/s)
ρ (density)	1028	998	(kg/m^3)
μ (dynamic viscosity)	$1.88\text{x}10^{-3}$	$1.00\text{x}10^{-3}$	Pa·s (N s/m^2)
F (force)	*calculate*	*to be measured*	N (kg m/s^2)

This example has five independent variables and three fundamental units. The fundamental units are: metre, kilogram, second.

Invoking the Buckingham π theorem shows that the system can be described with two dimensionless numbers and one independent variable.

Dimensional analysis is used to re-arrange the units to form the Reynolds number (R_e) and pressure coefficient (C_p). These dimensionless numbers account for all the variables listed above except F, which will be the test measurement. Since the dimensionless parameters will stay constant for both the test and the real application, they will be used to formulate scaling laws for the test.

Scaling laws:

$$R_e = \left(\frac{\rho V L}{\mu}\right) \longrightarrow V_{model} = V_{application} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{L_a}{L_m}\right) \times \left(\frac{\mu_m}{\mu_a}\right)$$

$$C_p = \left(\frac{2\Delta P}{\rho V^2}\right), F = \Delta P L^2 \longrightarrow F_{application} = F_{model} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{V_a}{V_m}\right)^2 \times \left(\frac{L_a}{L_m}\right)^2$$

This gives a required test velocity of:

$$V_{model} = V_{application} \times 21.9.$$

The force measured from the model at that velocity is then scaled to find the force that can be expected for the real application:

$$F_{application} = F_{model} \times 3.44$$

The power P in watts required by the submarine is then:

$$P[\mathrm{W}] = F_{application} \times V_{application} = F_{model}[\mathrm{N}] \times 17.2 \text{ m/s}$$

Note that even though the model is scaled smaller, the water velocity needs to be increased for testing. This remarkable result shows how similitude in nature is often counterintuitive.

Typical applications

Similitude has been well documented for a large number of engineering problems and is the basis of many textbook formulas and dimensionless quantities. These formulas and quantities are easy to use without having to repeat the laborious task of dimensional analysis and formula derivation. Simplification of the formulas (by neglecting some aspects of similitude) is common, and needs to be reviewed by the engineer for each application.

Similitude can be used to predict the performance of a new design based on data from an existing, similar design. In this case, the model is the existing design. Another use of similitude and models is in validation of computer simulations with the ultimate goal of eliminating the need for physical models altogether.

Another application of similitude is to replace the operating fluid with a different test fluid. Wind tunnels, for example, have trouble with air liquefying in certain conditions so helium is sometimes used. Other applications may operate in dangerous or expensive fluids so the testing is carried out in a more convenient substitute.

Some common applications of similitude and associated dimensionless numbers;

Incompressible flow -	Reynolds number, Pressure coefficient, (Froude number and Weber number for open channel hydraulics)
Compressible flows -	Reynolds number, Mach number, Prandtl number, Specific heat ratio
Flow excited vibration	Strouhal number
Centrifugal compressors -	Reynolds number, Mach number, Pressure coefficient, Velocity ratio

Chapter 8

Units Conversion by Factor-Label

Many, if not most, parameters and measurements in the physical sciences and engineering are expressed as a numerical quantity and a corresponding dimensional unit; for example: 1000 kg/m³, 100 kPa/bar, 50 miles per hour, 1000 Btu/lb. Converting from one dimensional unit to another is often somewhat complex and being able to perform such conversions is an important skill to acquire. The **factor-label method**, also known as the **unit-factor method** or **dimensional analysis**, is a widely used approach for performing such conversions. It is also used for determining whether the two sides of a mathematical equation involving dimensions have the same dimensional units.

The factor-label method for converting units

The factor-label method is the sequential application of conversion factors expressed as fractions and arranged so that any dimensional unit appearing in both the numerator and denominator of any of the fractions can be cancelled out until only the desired set of dimensional units is obtained. For example, 10 miles per hour can be converted to meters per second by using a sequence of conversion factors as shown below:

```
10 mile   1609 meter      1 hour          meter
-- ---- × ---- ----- × ---- ------  = 4.47 ------
 1 hour      1 mile    3600 second         second
```

It can be seen that each conversion factor is equivalent to the value of one. For example, starting with 1 mile = 1609 meters and dividing both sides of the equation by 1 mile yields 1 mile / 1 mile = 1609 meters / 1 mile, which when simplified yields 1 = 1609 meters / 1 mile.

So, when the units *mile* and *hour* are cancelled out and the arithmetic is done, 10 miles per hour converts to 4.47 meters per second.

As a more complex example, the concentration of nitrogen oxides (i.e., NOx) in the flue gas from an industrial furnace can be converted to a mass flow rate expressed in grams per hour (i.e., g/h) of NOx by using the following information as shown below:

NOx concentration

= 10 parts per million by volume = 10 ppmv = 10 volumes/10^6 volumes

NOx molar mass

= 46 kg/kgmol (sometimes also expressed as 46 kg/kmol)

Flow rate of flue gas

= 20 cubic meters per minute = 20 m³/min

The flue gas exits the furnace at 0 °C temperature and 101.325 kPa absolute pressure.

The molar volume of a gas at 0 °C temperature and 101.325 kPa is 22.414 m³/kgmol.

$$\frac{10\ \cancel{m^3\ NOx}}{10^6\ \cancel{m^3\ gas}} \times \frac{20\ \cancel{m^3\ gas}}{1\ \cancel{minute}} \times \frac{60\ \cancel{minute}}{1\ hour} \times \frac{1\ \cancel{kgmol\ NOx}}{22.414\ \cancel{m^3\ NOx}} \times \frac{46\ \cancel{kg}\ NOx}{1\ \cancel{kgmol\ NOx}} \times \frac{1000\ g}{1\ \cancel{kg}} = 24.63\ \frac{g\ NOx}{hour}$$

After cancelling out any dimensional units that appear both in the numerators and denominators of the fractions in the above equation, the NOx concentration of 10 ppm_v converts to mass flow rate of 24.63 grams per hour.

Checking equations that involve dimensions

The factor-label method can also be used on any mathematical equation to check whether or not the dimensional units on the left hand side of the equation are the same as the dimensional units on the right hand side of the equation. Having the same units on both sides of an equation does not guarantee that the equation is correct, but having different units on the two sides of an equation does guarantee that the equation is wrong.

For example, check the Universal Gas Law equation of $P \cdot V = n \cdot R \cdot T$, when:

- the pressure P is in pascals (Pa)
- the volume V is in cubic meters (m³)
- the amount of substance n is in moles (mol)
- the universal gas law constant R is 8.3145 Pa·m³/(mol·K)
- the temperature T is in kelvins (K)

$$(Pa)(m^3) = \frac{\cancel{mol}}{1} \times \frac{(Pa)(m^3)}{(\cancel{mol})(\cancel{K})} \times \frac{\cancel{K}}{1}$$

As can be seen, when the dimensional units appearing in the numerator and denominator of the equation's right hand side are cancelled out, both sides of the equation have the same dimensional units.

Limitations

The factor-label method can convert only unit quantities for which the units are in a linear relationship intersecting at 0. Most units fit this paradigm. An example for which it cannot be used is the conversion between degrees Celsius and kelvins (or Fahrenheit). Between degrees Celsius and kelvins, there is a constant difference rather than a constant ratio, while between Celsius and Fahrenheit there is both a constant difference and a constant ratio. Instead of multiplying the given quantity by a single conversion factor to obtain the converted quantity, it is more logical to think of the original quantity being divided by its unit, being added or subtracted by the constant difference, and the entire operation being multiplied by the new unit. Mathematically, this is an affine transform ($ax + b$), not a linear transform (ax). Formally, one starts with a displacement (in some units) from one point, and ends with a displacement (in some other units) from some other point.

For instance, the freezing point of water is 0° in Celsius and 32° in Fahrenheit, and a 5° change in Celsius correspond to a 9° change in Fahrenheit. Thus to convert from Fahrenheit to Celsius one subtracts 32° (displacement from one point), multiplies by 5 and divides by 9 (scales by the ratio of units), and adds 0 (displacement from new point). Reversing this yields the formula for Celsius; one could have started with the equivalence between 100° Celsius and 212° Fahrenheit, though this would yield the same formula at the end.

[°F = 1.8(°C) + 32°]

To convert Celsius to Fahrenheit, simply plug in the known numbers in the above formula.

[°C = (°F-32°) ÷ 1.8]

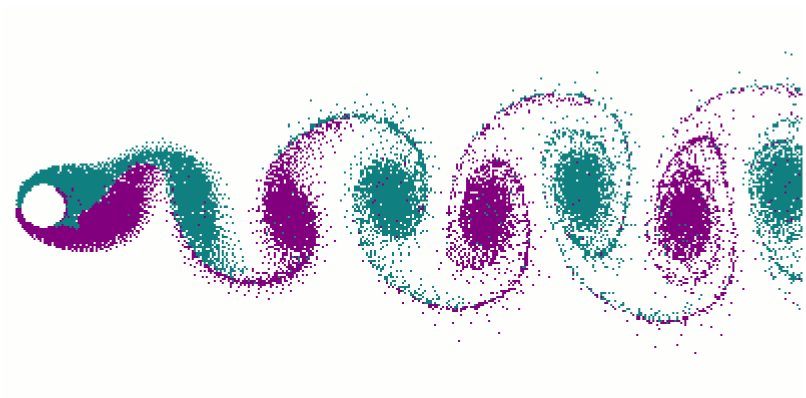
To convert Fahrenheit to Celsius (Centigrade), plug the known temperature into the above formula.

EX. °F = 1.8(-40°C) + 32° = -40°F (Identical temperature point in °C and °F)

EX. °C = (98.6°F-32°) ÷ 1.8 = 37°C (Known standard body temperature in °C and °F)

Chapter 9

Reynolds Number



A vortex street around a cylinder. This occurs around cylinders, independently of the fluid, the cylinder size and the fluid speed, provided that there is a Reynolds number of between 250 and 200,000. Picture courtesy, Cesareo de La Rosa Siqueira.

In fluid mechanics, the **Reynolds number** Re is a dimensionless number that gives a measure of the ratio of inertial forces $\rho v^2/L$ to viscous forces $\mu v/L^2$ and consequently quantifies the relative importance of these two types of forces for given flow conditions. The concept was introduced by George Gabriel Stokes in 1851, but the Reynolds number is named after Osborne Reynolds (1842–1912), who popularized its use in 1883.

Reynolds numbers frequently arise when performing dimensional analysis of fluid dynamics problems, and as such can be used to determine dynamic similitude between different experimental cases. They are also used to characterize different flow regimes, such as laminar or turbulent flow: laminar flow occurs at low Reynolds numbers, where viscous forces are dominant, and is characterized by smooth, constant fluid motion, while turbulent flow occurs at high Reynolds numbers and is dominated by inertial forces, which tend to produce chaotic eddies, vortices and other flow instabilities.

Definition

Reynolds number can be defined for a number of different situations where a fluid is in relative motion to a surface (the definition of the Reynolds number is not to be confused with the Reynolds Equation or lubrication equation). These definitions generally include the fluid properties of density and viscosity, plus a velocity and a *characteristic length* or *characteristic dimension*. This dimension is a matter of convention - for example a radius or diameter are equally valid for spheres or circles, but one is chosen by convention. For aircraft or ships, the length or width can be used. For flow in a pipe or a sphere moving in a fluid the internal diameter is generally used today. Other shapes (such as rectangular pipes or non-spherical objects) have an *equivalent diameter* defined. For fluids of variable density (e.g. compressible gases) or variable viscosity (non-Newtonian fluids) special rules apply. The velocity may also be a matter of convention in some circumstances, notably stirred vessels.

$$\mathrm{Re} = \frac{\rho \mathrm{V} L}{\mu} = \frac{\mathrm{V} L}{\nu}$$

where:

- V is the mean velocity of the object relative to the fluid (SI units: m/s)
- L is a characteristic linear dimension, (travelled length of the fluid; hydraulic diameter when dealing with river systems) (m)
- μ is the dynamic viscosity of the fluid (Pa·s or N·s/m² or kg/(m·s))
- ν is the kinematic viscosity ($\nu = \mu / \rho$) (m²/s)
- ρ is the density of the fluid (kg/m³)

Note that this is equal to the ratio between $\frac{\rho \mathrm{V}^2}{L}$, which is the drag (up to a numerical factor, half the drag coefficient), and $\frac{\mu \mathrm{V}}{L^2}$, which is the force due to viscosity (up to a numerical factor depending on the form of the flow).

Significance

$$\mathrm{Re} = \frac{\text{total momentum transfer}}{\text{molecular momentum transfer}}$$

Flow in Pipe

For flow in a pipe or tube, the Reynolds number is generally defined as:

$$\mathrm{Re} = \frac{\rho \mathrm{V} D_H}{\mu} = \frac{\mathrm{V} D_H}{\nu} = \frac{\mathrm{Q} D_H}{\nu A}$$

where:

- D_H is the hydraulic diameter of the pipe (m).
- Q is the volumetric flow rate (m^3/s)
- A is the pipe *cross-sectional* area (m²).

Flow in a non-circular duct (annulus)

For shapes such as squares, rectangular or annular ducts (where the height and width are comparable) the characteristic dimension for internal flow situations is taken to be the *hydraulic diameter*, D_H, defined as 4 times the cross-sectional area (of the fluid), divided by the **wetted perimeter**. The wetted perimeter for a channel is the total perimeter of all channel walls that are in contact with the flow. This means the length of the water exposed to air is NOT included in the wetted perimeter

$$D_H = \frac{4A}{P}.$$

For a circular pipe, the hydraulic diameter is exactly equal to the inside pipe diameter, as can be shown mathematically.

For an annular duct, such as the outer channel in a tube-in-tube heat exchanger, the hydraulic diameter can be shown algebraically to reduce to

$$D_{H,\text{annulus}} = D_o - D_i$$

where

D_o is the inside diameter of the outside pipe, and
D_i is the outside diameter of the inside pipe.

For calculations involving flow in non-circular ducts, the hydraulic diameter can be substituted for the diameter of a circular duct, with reasonable accuracy.

Flow in a Wide Duct

For a fluid moving between two plane parallel surfaces (where the width is much greater than the space between the plates) then the characteristic dimension is twice the distance between the plates.

Flow in an Open Channel

For flow of liquid with a free surface, the *hydraulic radius* must be determined. This is the cross-sectional area of the channel divided by the wetted perimeter. For a semi-circular channel, it is half the radius. For a rectangular channel, the hydraulic radius is the cross-sectional area divided by the wetted perimeter. Some texts then use a characteristic dimension that is 4 times the hydraulic radius (chosen because it gives the same value of Re for the onset of turbulence as in pipe flow), while others the hydraulic radius as the characteristic length-scale with consequently different values of Re for transition and turbulent flow.

Object in a fluid

The Reynolds number for an object in a fluid, called the particle Reynolds number and often denoted $\mathbf{Re_p}$, is important when considering the nature of flow around that grain, whether or not vortex shedding will occur, and its fall velocity.

Sphere in a fluid

For a sphere in a fluid, the characteristic length-scale is the diameter of the sphere and the characteristic velocity is that of the sphere relative to the fluid some distance away from the sphere (such that the motion of the sphere does not disturb that reference parcel of fluid). The density and viscosity are those belonging to the fluid. Note that purely laminar flow only exists up to Re = 0.1 under this definition.

Under the condition of low Re, the relationship between force and speed of motion is given by Stokes' law.

Oblong object in a fluid

The equation for an oblong object is identical to that of a sphere, with the object being approximated as an ellipsoid and the axis of length being chosen as the characteristic length scale. Such considerations are important in natural streams, for example, where there are few perfectly spherical grains. For grains in which measurement of each axis is impractical (e.g., because they are too small), sieve diameters are used instead as the characteristic particle length-scale. Both approximations alter the values of the critical Reynolds number.

Fall velocity

The particle Reynolds number is important in determining the fall velocity of a particle. When the particle Reynolds number indicates laminar flow, Stokes' law can be used to calculate its fall velocity. When the particle Reynolds number indicates turbulent flow, a turbulent drag law must be constructed to model the appropriate settling velocity.

Packed Bed

For flow of fluid through a bed of approximately spherical particles of diameter $\boldsymbol{D}$ in contact, if the voidage (fraction of the bed not filled with particles) is $\boldsymbol{\varepsilon}$ and the superficial velocity $\mathbf{V}$ (i.e. the velocity through the bed as if the particles were not there - the actual velocity will be higher) then a Reynolds number can be defined as:

$$\mathrm{Re} = \frac{\rho \mathrm{V} D}{\mu(1-\epsilon)}.$$

Laminar conditions apply up to Re = 10, fully turbulent from 2000.

Stirred Vessel

In a cylindrical vessel stirred by a central rotating paddle, turbine or propellor, the characteristic dimension is the diameter of the agitator *D*. The velocity is *ND* where *N* is the rotational speed (revolutions per second). Then the Reynolds number is:

$$\mathrm{Re} = \frac{\rho N D^2}{\mu}.$$

The system is fully turbulent for values of Re above 10 000.

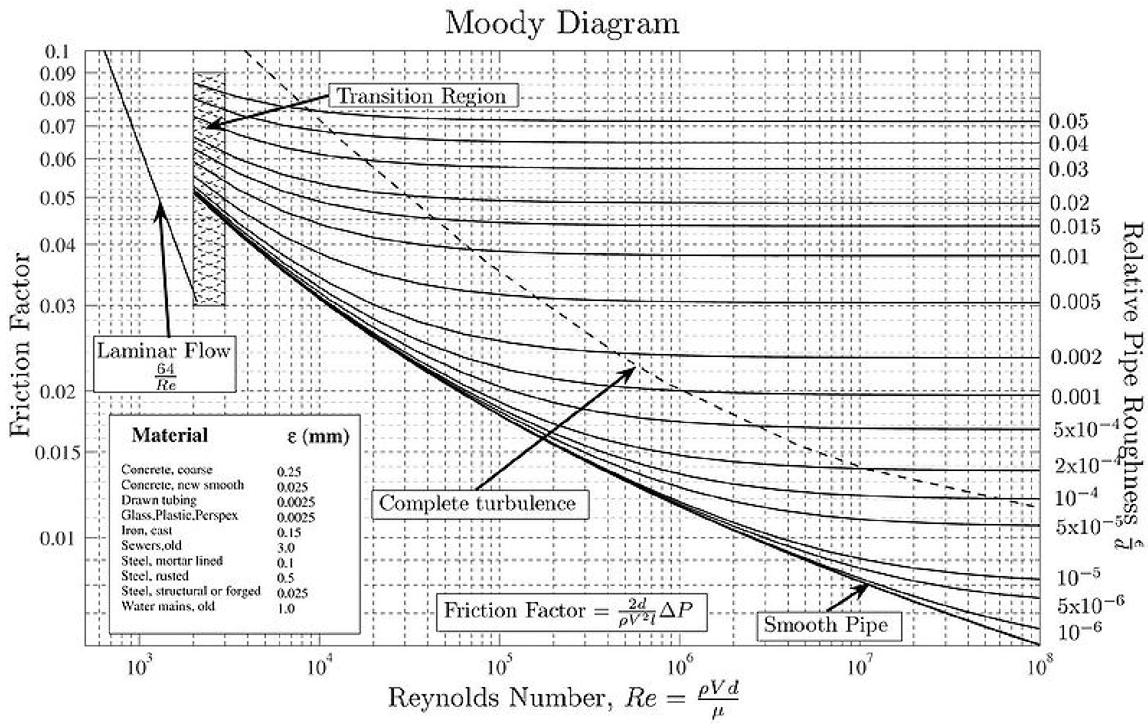
Transition Reynolds number

In boundary layer flow over a flat plate, experiments can confirm that, after a certain length of flow, a laminar boundary layer will become unstable and become turbulent. This instability occurs across different scales and with different fluids, usually when $\mathrm{Re}_x \approx 5 \times 10^5$, where *x* is the distance from the leading edge of the flat plate, and the flow velocity is the freestream velocity of the fluid outside the boundary layer.

For flow in a pipe of diameter *D*, experimental observations show that for 'fully developed' flow (Note:), laminar flow occurs when $\mathrm{Re}_D < 2300$ and turbulent flow occurs when $\mathrm{Re}_D > 4000$. In the interval between 2300 and 4000, laminar and turbulent flows are possible ('transition' flows), depending on other factors, such as pipe roughness and flow uniformity). This result is generalised to non-circular channels using the hydraulic diameter, allowing a transition Reynolds number to be calculated for other shapes of channel.

These transition Reynolds numbers are also called *critical Reynolds numbers*, and were studied by Osborne Reynolds around 1895.

Reynolds number in pipe friction



Pressure drops seen for fully-developed flow of fluids through pipes can be predicted using the Moody diagram which plots the Darcy–Weisbach friction factor *f* against Reynolds number Re and relative roughness ε / *D*. The diagram clearly shows the laminar, transition, and turbulent flow regimes as Reynolds number increases. The nature of pipe flow is strongly dependent on whether the flow is laminar or turbulent

The similarity of flows

In order for two flows to be similar they must have the same geometry, and have equal Reynolds numbers and Euler numbers. When comparing fluid behaviour at corresponding points in a model and a full-scale flow, the following holds:

$$\mathrm{Re}_m = \mathrm{Re}$$
$$\mathrm{Eu}_m = \mathrm{Eu} \qquad \text{i.e.} \quad \frac{p_m}{\varrho_m {v_m}^2} = \frac{p}{\varrho v^2} \, ,$$

quantities marked with 'm' concern the flow around the model and the others the actual flow. This allows engineers to perform experiments with reduced models in water channels or wind tunnels, and correlate the data to the actual flows, saving on costs during experimentation and on lab time. Note that true dynamic similitude may require matching other dimensionless numbers as well, such as the Mach number used in compressible flows, or the Froude number that governs open-channel flows. Some flows involve more dimensionless parameters than can be practically satisfied with the available apparatus and fluids (for example air or water), so one is forced to decide which parameters are most important. For experimental flow modeling to be useful, it requires a fair amount of experience and judgement of the engineer.

Typical values of Reynolds number

- Ciliate ~ 1 x 10^{-1}
- Smallest Fish ~ 1

- Blood flow in brain ~ 1×10^2
- Blood flow in aorta ~ 1×10^3

Onset of turbulent flow ~ 2.3×10^3 to 5.0×10^4 for pipe flow to 10^6 for boundary layers

- Typical pitch in Major League Baseball ~ 2×10^5
- Person swimming ~ 4×10^6
- Fastest Fish ~ 1 x 10^6
- Blue Whale ~ 3×10^8
- A large ship (RMS Queen Elizabeth 2) ~ 5×10^9

Reynolds number sets the smallest scales of turbulent motion

In a turbulent flow, there is a range of scales of the time-varying fluid motion. The size of the largest scales of fluid motion (sometimes called eddies) are set by the overall geometry of the flow. For instance, in an industrial smoke stack, the largest scales of fluid motion are as big as the diameter of the stack itself. The size of the smallest scales is set by the Reynolds number. As the Reynolds number increases, smaller and smaller scales of the flow are visible. In a smoke stack, the smoke may appear to have many very small velocity perturbations or eddies, in addition to large bulky eddies. In this sense, the Reynolds number is an indicator of the range of scales in the flow. The higher the Reynolds number, the greater the range of scales. The largest eddies will always be the same size; the smallest eddies are determined by the Reynolds number.

What is the explanation for this phenomenon? A large Reynolds number indicates that viscous forces are not important at large scales of the flow. With a strong predominance of inertial forces over viscous forces, the largest scales of fluid motion are undamped—there is not enough viscosity to dissipate their motions. The kinetic energy must "cascade" from these large scales to progressively smaller scales until a level is reached for which the scale is small enough for viscosity to become important (that is, viscous forces become of the order of inertial ones). It is at these small scales where the dissipation of energy by viscous action finally takes place. The Reynolds number indicates at what scale this viscous dissipation occurs. Therefore, since the largest eddies are dictated by the flow geometry and the smallest scales are dictated by the viscosity, the Reynolds number can be understood as the ratio of the largest scales of the turbulent motion to the smallest scales.

Example of the importance of the Reynolds number

If an airplane wing needs testing, one can make a scaled down model of the wing and test it in a wind tunnel using the same Reynolds number that the actual airplane is subjected to. If for example the scale model has linear dimensions one quarter of full size, the flow velocity of the model would have to be multiplied by a factor of 4 to obtain similar flow behavior.

Alternatively, tests could be conducted in a water tank instead of in air (provided the compressibility effects of air are not significant). As the kinematic viscosity of water is around 13 times less than that of air at 15 °C, in this case the scale model would need to be about one thirteenth the size in all dimensions to maintain the same Reynolds number, assuming the full-scale flow velocity was used.

The results of the laboratory model will be similar to those of the actual plane wing results. Thus there is no need to bring a full scale plane into the lab and actually test it. This is an example of "dynamic similarity".

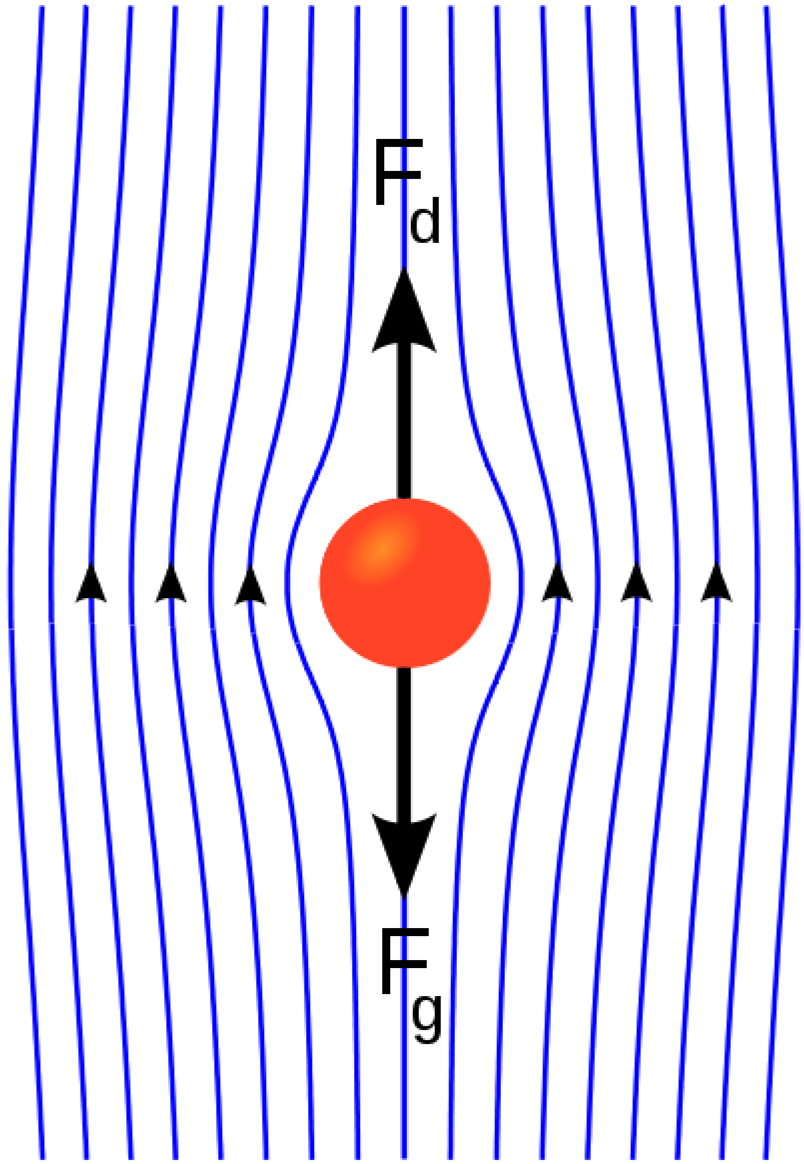
Reynolds number is important in the calculation of a body's drag characteristics. A notable example is that of the flow around a cylinder. Above roughly 3×10^6 Re the drag coefficient drops considerably. This is important when calculating the optimal cruise speeds for low drag (and therefore long range) profiles for airplanes.

Reynolds number in physiology

Poiseuille's law on blood circulation in the body is dependent on laminar flow. In turbulent flow the flow rate is proportional to the square root of the pressure gradient, as opposed to its direct proportionality to pressure gradient in laminar flow.

Using the definition of the Reynolds number we can see that a large diameter with rapid flow, where the density of the blood is high, tends towards turbulence. Rapid changes in vessel diameter may lead to turbulent flow, for instance when a narrower vessel widens to a larger one. Furthermore, an atheroma may be the cause of turbulent flow, and as such detecting turbulence with a stethoscope may be a sign of such a condition.

Reynolds number in viscous fluids



Creeping flow past a sphere: streamlines, drag force F_d and force by gravity F_g.

Where the viscosity is naturally high, such as polymer solutions and polymer melts, flow is normally laminar. The Reynolds number is very small and Stokes' Law can be used to measure the viscosity of the fluid. Spheres are allowed to fall through the fluid and they reach the terminal velocity quickly, from which the viscosity can be determined.

The laminar flow of polymer solutions is exploited by animals such as fish and dolphins, who exude viscous solutions from their skin to aid flow over their bodies while swimming. It has been used in yacht racing by owners who want to gain a speed advantage by pumping a polymer solution such as low molecular weight polyoxyethylene in water, over the wetted surface of the hull. It is however, a problem for mixing of polymers, because turbulence is needed to distribute fine filler (for example) through the material. Inventions such as the "cavity transfer mixer" have been developed to produce multiple folds into a moving melt so as to improve mixing efficiency. The device can be fitted onto extruders to aid mixing.

Where does it come from?

The Reynolds number can be obtained when one uses the nondimensional form of the incompressible Navier-Stokes equations:

$$\varrho\left(\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}\right)=-\nabla p+\mu\nabla^2\mathbf{v}+\mathbf{f}.$$

Each term in the above equation has the units of a volume force or, equivalently, an acceleration times a density. Each term is thus dependent on the exact measurements of a flow. When one renders the equation nondimensional, that is that we multiply it by a factor with inverse units of the base equation, we obtain a form which does not depend directly on the physical sizes. One possible way to obtain a nondimensional equation is to multiply the whole equation by the following factor:

$$\frac{D}{\varrho V^2}$$

where the symbols are the same as those used in the definition of the Reynolds number. If we now set:

$$\mathbf{v}'=\frac{\mathbf{v}}{V},\ p'=p\frac{1}{\varrho V^2},\ \mathbf{f}'=\mathbf{f}\frac{D}{\varrho V^2},\ \frac{\partial}{\partial t'}=\frac{D}{V}\frac{\partial}{\partial t},\ \nabla'=D\nabla$$

we can rewrite the Navier-Stokes equation without dimensions:

$$\frac{\partial \mathbf{v}'}{\partial t'}+\mathbf{v}'\cdot\nabla'\mathbf{v}'=-\nabla'p'+\frac{\mu}{\varrho DV}\nabla'^2\mathbf{v}'+\mathbf{f}'$$

where the term : $\dfrac{\mu}{\varrho DV}=\dfrac{1}{\mathrm{Re}}.$

Finally, dropping the primes for ease of reading:

$$\frac{\partial \mathbf{v}}{\partial t}+\mathbf{v}\cdot\nabla\mathbf{v}=-\nabla p+\frac{1}{\mathrm{Re}}\nabla^2\mathbf{v}+\mathbf{f}.$$

This is why mathematically all flows with the same Reynolds number are comparable. Notice also, in the above equation, as : $\mathrm{Re} \to \infty$ the viscous terms vanish. Thus, high Reynolds number flows are approximately inviscid in the free-stream.

Chapter 10

Affine Space

In mathematics, an **affine space** is a geometric structure that generalizes the affine properties of Euclidean space. In an affine space, one can subtract points to get vectors, or add a vector to a point to get another point, but one cannot add points. In particular, there is no distinguished point that serves as an origin. The solution set of an inhomogeneous linear equation is either empty or an affine space.

Informal descriptions

The following characterization may be easier to understand than the usual formal definition: an affine space is what is left of a vector space after you've forgotten which point is the origin (or, in the words of the great French mathematician Marcel Berger, "An affine space is nothing more than a vector space whose origin we try to forget about, by adding translations to the linear maps") . Imagine that Smith knows that a certain point is the true origin, and Jones believes that another point — call it $\boldsymbol{p}$ — is the origin. Two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are to be added. Jones draws an arrow from $\boldsymbol{p}$ to $\boldsymbol{a}$ and another arrow from $\boldsymbol{p}$ to $\boldsymbol{b}$, and completes the parallelogram to find what Jones thinks is $\boldsymbol{a} + \boldsymbol{b}$, but Smith knows that it is actually

$$\boldsymbol{p} + (\boldsymbol{a} - \boldsymbol{p}) + (\boldsymbol{b} - \boldsymbol{p}).$$

Similarly, Jones and Smith may evaluate any linear combination of $\boldsymbol{a}$ and $\boldsymbol{b}$, or of any finite set of vectors, and will generally get different answers. However, importantly if the sum of the coefficients in a linear combination is 1, then Smith and Jones will arrive at the same answer.

Extending our previous example: Smith, knowing the true origin, can deduce that the point that Jones thinks is

$$\lambda\boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$$

is actually

$$\boldsymbol{p} + \lambda(\boldsymbol{a} - \boldsymbol{p}) + (1 - \lambda)(\boldsymbol{b} - \boldsymbol{p}) = \lambda\boldsymbol{a} + (1 - \lambda)\boldsymbol{b}.$$

Since $\lambda + (1 - \lambda) = 1$, Smith and Jones describe the same point with the same linear combination in their respective frames of reference.

Here is the punch line: Smith knows the "linear structure", but both Smith and Jones know the "affine structure"—i.e. the values of affine combinations, defined as linear combinations in which the sum of the coefficients is 1. An underlying set with an affine structure is an affine space.

Definition

An affine space is a set A together with a group action of a vector space V, considered as Abelian Lie group acting on the left such that the only vector acting with a fixpoint is 0 and there is a single orbit (i.e. the action is free and transitive). It is an example of (left) G-set, where G is the additive group V.

Explicitly, an affine space is a point set A together with a map

$l: V \times A \to A$, written as $(v, a) \mapsto v + a$

with the following properties:

1. for every a in A one has $0 + a = a$,
2. for every v, w in V and a in A one has $v + (w + a) = (v + w) + a$,
3. for every a in A the map $V \to A: v \mapsto v + a$ is a bijection.

By choosing an origin a one can thus identify A with V, hence turn A into a vector space.

Conversely, any vector space V is an affine space over itself.

If o, a and b are points in A and λ is a scalar, then

$$\lambda a + (1 - \lambda)b = o + \lambda(a - o) + (1 - \lambda)(b - o)$$

is independent of o. Instead of arbitrary linear combinations, only such affine combinations of points have meaning.

By noting that one can define subtraction of points of an affine space as follows:

$a - b$ is the unique vector in V such that $(a - b) + b = a$,

one can equivalently define an affine space as a point set A, together with a vector space V, and a subtraction map ϕ which sends (a,b) in $A \times A$ to $b - a$ in V

written as $(a, b) \mapsto b - a \equiv \overrightarrow{ab}$ with the following properties:

1. for every point $p \in A$ and vector $v \in V$ there is a unique point $q \in A$ such that $q - p = v$ and
2. for every p, q and r in A one has $(q - p) + (r - q) = r - p.$

Examples

- Any coset of a subspace V of a vector space is an affine space over V.
- If A is a matrix and b lies in its column space, the set of solutions of the equation $Ax = b$ is an affine space over the subspace of solutions of $Ax = 0$.
- The solutions of an inhomogeneous linear differential equation form an affine space over the solutions of the corresponding homogeneous linear equation.
- Generalizing all of the above, if $T\colon V \to W$ is a linear mapping and y lies in its image, the set of solutions $x \in V$ to the equation $T(x) = y$ is a coset of the kernel of T, and is therefore an affine space over Ker(T).

Affine subspaces

An **affine subspace** (sometimes called a **linear manifold** or **linear variety**) of a vector space V is a subset closed under affine combinations of vectors in the space. For example, the set

$$A = \Big\{ \sum_i^N \alpha_i \mathbf{v}_i \Big| \sum_i^N \alpha_i = 1 \Big\}$$

is an affine space, where $\{\mathbf{v}_i\}_i$ is a family of vectors in V – this space is the *affine span* of these point. To see that this is indeed an affine space, observe that this set carries a transitive action of the vector subspace W of V

$$W = \Big\{ \sum_i^N \beta_i \mathbf{v}_i \Big| \sum_i^N \beta_i = 0 \Big\}.$$

This affine subspace can be equivalently described as the coset of the W-action

$$S = \mathbf{p} + W,$$

where $\mathbf{p}$ is any element of A, or equivalently as any level set of the quotient map $V \to V/W$. A choice of $\mathbf{p}$ gives a base point of A and an identification of W with A, but there is no natural choice, nor a natural identification of W with A.

A linear transformation is a function that preserves all linear combinations; an affine transformation is a function that preserves all affine combinations. A linear subspace is an affine subspace containing the origin, or, equivalently, a subspace that is closed under linear combinations.

For example, in R^3, the origin, lines and planes through the origin and the whole space are linear subspaces, while points, lines and planes in general as well as the whole space are the affine subspaces.

Affine combinations and affine dependence

An **affine combination** is a linear combination in which the sum of the coefficients is 1. Just as members of a set of vectors are linearly independent if none is a linear combination of the others, so also they are **affinely independent** if none is an affine combination of the others. The set of linear combinations of a set of vectors is their "linear span" and is always a linear subspace; the set of all affine combinations is their "affine span" and is always an affine subspace. For example, the affine span of a set of two points is the line that contains both; the affine span of a set of three non-collinear points is the plane that contains all three. Vectors

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

are linearly dependent if there exist scalars $a_1, a_2, \ldots, a_n$, not all zero, for which

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \ldots + a_n\mathbf{v}_n = \mathbf{0} \qquad (1)$$

Similarly they are **affinely dependent** if the same is true and also

$$\sum_{i=1}^{n} a_i = 0.$$

Equation (**1**) is an **affine relation** among the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

Axioms

Affine space is usually studied as analytic geometry using coordinates, or equivalently vector spaces. It can also be studied as synthetic geometry by writing down axioms, though this approach is much less common. There are several different systems of axioms for affine space.

Coxeter (1969, p.192) axiomatizes affine geometry (over the reals) as ordered geometry together with an affine form of Desargues's theorem and an axiom stating that in a plane there is at most one line through a given point not meeting a given line.

Affine planes satisfy the following axioms (Cameron 1991, chapter 2): (in which two lines are called parallel if they are equal or disjoint):

- Any two distinct points lie on a unique line.
- Given a point and line there is a unique line which contains the point and is parallel to the line
- There exist three non-collinear points.

As well as affine planes over fields (or division rings), there are also many non-Desarguesian planes satisfying these axioms. (Cameron 1991, chapter 3) gives axioms for higher dimensional affine spaces.

Relation to projective spaces

$$\begin{tikzcd} & K^{n+1} \arrow[d, two heads] \\ \mathbf{A}^n \arrow[r, hook] & \mathbf{P}^n \end{tikzcd}$$

An affine space is a subspace of projective space, which is in turn a quotient of a vector space.

Affine spaces are subspaces of projective spaces: an affine plane can be obtained from any projective plane by removing a line and all the points on it, and conversely any affine plane can be used to construct a projective plane as a closure by adding a "line at infinity" whose points correspond to equivalence classes of parallel lines.

Further, transformations of projective space that preserve affine space (equivalently, that preserve the points at infinity as a set) yield transformations of affine space, and conversely any affine linear transformation extends uniquely to a projective linear transformations, so affine transformations are a subset of projective transforms. Most familiar is that Möbius transformations (transformations of the projective line, or Riemann sphere) are affine (transformations of the complex plane) if and only if they fix the point at infinity.

However, one cannot take the projectivization of an affine space, so projective spaces are not naturally *quotients* of affine spaces: one can only take the projectivization of a *vector* space, since the projective space is lines *through a given point,* and there is no distinguished point in an affine space. If one chooses a base point (as zero), then an affine space becomes a vector space, which one may then projectivize, but this requires a choice.