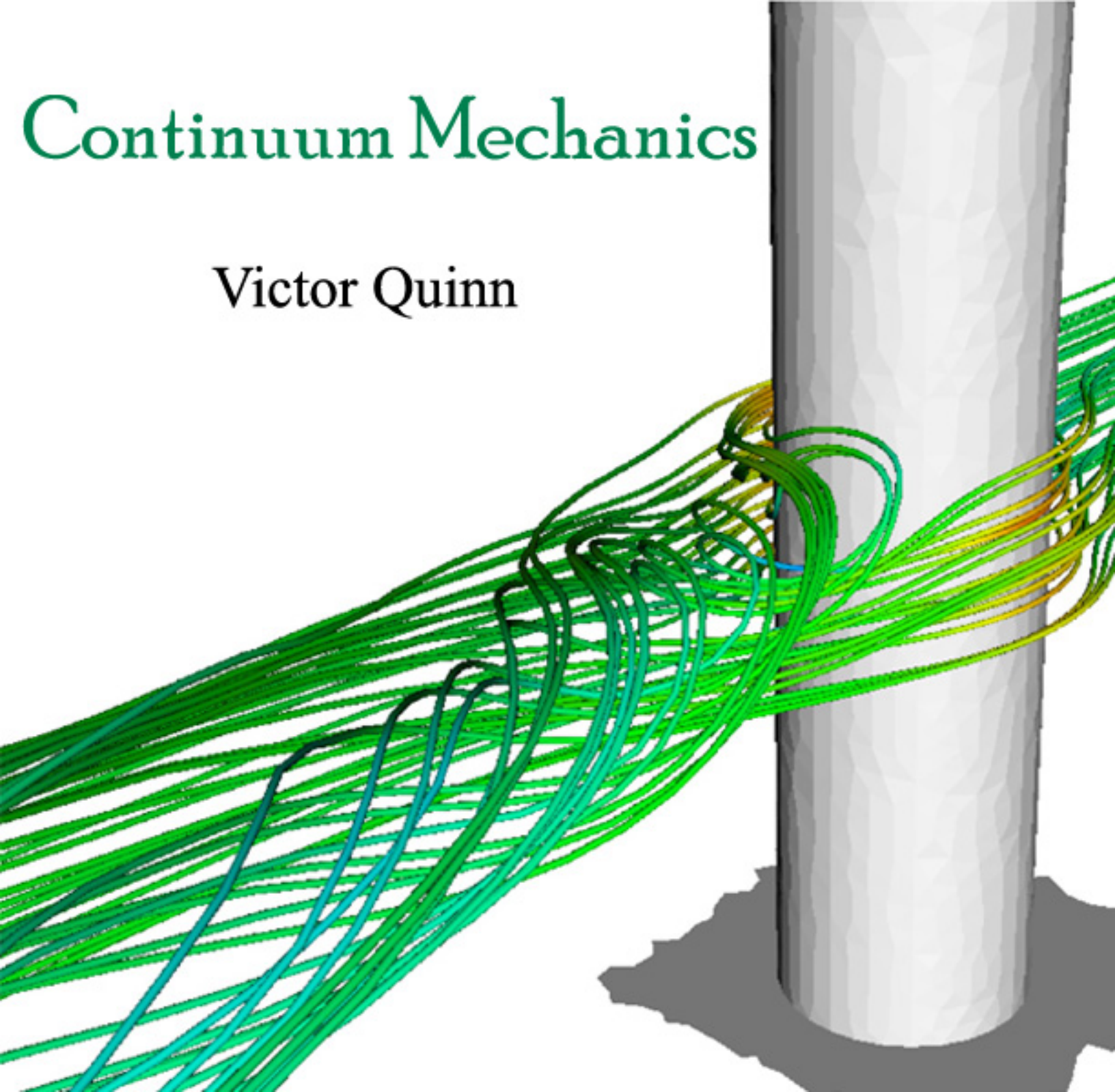
Continuum Mechanics

Victor Quinn



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Chapter 1

Continuum Mechanics

Continuum mechanics is a branch of mechanics that deals with the analysis of the kinematics and the mechanical behavior of materials modeled as a continuous mass rather than as discrete particles. The French mathematician Augustin Louis Cauchy was the first to formulate such models in the 19th century, but research in the area continues today.

Modeling an object as a continuum assumes that the substance of the object completely fills the space it occupies. Modeling objects in this way ignores the fact that matter is made of atoms, and so is not continuous; however, on length scales much greater than that of inter-atomic distances, such models are highly accurate. Fundamental physical laws such as the conservation of mass, the conservation of momentum, and the conservation of energy may be applied to such models to derive differential equations describing the behavior of such objects, and some information about the particular material studied is added through a constitutive relation.

Continuum mechanics deals with physical properties of solids and fluids which are independent of any particular coordinate system in which they are observed. These physical properties are then represented by tensors, which are mathematical objects that have the required property of being independent of coordinate system. These tensors can be expressed in coordinate systems for computational convenience.

The concept of a continuum

Materials, such as solids, liquids and gases, are composed of molecules separated by empty space. On a macroscopic scale, materials have cracks and discontinuities. However, certain physical phenomena can be modeled assuming the materials exist as a **continuum, meaning the matter in the body is continuously distributed and fills the entire region of space it occupies**. A continuum is a body that can be continually sub-divided into infinitesimal elements with properties being those of the bulk material.

The validity of the continuum assumption may be verified by a theoretical analysis, in which either some clear periodicity is identified or statistical homogeneity and ergodicity of the microstructure exists. More specifically, the continuum hypothesis/assumption hinges on the concepts of a *representative volume element* (RVE) (sometimes called "representative elementary volume") and *separation of scales* based on the Hill-Mandel condition. This condition provides a link between an experimentalist's and a theoretician's viewpoint on constitutive equations (linear and nonlinear elastic/inelastic or coupled fields) as well as a way of spatial and statistical averaging of the microstructure.

When the separation of scales does not hold, or when one wants to establish a continuum of a finer resolution than that of the RVE size, one employs a *statistical volume element* (SVE), which, in turn, leads to random continuum fields. The latter then provide a micromechanics basis for stochastic finite elements (SFE). The levels of SVE and RVE link continuum mechanics to statistical mechanics. The RVE may be assessed only in a limited way via experimental testing: when the constitutive response becomes spatially homogeneous.

Specifically for fluids, the Knudsen number is used to assess to what extent the approximation of continuity can be made.

Major areas of continuum mechanics

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<tr><td rowspan="4">Continuum mechanics
The study of the physics of continuous materials</td><td rowspan="2">Solid mechanics
The study of the physics of continuous materials with a defined rest shape.</td><td colspan="2">Elasticity
Describes materials that return to their rest shape after an applied stress.</td></tr>
<tr><td>Plasticity
Describes materials that permanently deform after a sufficient applied stress.</td><td rowspan="2">Rheology
The study of materials with both solid and fluid characteristics.</td></tr>
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The study of the physics of continuous materials which take the shape of their container.</td><td>Non-Newtonian fluids</td></tr>
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Formulation of models

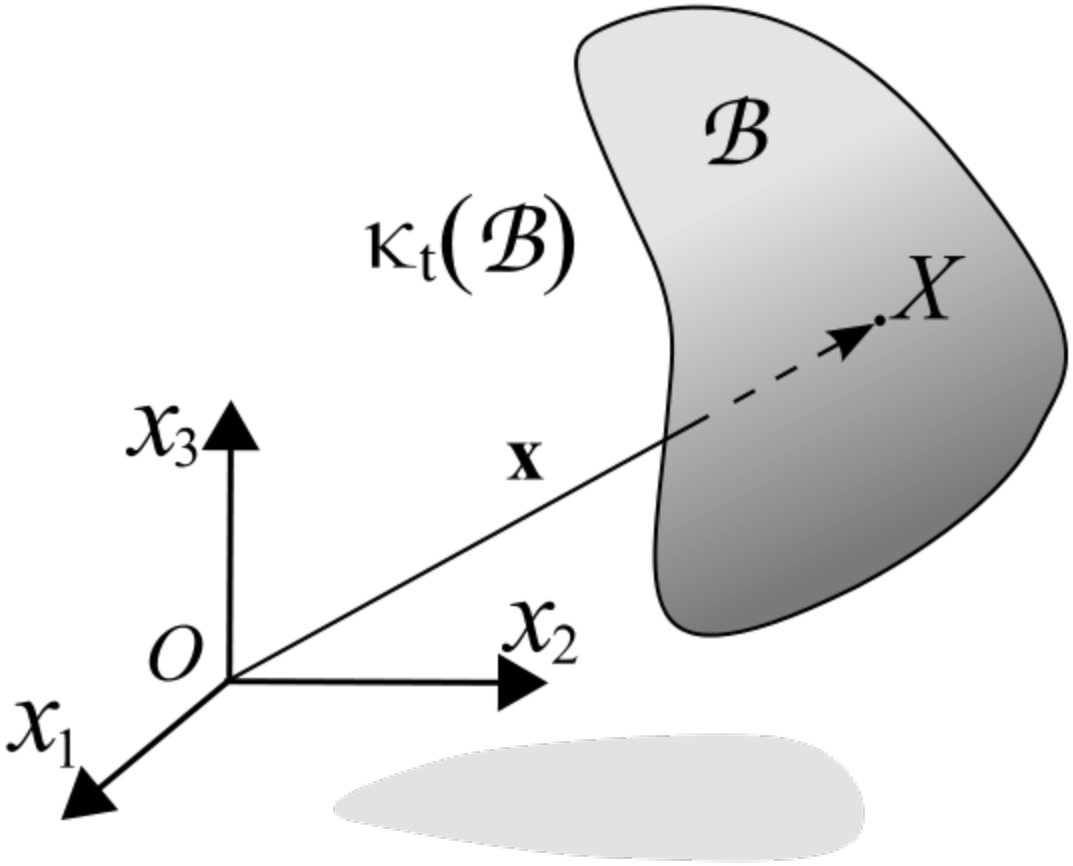


Figure 1. Configuration of a continuum body

Continuum mechanics models begin by assigning a region in three dimensional Euclidean space to the material body $\mathcal{B}$being modeled. The points within this region are called particles or material points. Different *configurations* or states of the body correspond to different regions in Euclidean space. The region corresponding to the body's configuration at time $t$is labeled $\kappa_t(\mathcal{B})$.

A particular particle within the body in a particular configuration is characterized by a position vector

$$\mathbf{x} = \sum_{i=1}^{3} x_i \mathbf{e}_i \quad ,$$

where $\mathbf{e}_i$ are the coordinate vectors in some frame of reference chosen for the problem. This vector can be expressed as a function of the particle position $\mathbf{X}$ in some *reference configuration*, for example the configuration at the initial time, so that

$$\mathbf{x} = \kappa_t(\mathbf{X}).$$

This function needs to have various properties so that the model makes physical sense. $\kappa_t(\cdot)$ needs to be:

- continuous in time, so that the body changes in a way which is realistic,
- globally invertible at all times, so that the body cannot intersect itself,
- orientation-preserving, as transformations which produce mirror reflections are not possible in nature.

For the mathematical formulation of the model, $\kappa_t(\cdot)$ is also assumed to be twice continuously differentiable, so that differential equations describing the motion may be formulated.

Forces in a continuum

Continuum mechanics deals with deformable bodies, as opposed to rigid bodies. A solid is a deformable body that possesses shear strength, *sc.* a solid can support shear forces (forces parallel to the material surface on which they act). Fluids, on the other hand, do not sustain shear forces. For the study of the mechanical behavior of solids and fluids these are assumed to be continuous bodies, which means that the matter fills the entire region of space it occupies, despite the fact that matter is made of atoms, has voids, and is discrete. Therefore, when continuum mechanics refers to a point or particle in a continuous body it does not describe a point in the interatomic space or an atomic particle, rather an idealized part of the body occupying that point.

Following the classical dynamics of Newton and Euler, the motion of a material body is produced by the action of externally applied forces which are assumed to be of two kinds: surface forces $\mathbf{F}_C$ and body forces $\mathbf{F}_B$. Thus, the total force $\mathcal{F}$ applied to a body or to a portion of the body can be expressed as:

$$\mathcal{F} = \mathbf{F}_B + \mathbf{F}_C$$

Surface forces or *contact forces*, expressed as force per unit area, can act either on the bounding surface of the body, as a result of mechanical contact with other bodies, or on imaginary internal surfaces that bound portions of the body, as a result of the mechanical interaction between the parts of the body to either side of the surface (Euler-Cauchy's stress principle). When a body is acted upon by external contact forces, internal contact forces are then transmitted from point to point inside the body to balance their action, according to Newton's second law of motion of conservation of linear momentum and angular momentum (for continuous bodies these laws are called the Euler's equations of

motion). The internal contact forces are related to the body's deformation through constitutive equations. The internal contact forces may be mathematically described by how they relate to the motion of the body, independent of the body's material makeup.

The distribution of internal contact forces throughout the volume of the body is assumed to be continuous. Therefore, there exists a *contact force density* or *Cauchy traction field* $\mathbf{T}(\mathbf{n}, \mathbf{x}, t)$ that represents this distribution in a particular configuration of the body at a given time t. It is not a vector field because it depends not only on the position $\mathbf{x}$ of a particular material point, but also on the local orientation of the surface element as defined by its normal vector $\mathbf{n}$.

Any differential area dS with normal vector $\mathbf{n}$ of a given internal surface area S, bounding a portion of the body, experiences a contact force $d\mathbf{F}_C$ arising from the contact between both portions of the body on each side of S, and it is given by

$$d\mathbf{F}_C = \mathbf{T}^{(\mathbf{n})}\, dS$$

where $\mathbf{T}^{(\mathbf{n})}$ is the *surface traction*, also called *stress vector*, *traction*, or *traction vector*,. The stress vector is a frame-indifferent vector.

The total contact force on the particular internal surface S is then expressed as the sum (surface integral) of the contact forces on all differential surfaces dS:

$$\mathbf{F}_C = \int_S \mathbf{T}^{(\mathbf{n})}\, dS$$

In continuum mechanics a body is considered stress-free if the only forces present are those inter-atomic forces (ionic, metallic, and van der Waals forces) required to hold the body together and to keep its shape in the absence of all external influences, including gravitational attraction. Stresses generated during manufacture of the body to a specific configuration are also excluded when considering stresses in a body. Therefore, the stresses considered in continuum mechanics are only those produced by deformation of the body, *sc.* only relative changes in stress are considered, not the absolute values of stress.

Body forces are forces originating from sources outside of the body that act on the volume (or mass) of the body. Saying that body forces are due to outside sources implies that the interaction between different parts of the body (internal forces) are manifested through the contact forces alone. These forces arise from the presence of the body in force fields, *e.g.* gravitational field (gravitational forces) or electromagnetic field (electromagnetic forces), or from inertial forces when bodies are in motion. As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed. Thus, body forces are specified by vector fields which are assumed to be continuous over the entire volume of the body, *i.e.* acting

on every point in it. Body forces are represented by a body force density $\mathbf{b}(\mathbf{x}, t)$(per unit of mass), which is a frame-indifferent vector field.

In the case of gravitational forces, the intensity of the force depends on, or is proportional to, the mass density $\rho(\mathbf{x}, t)$of the material, and it is specified in terms of force per unit mass (b_i) or per unit volume (p_i). These two specifications are related through the material density by the equation $\rho b_i = p_i$. Similarly, the intensity of electromagnetic forces depends upon the strength (electric charge) of the electromagnetic field.

The total body force applied to a continuous body is expressed as

$$\mathbf{F}_B = \int_V \mathbf{b}\, dm = \int_V \rho \mathbf{b}\, dV$$

Body forces and contact forces acting on the body lead to corresponding moments of force (torques) relative to a given point. Thus, the total applied torque $\mathcal{M}$about the origin is given by

$$\mathcal{M} = \mathbf{M}_B + \mathbf{M}_C$$

In certain situations, not commonly considered in the analysis of the mechanical behavior or materials, it becomes necessary to include two other types of forces: these are *body moments* and *couple stresses* (surface couples, contact torques). Body moments, or body couples, are moments per unit volume or per unit mass applied to the volume of the body. Couple stresses are moments per unit area applied on a surface. Both are important in the analysis of stress for a polarized dielectric solid under the action of an electric field, materials where the molecular structure is taken into consideration (*e.g.* bones), solids under the action of an external magnetic field, and the dislocation theory of metals.

Materials that exhibit body couples and couple stresses in addition to moments produced exclusively by forces are called *polar materials*. *Non-polar materials* are then those materials with only moments of forces. In the classical branches of continuum mechanics the development of the theory of stresses is based on non-polar materials.

Thus, the sum of all applied forces and torques (with respect to the origin of the coordinate system) in the body can be given by

$$\mathcal{F} = \int_V \mathbf{a}\, dm = \int_S \mathbf{T}\, dS + \int_V \rho \mathbf{b}\, dV$$
$$\mathcal{M} = \int_S \mathbf{r} \times \mathbf{T}\, dS + \int_V \mathbf{r} \times \rho \mathbf{b}\, dV$$

Kinematics: deformation and motion

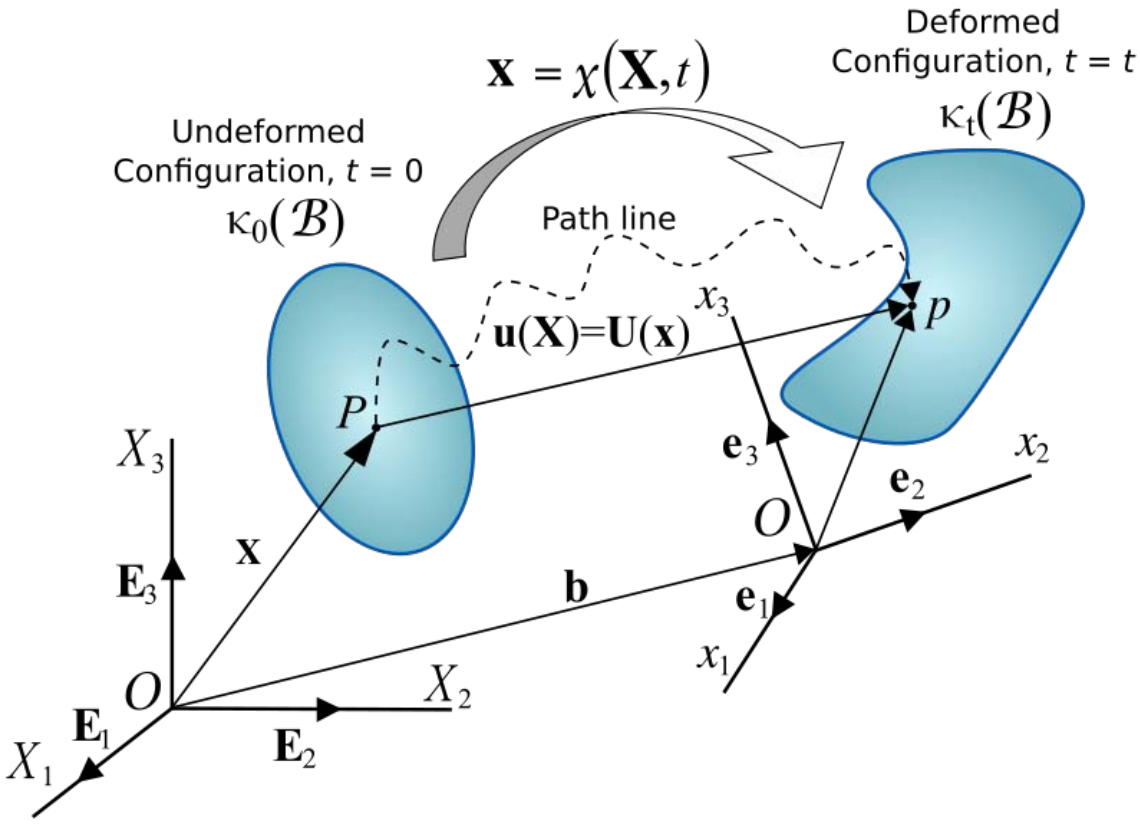


Figure 2. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consist of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 2).

The motion of a continuum body is a continuous time sequence of displacements. Thus, the material body will occupy different configurations at different times so that a particle occupies a series of points in space which describe a pathline.

There is continuity during deformation or motion of a continuum body in the sense that:

- The material points forming a closed curve at any instant will always form a closed curve at any subsequent time.

- The material points forming a closed surface at any instant will always form a closed surface at any subsequent time and the matter within the closed surface will always remain within.

It is convenient to identify a reference configuration or initial condition which all subsequent configurations are referenced from. The reference configuration need not be one that the body will ever occupy. Often, the configuration at $t = 0$ is considered the reference configuration , $\kappa_0(\mathcal{B})$. The components X_i of the position vector $\mathbf{X}$ of a particle, taken with respect to the reference configuration, are called the material or reference coordinates.

When analyzing the deformation or motion of solids, or the flow of fluids, it is necessary to describe the sequence or evolution of configurations throughout time. One description for motion is made in terms of the material or referential coordinates, called material description or Lagrangian description.

Lagrangian description

In the Lagrangian description the position and physical properties of the particles are described in terms of the material or referential coordinates and time. In this case the reference configuration is the configuration at $t = 0$. An observer standing in the referential frame of reference observes the changes in the position and physical properties as the material body moves in space as time progresses. The results obtained are independent of the choice of initial time and reference configuration, $\kappa_0(\mathcal{B})$. This description is normally used in solid mechanics.

In the Lagrangian description, the motion of a continuum body is expressed by the mapping function $\chi(\cdot)$ (Figure 2),

$$\mathbf{x} = \chi(\mathbf{X}, t)$$

which is a mapping of the initial configuration $\kappa_0(\mathcal{B})$ onto the current configuration $\kappa_t(\mathcal{B})$, giving a geometrical correspondence between them, i.e. giving the position vector $\mathbf{x} = x_i \mathbf{e}_i$ that a particle X, with a position vector $\mathbf{X}$ in the undeformed or reference configuration $\kappa_0(\mathcal{B})$, will occupy in the current or deformed configuration $\kappa_t(\mathcal{B})$ at time t. The components x_i are called the spatial coordinates.

Physical and kinematic properties $P_{ij\ldots}$, i.e. thermodynamic properties and velocity, which describe or characterize features of the material body, are expressed as continuous functions of position and time, i.e. $P_{ij\ldots} = P_{ij\ldots}(\mathbf{X}, t)$.

The material derivative of any property $P_{ij\ldots}$ of a continuum, which may be a scalar, vector, or tensor, is the time rate of change of that property for a specific group of

particles of the moving continuum body. The material derivative is also known as the *substantial derivative*, or *comoving derivative*, or *convective derivative*. It can be thought as the rate at which the property changes when measured by an observer traveling with that group of particles.

In the Lagrangian description, the material derivative of $P_{ij\ldots}$ is simply the partial derivative with respect to time, and the position vector $\mathbf{X}$ is held constant as it does not change with time. Thus, we have

$$\frac{d}{dt}[P_{ij\ldots}(\mathbf{X},t)] = \frac{\partial}{\partial t}[P_{ij\ldots}(\mathbf{X},t)]$$

The instantaneous position $\mathbf{x}$ is a property of a particle, and its material derivative is the *instantaneous velocity* $\mathbf{v}$ of the particle. Therefore, the velocity field of the continuum is given by

$$\mathbf{v} = \dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \frac{\partial \chi(\mathbf{X},t)}{\partial t}$$

Similarly, the acceleration field is given by

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{dt^2} = \frac{\partial^2 \chi(\mathbf{X},t)}{\partial t^2}$$

Continuity in the Lagrangian description is expressed by the spatial and temporal continuity of the mapping from the reference configuration to the current configuration of the material points. All physical quantities characterizing the continuum are described this way. In this sense, the function $\chi(\cdot)$ and $P_{ij\ldots}(\cdot)$ are single-valued and continuous, with continuous derivatives with respect to space and time to whatever order is required, usually to the second or third.

Eulerian description

Continuity allows for the inverse of $\chi(\cdot)$ to trace backwards where the particle currently located at $\mathbf{x}$ was located in the initial or referenced configuration $\kappa_0(\mathcal{B})$. In this case the description of motion is made in terms of the spatial coordinates, in which case is called the spatial description or Eulerian description, i.e. the current configuration is taken as the reference configuration.

The Eulerian description, introduced by d'Alembert, focuses on the current configuration $\kappa_t(\mathcal{B})$, giving attention to what is occurring at a fixed point in space as time progresses, instead of giving attention to individual particles as they move through space and time. This approach is conveniently applied in the study of fluid flow where the kinematic

property of greatest interest is the rate at which change is taking place rather than the shape of the body of fluid at a reference time.

Mathematically, the motion of a continuum using the Eulerian description is expressed by the mapping function

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$$

which provides a tracing of the particle which now occupies the position $\mathbf{x}$ in the current configuration $\kappa_t(\mathcal{B})$ to its original position $\mathbf{X}$ in the initial configuration $\kappa_0(\mathcal{B})$.

A necessary and sufficient condition for this inverse function to exist is that the determinant of the Jacobian Matrix, often referred to simply as the Jacobian, should be different from zero. Thus,

$$J = \left| \frac{\partial \chi_i}{\partial X_J} \right| = \left| \frac{\partial x_i}{\partial X_J} \right| \neq 0$$

In the Eulerian description, the physical properties $P_{ij\cdots}$ are expressed as

$$P_{ij\cdots} = P_{ij\cdots}(\mathbf{X}, t) = P_{ij\cdots}[\chi^{-1}(\mathbf{x}, t), t] = p_{ij\cdots}(\mathbf{x}, t)$$

where the functional form of $P_{ij\cdots}$ in the Lagrangian description is not the same as the form of $p_{ij\cdots}$ in the Eulerian description.

The material derivative of $p_{ij\cdots}(\mathbf{x}, t)$, using the chain rule, is then

$$\frac{d}{dt}[p_{ij\cdots}(\mathbf{x}, t)] = \frac{\partial}{\partial t}[p_{ij\cdots}(\mathbf{x}, t)] + \frac{\partial}{\partial x_k}[p_{ij\cdots}(\mathbf{x}, t)]\frac{dx_k}{dt}$$

The first term on the right-hand side of this equation gives the *local rate of change* of the property $p_{ij\cdots}(\mathbf{x}, t)$ occurring at position $\mathbf{x}$. The second term of the right-hand side is the *convective rate of change* and expresses the contribution of the particle changing position in space (motion).

Continuity in the Eulerian description is expressed by the spatial and temporal continuity and continuous differentiability of the velocity field. All physical quantities are defined this way at each instant of time, in the current configuration, as a function of the vector position $\mathbf{x}$.

Displacement field

The vector joining the positions of a particle P in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i$, in the Lagrangian description, or $\mathbf{U}(\mathbf{x},t) = U_J\mathbf{E}_J$, in the Eulerian description.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field, In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X},t) = \mathbf{b} + \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ}b_J + x_i - \alpha_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{b} + \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji}x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ}U_J \qquad \text{or} \qquad U_J = \alpha_{Ji}u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ}\mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i = u_i(\alpha_{iJ}\mathbf{E}_J) = U_J\mathbf{E}_J = \mathbf{U}(\mathbf{x},t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in $\mathbf{b} = 0$, and the direction cosines become Kronecker deltas, i.e.

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}$$

Thus, we have

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = \delta_{Ji}x_i - X_J$$

Governing equations

Continuum mechanics deals with the behavior of materials that can be approximated as continuous for certain length and time scales. The equations that govern the mechanics of such materials include the balance laws for mass, momentum, and energy. Kinematic relations and constitutive equations are needed to complete the system of governing equations. Physical restrictions on the form of the constitutive relations can be applied by requiring that the second law of thermodynamics be satisfied under all conditions. In the continuum mechanics of solids, the second law of thermodynamics is satisfied if the Clausius–Duhem form of the entropy inequality is satisfied.

The balance laws express the idea that the rate of change of a quantity (mass, momentum, energy) in a volume must arise from three causes:

1. the physical quantity itself flows through the surface that bounds the volume,
2. there is a source of the physical quantity on the surface of the volume, or/and,
3. there is a source of the physical quantity inside the volume.

Let Ω be the body (an open subset of Euclidean space) and let $\partial\Omega$ be its surface (the boundary of Ω).

Let the motion of material points in the body be described by the map

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \mathbf{x}(\mathbf{X})$$

where $\mathbf{X}$ is the position of a point in the initial configuration and $\mathbf{x}$ is the location of the same point in the deformed configuration.

The deformation gradient is given by

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{x} \cdot \nabla \ .$$

Balance laws

Let $f(\mathbf{x},t)$ be a physical quantity that is flowing through the body. Let $g(\mathbf{x},t)$ be sources on the surface of the body and let $h(\mathbf{x},t)$ be sources inside the body. Let $\mathbf{n}(\mathbf{x},t)$ be the outward unit normal to the surface $\partial\Omega$. Let $\mathbf{v}(\mathbf{x},t)$ be the velocity of the

physical particles that carry the physical quantity that is flowing. Also, let the speed at which the bounding surface $\partial\Omega$ is moving be u_n (in the direction $\mathbf{n}$).

Then, balance laws can be expressed in the general form

$$\frac{d}{dt}\left[\int_\Omega f(\mathbf{x},t)\ \mathrm{dV}\right] = \int_{\partial\Omega} f(\mathbf{x},t)[u_n(\mathbf{x},t) - \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x},t)]\ \mathrm{dA} + \int_{\partial\Omega} g(\mathbf{x},t)\ \mathrm{dA} + \int_\Omega h(\mathbf{x},t)\ \mathrm{dV}\ .$$

Note that the functions $f(\mathbf{x},t)$, $g(\mathbf{x},t)$, and $h(\mathbf{x},t)$ can be scalar valued, vector valued, or tensor valued - depending on the physical quantity that the balance equation deals with. If there are internal boundaries in the body, jump discontinuities also need to be specified in the balance laws.

If we take the Lagrangian point of view, it can be shown that the balance laws of mass, momentum, and energy for a solid can be written as

$$\begin{aligned}
\dot\rho + \rho\ \boldsymbol{\nabla}\cdot\mathbf{v} &= 0 && \text{Balance of Mass}\\
\rho\ \dot{\mathbf{v}} - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} - \rho\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\
\boldsymbol{\sigma} &= \boldsymbol{\sigma}^T && \text{Balance of Angular Momentum}\\
\rho\ \dot e - \boldsymbol{\sigma} : (\boldsymbol{\nabla}\mathbf{v}) + \boldsymbol{\nabla}\cdot\mathbf{q} - \rho\ s &= 0 && \text{Balance of Energy.}
\end{aligned}$$

In the above equations $\rho(\mathbf{x},t)$ is the mass density (current), $\dot\rho$ is the material time derivative of ρ, $\mathbf{v}(\mathbf{x},t)$ is the particle velocity, $\dot{\mathbf{v}}$ is the material time derivative of $\mathbf{v}$, $\boldsymbol{\sigma}(\mathbf{x},t)$ is the Cauchy stress tensor, $\mathbf{b}(\mathbf{x},t)$ is the body force density, $e(\mathbf{x},t)$ is the internal energy per unit mass, $\dot e$ is the material time derivative of e, $\mathbf{q}(\mathbf{x},t)$ is the heat flux vector, and $s(\mathbf{x},t)$ is an energy source per unit mass.

With respect to the reference configuration, the balance laws can be written as

$$\begin{aligned}
\rho\ \det(\boldsymbol{F}) - \rho_0 &= 0 && \text{Balance of Mass}\\
\rho_0\ \ddot{\mathbf{x}} - \boldsymbol{\nabla}_\circ\cdot\boldsymbol{P}^T - \rho_0\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\
\boldsymbol{F}\cdot\boldsymbol{P}^T &= \boldsymbol{P}\cdot\boldsymbol{F}^T && \text{Balance of Angular Momentum}\\
\rho_0\ \dot e - \boldsymbol{P}^T : \dot{\boldsymbol{F}} + \boldsymbol{\nabla}_\circ\cdot\mathbf{q} - \rho_0\ s &= 0 && \text{Balance of Energy.}
\end{aligned}$$

In the above, $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, and ρ_0 is the mass density in the reference configuration. The first Piola-Kirchhoff stress tensor is related to the Cauchy stress tensor by

$$\boldsymbol{P} = J\ \boldsymbol{\sigma}\cdot\boldsymbol{F}^{-T} \text{ where } J = \det(\boldsymbol{F})$$

We can alternatively define the nominal stress tensor $\boldsymbol{N}$ which is the transpose of the first Piola-Kirchhoff stress tensor such that

$$\boldsymbol{N} = \boldsymbol{P}^T = J\ \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma}\ .$$

Then the balance laws become

$$\begin{aligned}
\rho\ \det(\boldsymbol{F}) - \rho_0 &= 0 && \text{Balance of Mass}\\
\rho_0\ \ddot{\mathbf{x}} - \boldsymbol{\nabla}_\circ \cdot \boldsymbol{N} - \rho_0\ \mathbf{b} &= 0 && \text{Balance of Linear Momentum}\\
\boldsymbol{F} \cdot \boldsymbol{N} &= \boldsymbol{N}^T \cdot \boldsymbol{F}^T && \text{Balance of Angular Momentum}\\
\rho_0\ \dot{e} - \boldsymbol{N} : \dot{\boldsymbol{F}} + \boldsymbol{\nabla}_\circ \cdot \mathbf{q} - \rho_0\ s &= 0 && \text{Balance of Energy.}
\end{aligned}$$

The operators in the above equations are defined as such that

$$\boldsymbol{\nabla}\mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \mathbf{e}_i \otimes \mathbf{e}_j = v_{i,j} \mathbf{e}_i \otimes \mathbf{e}_j\ ;\ \ \boldsymbol{\nabla} \cdot \mathbf{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial x_i} = v_{i,i}\ ;\ \ \boldsymbol{\nabla} \cdot \boldsymbol{S} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial x_j}\, \mathbf{e}_i = \sigma_{ij,j}\, \mathbf{e}_i\ .$$

where $\mathbf{v}$ is a vector field, $\boldsymbol{S}$ is a second-order tensor field, and $\mathbf{e}_i$ are the components of an orthonormal basis in the current configuration. Also,

$$\boldsymbol{\nabla}_\circ \mathbf{v} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial X_j} \mathbf{E}_i \otimes \mathbf{E}_j = v_{i,j} \mathbf{E}_i \otimes \mathbf{E}_j\ ;\ \ \boldsymbol{\nabla}_\circ \cdot \mathbf{v} = \sum_{i=1}^{3} \frac{\partial v_i}{\partial X_i} = v_{i,i}\ ;\ \ \boldsymbol{\nabla}_\circ \cdot \boldsymbol{S} = \sum_{i,j=1}^{3} \frac{\partial S_{ij}}{\partial X_j}\, \mathbf{E}_i = S_{ij,j}\, \mathbf{E}_i$$

where $\mathbf{v}$ is a vector field, $\boldsymbol{S}$ is a second-order tensor field, and $\mathbf{E}_i$ are the components of an orthonormal basis in the reference configuration.

The inner product is defined as

$$\boldsymbol{A} : \boldsymbol{B} = \sum_{i,j=1}^{3} A_{ij}\ B_{ij} = trace(\boldsymbol{A}\boldsymbol{B}^T)\ .$$

The Clausius–Duhem inequality

The Clausius–Duhem inequality can be used to express the second law of thermodynamics for elastic-plastic materials. This inequality is a statement concerning the irreversibility of natural processes, especially when energy dissipation is involved.

Just like in the balance laws in the previous section, we assume that there is a flux of a quantity, a source of the quantity, and an internal density of the quantity per unit mass. The quantity of interest in this case is the entropy. Thus, we assume that there is an entropy flux, an entropy source, and an internal entropy density per unit mass (η) in the region of interest.

Let Ω be such a region and let $\partial\Omega$ be its boundary. Then the second law of thermodynamics states that the rate of increase of η in this region is greater than or equal

to the sum of that supplied to Ω (as a flux or from internal sources) and the change of the internal entropy density due to material flowing in and out of the region.

Let $\partial\Omega$ move with a velocity u_n and let particles inside Ω have velocities $\mathbf{v}$. Let $\mathbf{n}$ be the unit outward normal to the surface $\partial\Omega$. Let ρ be the density of matter in the region, $\bar{q}$ be the entropy flux at the surface, and r be the entropy source per unit mass. Then the entropy inequality may be written as

$$\frac{d}{dt}\left(\int_\Omega \rho\ \eta\ \mathrm{dV}\right) \geq \int_{\partial\Omega} \rho\ \eta\ (u_n - \mathbf{v}\cdot\mathbf{n})\ \mathrm{dA} + \int_{\partial\Omega} \bar{q}\ \mathrm{dA} + \int_\Omega \rho\ r\ \mathrm{dV}.$$

The scalar entropy flux can be related to the vector flux at the surface by the relation $\bar{q} = -\boldsymbol{\psi}(\mathbf{x})\cdot\mathbf{n}$. Under the assumption of incrementally isothermal conditions, we have

$$\boldsymbol{\psi}(\mathbf{x}) = \frac{\mathbf{q}(\mathbf{x})}{T}\ ;\quad r = \frac{s}{T}$$

where $\mathbf{q}$ is the heat flux vector, s is a energy source per unit mass, and T is the absolute temperature of a material point at $\mathbf{x}$ at time t.

We then have the Clausius–Duhem inequality in integral form:

$$\frac{d}{dt}\left(\int_\Omega \rho\ \eta\ \mathrm{dV}\right) \geq \int_{\partial\Omega} \rho\ \eta\ (u_n - \mathbf{v}\cdot\mathbf{n})\ \mathrm{dA} - \int_{\partial\Omega} \frac{\mathbf{q}\cdot\mathbf{n}}{T}\ \mathrm{dA} + \int_\Omega \frac{\rho\ s}{T}\ \mathrm{dV}.$$

We can show that the entropy inequality may be written in differential form as

$$\rho\ \dot{\eta} \geq -\boldsymbol{\nabla}\cdot\left(\frac{\mathbf{q}}{T}\right) + \frac{\rho\ s}{T}.$$

In terms of the Cauchy stress and the internal energy, the Clausius–Duhem inequality may be written as

$$\rho\ (\dot{e} - T\ \dot{\eta}) - \boldsymbol{\sigma} : \boldsymbol{\nabla}\mathbf{v} \leq -\frac{\mathbf{q}\cdot\boldsymbol{\nabla}T}{T}.$$

Chapter 2

Fluid Mechanics

Fluid mechanics is the study of fluids and the forces on them. (Fluids include liquids, gases, and plasmas.) Fluid mechanics can be divided into fluid kinematics, the study of fluid motion, and fluid dynamics, the study of the effect of forces on fluid motion, which can further be divided into fluid statics, the study of fluids at rest, and fluid kinetics, the study of fluids in motion. It is a branch of continuum mechanics, a subject which models matter without using the information that it is made out of atoms, that is, it models matter from a macroscopic viewpoint rather than from a microscopic viewpoint. Fluid mechanics, especially fluid dynamics, is an active field of research with many unsolved or partly solved problems. Fluid mechanics can be mathematically complex. Sometimes it can best be solved by numerical methods, typically using computers. A modern discipline, called computational fluid dynamics (CFD), is devoted to this approach to solving fluid mechanics problems. Also taking advantage of the highly visual nature of fluid flow is particle image velocimetry, an experimental method for visualizing and analyzing fluid flow.

Brief history

The study of fluid mechanics goes back at least to the days of ancient Greece, when Archimedes investigated fluid statics and buoyancy and formulated his famous law known now as the Archimedes Principle. Rapid advancement in fluid mechanics began with Leonardo da Vinci (observation and experiment), Evangelista Torricelli (barometer), Isaac Newton (viscosity) and Blaise Pascal (hydrostatics), and was continued by Daniel Bernoulli with the introduction of mathematical fluid dynamics in *Hydrodynamica* (1738). Inviscid flow was further analyzed by various mathematicians (Leonhard Euler, d'Alembert, Lagrange, Laplace, Poisson) and viscous flow was explored by a multitude of engineers including Poiseuille and Gotthilf Heinrich Ludwig Hagen. Further mathematical justification was provided by Claude-Louis Navier and George Gabriel Stokes in the Navier–Stokes equations, and boundary layers were investigated (Ludwig

Prandtl), while various scientists (Osborne Reynolds, Andrey Kolmogorov, Geoffrey Ingram Taylor) advanced the understanding of fluid viscosity and turbulence.

Relationship to continuum mechanics

Fluid mechanics is a subdiscipline of continuum mechanics, as illustrated in the following table.

<table>
<tr><td rowspan="4">Continuum mechanics
The study of the physics of continuous materials</td><td rowspan="2">Solid mechanics
The study of the physics of continuous materials with a defined rest shape.</td><td colspan="2">Elasticity
Describes materials that return to their rest shape after an applied stress.</td></tr>
<tr><td>Plasticity
Describes materials that permanently deform after a sufficient applied stress.</td><td rowspan="2">Rheology
The study of materials with both solid and fluid characteristics.</td></tr>
<tr><td rowspan="2">Fluid mechanics
The study of the physics of continuous materials which take the shape of their container.</td><td>Non-Newtonian fluids</td></tr>
<tr><td colspan="2">Newtonian fluids</td></tr>
</table>

In a mechanical view, a fluid is a substance that does not support shear stress; that is why a fluid at rest has the shape of its containing vessel. A fluid at rest has no shear stress.

Assumptions

Like any mathematical model of the real world, fluid mechanics makes some basic assumptions about the materials being studied. These assumptions are turned into equations that must be satisfied if the assumptions are to be held true. For example, consider an incompressible fluid in three dimensions. The assumption that mass is conserved means that for any fixed closed surface (such as a sphere) the rate of mass passing from *outside* to *inside* the surface must be the same as rate of mass passing the other way. (Alternatively, the mass *inside* remains constant, as does the mass *outside*). This can be turned into an integral equation over the surface.

Fluid mechanics assumes that every fluid obeys the following:

- Conservation of mass
- Conservation of energy
- Conservation of momentum
- The *continuum hypothesis*, detailed below.

Further, it is often useful (at subsonic conditions) to assume a fluid is incompressible – that is, the density of the fluid does not change. Liquids can often be modelled as incompressible fluids, whereas gases cannot.

Similarly, it can sometimes be assumed that the viscosity of the fluid is zero (the fluid is *inviscid*). Gases can often be assumed to be inviscid. If a fluid is viscous, and its flow contained in some way (e.g. in a pipe), then the flow at the boundary must have zero velocity. For a viscous fluid, if the boundary is not porous, the shear forces between the fluid and the boundary results also in a zero velocity for the fluid at the boundary. This is called the no-slip condition. For a porous media otherwise, in the frontier of the containing vessel, the slip condition is not zero velocity, and the fluid has a discontinuous velocity field between the free fluid and the fluid in the porous media (this is related to the Beavers and Joseph condition).

The continuum hypothesis

Fluids are composed of molecules that collide with one another and solid objects. The continuum assumption, however, considers fluids to be continuous. That is, properties such as density, pressure, temperature, and velocity are taken to be well-defined at "infinitely" small points, defining a REV (Reference Element of Volume), at the geometric order of the distance between two adjacent molecules of fluid. Properties are assumed to vary continuously from one point to another, and are averaged values in the REV. The fact that the fluid is made up of discrete molecules is ignored.

The continuum hypothesis is basically an approximation, in the same way planets are approximated by point particles when dealing with celestial mechanics, and therefore results in approximate solutions. Consequently, assumption of the continuum hypothesis can lead to results which are not of desired accuracy. That said, under the right circumstances, the continuum hypothesis produces extremely accurate results.

Those problems for which the continuum hypothesis does not allow solutions of desired accuracy are solved using statistical mechanics. To determine whether or not to use conventional fluid dynamics or statistical mechanics, the Knudsen number is evaluated for the problem. The Knudsen number is defined as the ratio of the molecular mean free path length to a certain representative physical length scale. This length scale could be, for example, the radius of a body in a fluid. (More simply, the Knudsen number is how many times its own diameter a particle will travel on average before hitting another particle). Problems with Knudsen numbers at or above unity are best evaluated using statistical mechanics for reliable solutions.

Navier–Stokes equations

The **Navier–Stokes equations** (named after Claude-Louis Navier and George Gabriel Stokes) are the set of equations that describe the motion of fluid substances such as liquids and gases. These equations state that changes in momentum (force) of fluid particles depend only on the external pressure and internal viscous forces (similar to friction) acting on the fluid. Thus, the Navier–Stokes equations describe the balance of forces acting at any given region of the fluid.

The Navier–Stokes equations are differential equations which describe the motion of a fluid. Such equations establish relations among the rates of change of the variables of interest. For example, the Navier–Stokes equations for an ideal fluid with zero viscosity states that acceleration (the rate of change of velocity) is proportional to the derivative of internal pressure.

This means that solutions of the Navier–Stokes equations for a given physical problem must be sought with the help of calculus. In practical terms only the simplest cases can be solved exactly in this way. These cases generally involve non-turbulent, steady flow (flow does not change with time) in which the Reynolds number is small.

For more complex situations, such as global weather systems like El Niño or lift in a wing, solutions of the Navier–Stokes equations can currently only be found with the help of computers. This is a field of sciences by its own called computational fluid dynamics.

General form of the equation

The general form of the Navier–Stokes equations for the conservation of momentum is:

$$\rho \frac{D\mathbf{v}}{Dt} = \nabla \cdot \mathbb{P} + \rho \mathbf{f}$$

where

- ρ is the fluid density,
- $\frac{D}{Dt}$ is the substantive derivative (also called the material derivative),
- $\mathbf{v}$ is the velocity vector,
- $\mathbf{f}$ is the body force vector, and
- $\mathbb{P}$ is a tensor that represents the surface forces applied on a fluid particle (the stress tensor).

Unless the fluid is made up of spinning degrees of freedom like vortices, $\mathbb{P}$ is a symmetric tensor. In general, (in three dimensions) $\mathbb{P}$ has the form:

$$\mathbb{P} = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

where

- σ are normal stresses,
- τ are tangential stresses (shear stresses).

The above is actually a set of three equations, one per dimension. By themselves, these aren't sufficient to produce a solution. However, adding conservation of mass and appropriate boundary conditions to the system of equations produces a solvable set of equations.

Newtonian versus non-Newtonian fluids

A **Newtonian fluid** (named after Isaac Newton) is defined to be a fluid whose shear stress is linearly proportional to the velocity gradient in the direction perpendicular to the plane of shear. This definition means regardless of the forces acting on a fluid, it *continues to flow*. For example, water is a Newtonian fluid, because it continues to display fluid properties no matter how much it is stirred or mixed. A slightly less rigorous definition is that the drag of a small object being moved slowly through the fluid is proportional to the force applied to the object. (Compare friction). Important fluids, like water as well as most gases, behave — to good approximation — as a Newtonian fluid under normal conditions on Earth.

By contrast, stirring a non-Newtonian fluid can leave a "hole" behind. This will gradually fill up over time – this behaviour is seen in materials such as pudding, oobleck, or sand (although sand isn't strictly a fluid). Alternatively, stirring a non-Newtonian fluid can cause the viscosity to decrease, so the fluid appears "thinner" (this is seen in non-drip paints). There are many types of non-Newtonian fluids, as they are defined to be something that fails to obey a particular property — for example, most fluids with long molecular chains can react in a non-Newtonian manner.

Equations for a Newtonian fluid

The constant of proportionality between the shear stress and the velocity gradient is known as the viscosity. A simple equation to describe Newtonian fluid behaviour is

$$\tau = -\mu \frac{dv}{dy}$$

where

τ is the shear stress exerted by the fluid ("drag")
μ is the fluid viscosity – a constant of proportionality
$\frac{dv}{dy}$ is the velocity gradient perpendicular to the direction of shear.

For a Newtonian fluid, the viscosity, by definition, depends only on temperature and pressure, not on the forces acting upon it. If the fluid is incompressible and viscosity is constant across the fluid, the equation governing the shear stress (in Cartesian coordinates) is

$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

where

τ_{ij} is the shear stress on the i^{th} face of a fluid element in the j^{th} direction
v_i is the velocity in the i^{th} direction
x_j is the j^{th} direction coordinate.

If a fluid does not obey this relation, it is termed a non-Newtonian fluid, of which there are several types.

Among fluids, two rough broad divisions can be made: ideal and non-ideal fluids. An ideal fluid really does not exist, but in some calculations, the assumption is justifiable. An Ideal fluid is non viscous- offers no resistance whatsoever to a shearing force.

One can group real fluids into Newtonian and non-Newtonian. Newtonian fluids agree with Newton's law of viscosity. Non-Newtonian fluids can be either plastic, bingham plastic, pseudoplastic, dilatant, thixotropic, rheopectic, viscoelatic.

Chapter 3

Stress (Mechanics)

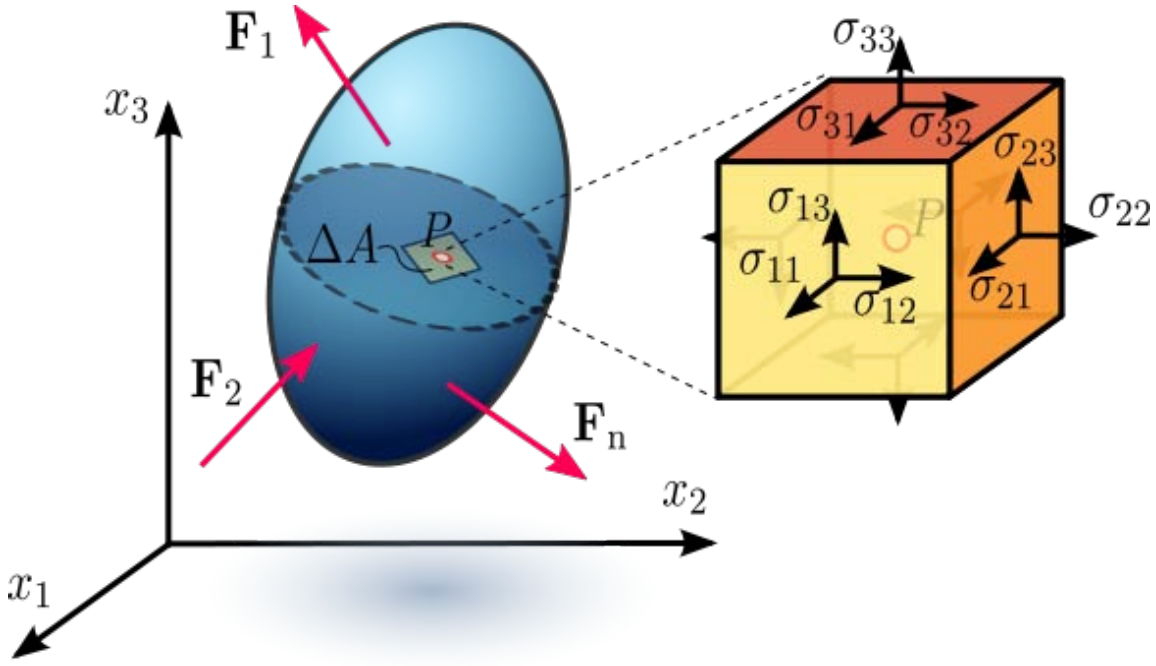


Figure 1.1 Stress in a loaded deformable material body assumed as a continuum.

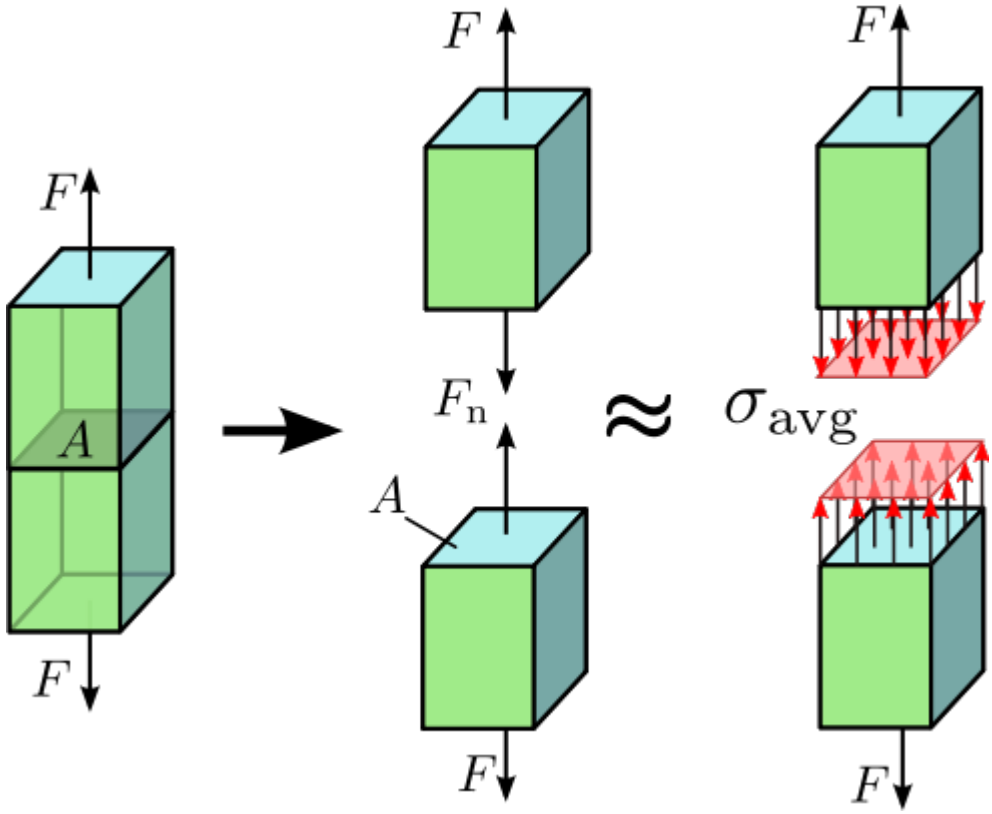


Figure 1.2 Axial stress in a prismatic bar axially loaded

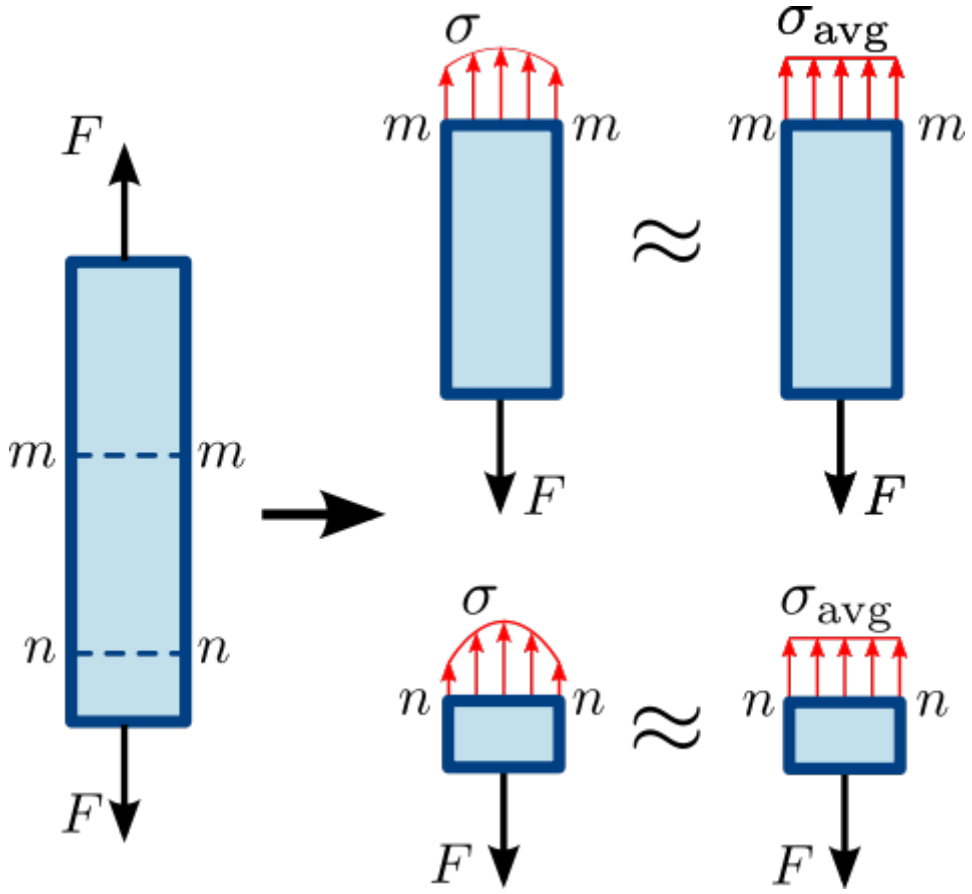


Figure 1.3 Normal stress in a prismatic (straight member of uniform cross-sectional area) bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. However, an average normal stress σ_{avg} can be used

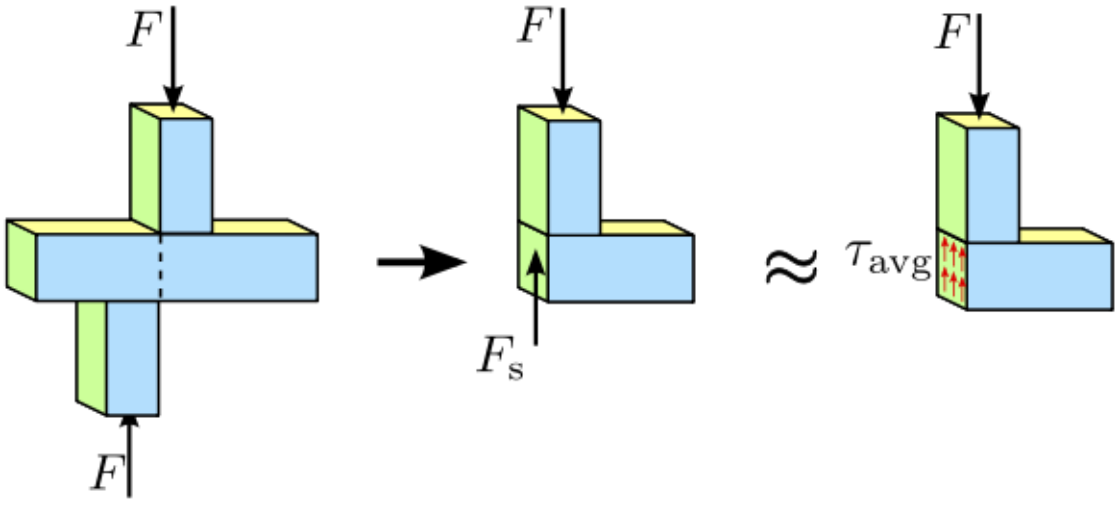


Figure 1.4 Shear stress in a prismatic bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. Nevertheless, an average shear stress τ_{avg} is a reasonable approximation.

In continuum mechanics, **stress** is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, and result in deformation of the body's shape. Beyond certain limits of material strength, this can lead to a permanent change of shape or physical failure.

However, treating physical force as a "one dimensional entity", as it is often done in mechanics, creates a few problems. Any model of continuum mechanics which explicitly expresses force as a variable generally fails to merge and describe deformation of matter and solid bodies, because the attributes of matter and solids are three dimensional. Classical models of continuum mechanics assume an average force and fail to properly incorporate "geometrical factors", which are important to describe stress distribution and accumulation of energy during the continuum.

The dimension of stress is that of pressure, and therefore the SI unit for stress is the pascal (symbol Pa), which is equivalent to one newton (force) per square meter (unit area), that is N/m^2. In Imperial units, stress is measured in pound-force per square inch, which is abbreviated as psi.

Introduction

Stress is a measure of the average force per unit area of a surface within a deformable body on which internal forces act. It is a measure of the intensity of the internal forces acting between particles of a deformable body across imaginary internal surfaces. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. External forces are either surface forces or body forces. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, *i.e.* the stress distribution in the body is expressed as a piecewise continuous function of space coordinates and time.

Normal , shear stresses and virial stresses

For the simple case of a body axially loaded, e.g., a prismatic bar subjected to tension or compression by a force passing through its centroid (Figures 1.2 and 1.3) the stress σ, or intensity of internal forces, can be obtained by dividing the total *normal force* F_{n}, determined from the equilibrium of forces, by the cross-sectional area A of the prism it is acting upon. The normal force can be a *tensile force* if acting outward from the plane, or *compressive force* if acting inward to the plane. In the case of a prismatic bar axially loaded, the stress σ is represented by a scalar called *engineering stress* or *nominal stress* that represents an average stress (σ_{avg}) over the area, meaning that the stress in the cross section is uniformly distributed. Thus, we have

$$\sigma_{\mathrm{avg}} = \frac{F_{\mathrm{n}}}{A} \approx \sigma$$

A different type of stress is obtained when transverse forces F are applied to the prismatic bar as shown in Figure 1.4. Considering the same cross-section as before, from static equilibrium the internal force has a magnitude equal to F_{s} and in opposite direction parallel to the cross-section. F_{s} is called the *shear force*. Dividing the shear force F_{s} by the area A of the cross section we obtain the *shear stress*. In this case the shear stress τ is a scalar quantity representing an average shear stress (τ_{avg}) in the section, *i.e.* the stress in the cross-section is uniformly distributed. In materials science and in engineering aspects the average of the ""scalar"" shear force (τ_{avg}) are true for crystallized materials during brittle fracture and operates through the fractured cross-section or stress plane.

$$\tau_{\mathrm{avg}} = \frac{F_{\mathrm{s}}}{A} \approx \tau$$

In Figure 1.3, the normal stress is observed in two planes $m - m$ and $n - n$ of the axially loaded prismatic bar. The stress on plane $n - n$, which is closer to the point of application of the load F, varies more across the cross-section than that of plane $m - m$. However, if the cross-sectional area of the bar is very small, *i.e.* the bar is slender, the variation of stress across the area is small and the normal stress can be approximated by σ_{avg}. On the other hand, the variation of shear stress across the section of a prismatic bar cannot be assumed to be uniform.

Virial stress is a measure of stress on an atomic scale. It is given by

$$\tau_{ij} = \frac{1}{\Omega} \sum_{k \in \Omega} \left(-m^{(k)} (u_i^{(k)} - \bar{u}_i)(u_j^{(k)} - \bar{u}_j) + \frac{1}{2} \sum_{\ell \in \Omega} (x_i^{(\ell)} - x_i^{(k)}) f_j^{(k\ell)} \right)$$

where

- k and ℓ are atoms in the domain,
- Ω is the volume of the domain,
- $m^{(k)}$ is the mass of atom k,
- $u_i^{(k)}$ is the i^{th} component of the velocity of atom k,
- $\bar{u}_j$ is the j^{th} component of the average velocity of atoms in the volume,
- $x_i^{(k)}$ is the i^{th} component of the position of atom k, and
- $f_i^{(k\ell)}$ is the i^{th} component of the force between atom k and ℓ.

At zero kelvin, all velocities are zero so we have

$$\tau_{ij} = \frac{1}{2\Omega} \sum_{k,\ell \in \Omega} (x_i^{(\ell)} - x_i^{(k)}) f_j^{(k\ell)}$$

.

This can be thought of as follows. The τ_{11} component of stress is the force in the 1 direction divided by the area of a plane perpendicular to that direction. Consider two adjacent volumes separated by such a plane. The 11-component of stress on that interface is the sum of all pairwise forces between atoms on the two sides....

Stress modeling (Cauchy)

In general, stress is not uniformly distributed over the cross-section of a material body, and consequently the stress at a point in a given region is different from the average stress over the entire area. Therefore, it is necessary to define the stress not over a given area but at a specific point in the body (Figure 1.1). According to Cauchy, the *stress at any point* in an object, assumed to behave as a continuum, is completely defined by the nine components σ_{ij} of a second-order tensor of type (0,2) known as the Cauchy stress tensor, $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

The Cauchy stress tensor obeys the tensor transformation law under a change in the system of coordinates. A graphical representation of this transformation law is the Mohr's circle of stress distribution.

The Cauchy stress tensor is used for stress analysis of material bodies experiencing small deformations where the differences in stress distribution in most cases can be neglected. For large deformations, also called finite deformations, other measures of stress, such as the first and second Piola-Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor, are required.

According to the principle of conservation of linear momentum, if a continuous body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations (Cauchy's equations of motion for zero acceleration). At the same time, according to the principle of conservation of angular momentum, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, thus having only six independent stress components instead of the original nine.

There are certain invariants associated with the stress tensor, whose values do not depend upon the coordinate system chosen or the area element upon which the stress tensor operates. These are the three eigenvalues of the stress tensor, which are called the

principal stresses. Solids, liquids, and gases have stress fields. Static fluids support normal stress but will flow under shear stress. Moving viscous fluids can support shear stress (dynamic pressure). Solids can support both shear and normal stress, with ductile materials failing under shear and brittle materials failing under normal stress. All materials have temperature dependent variations in stress-related properties, and non-Newtonian materials have rate-dependent variations.

Stress analysis

Stress analysis means the determination of the internal distribution of stresses in a structure. It is needed in engineering for the study and design of structures such as tunnels, dams, mechanical parts, and structural frames, under prescribed or expected loads. To determine the distribution of stress in a structure, the engineer needs to solve a boundary-value problem by specifying the boundary conditions. These are displacements and forces on the boundary of the structure.

Constitutive equations, such as Hooke's Law for linear elastic materials, describe the stress-strain relationship in these calculations.

When a structure is expected to deform elastically (and resume its original shape), a boundary-value problem based on the theory of elasticity is applied, with infinitesimal strains, under design loads.

When the applied loads permanently deform the structure, the theory of plasticity is used.

The stress analysis can be simplified when the physical dimensions and the distribution of loads allow the structure to be treated as one-dimensional or two-dimensional. For a two-dimensional analysis a plane stress or a plane strain condition can be assumed. Alternatively, experimental determination of stresses can be carried out.

Approximate computer-based solutions for boundary-value problems can be obtained through numerical methods such as the Finite Element Method, the Finite Difference Method, and the Boundary Element Method. Analytical or closed-form solutions can be obtained for simple geometries, constitutive relations, and boundary conditions.

Theoretical background

Continuum mechanics deals with deformable bodies, as opposed to rigid bodies. The stresses considered in continuum mechanics are only those produced by deformation of the body, *sc.* only relative changes in stress are considered, not the absolute values. A body is considered stress-free if the only forces present are those inter-atomic forces (ionic, metallic, and van der Waals forces) required to hold the body together and to keep its shape in the absence of all external influences, including gravitational attraction. Stresses generated during manufacture of the body to a specific configuration are also excluded.

Following the classical dynamics of Newton and Euler, the motion of a material body is produced by the action of externally applied forces which are assumed to be of two kinds: surface forces and body forces.

Surface forces, or contact forces, can act either on the bounding surface of the body, as a result of mechanical contact with other bodies, or on imaginary internal surfaces that bound portions of the body, as a result of the mechanical interaction between the parts of the body to either side of the surface (Euler-Cauchy's stress principle). When a body is acted upon by external contact forces, internal contact forces are then transmitted from point to point inside the body to balance their action, according to Newton's second law of motion of conservation of linear momentum and angular momentum (for continuous bodies these laws are called the Euler's equations of motion). The internal contact forces are related to the body's deformation through constitutive equations.

The concept of stress can then be thought as a measure of the intensity of the internal contact forces acting between particles of the body across imaginary internal surfaces. In other words, stress is a measure of the average quantity of force exerted per unit area of the surface on which these internal forces act. The intensity of contact forces is related, specifically in an inverse proportion, to the area of contact. For example, if a force applied to a small area is compared to a distributed load of the same resultant magnitude applied to a larger area, one finds that the effects or intensities of these two forces are locally different because the stresses are not the same.

Body forces are forces originating from sources outside of the body that act on the volume (or mass) of the body. Saying that body forces are due to outside sources implies that the *internal forces* are manifested through the contact forces alone. These forces arise from the presence of the body in force fields, (*e.g.*, a gravitational field). As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed. Thus, body forces are assumed to be continuous over the entire volume of the body.

The density of internal forces at every point in a deformable body are not necessarily equal, *i.e.* there is a distribution of stresses throughout the body. This variation of internal forces throughout the body is governed by Newton's second law of motion of conservation of linear momentum and angular momentum, which normally are applied to a mass particle but are extended in continuum mechanics to a body of continuously distributed mass. For continuous bodies these laws are called Euler’s equations of motion. If a body is represented as an assemblage of discrete particles, each governed by Newton’s laws of motion, then Euler’s equations can be derived from Newton’s laws. Euler’s equations can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

Euler–Cauchy stress principle

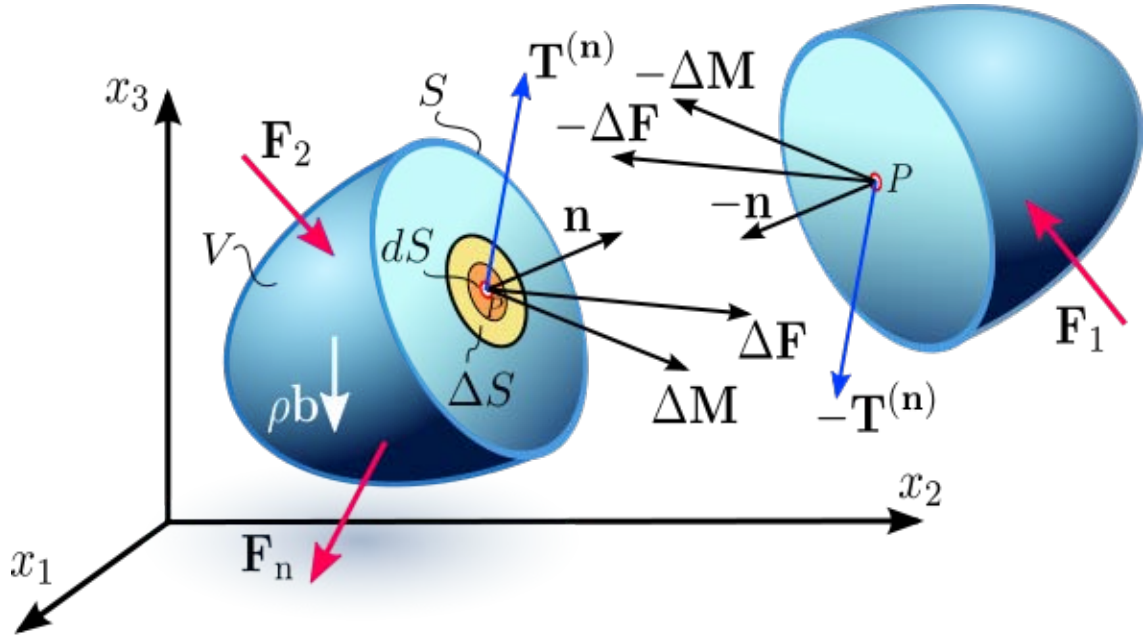


Figure 2.1a Internal distribution of contact forces and couple stresses on a differential dS of the internal surface S in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

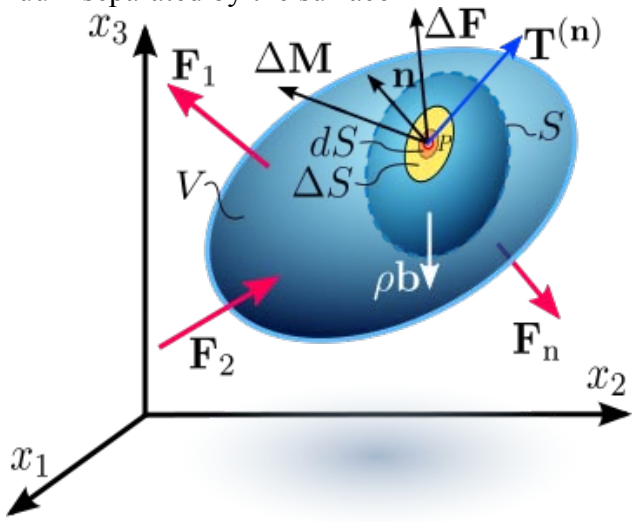


Figure 2.1b Internal distribution of contact forces and couple stresses on a differential dS of the internal surface S in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

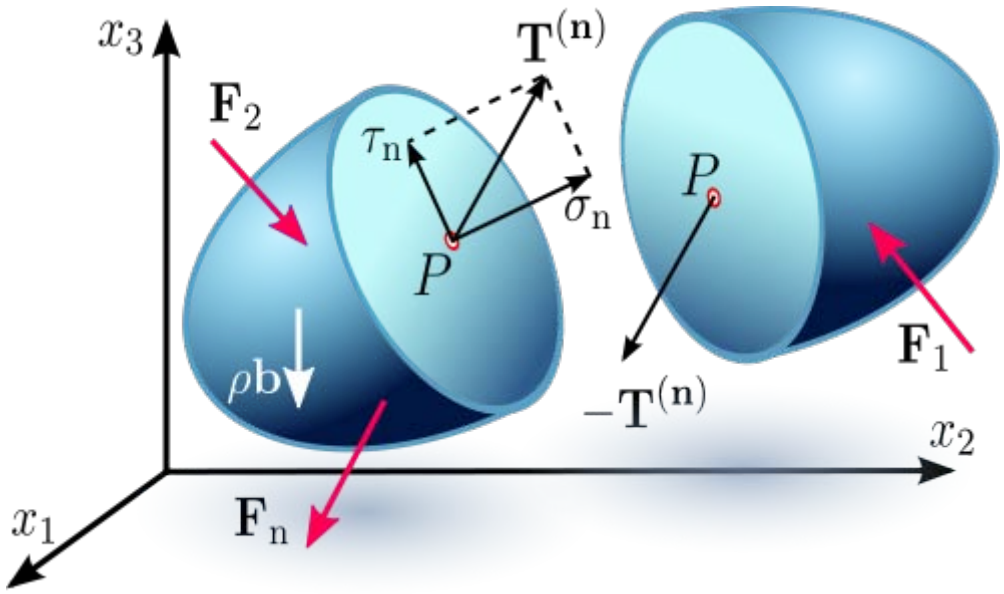


Figure 2.1c Stress vector on an internal surface S with normal vector n. Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to $\mathbf{n}$, and can be resolved into two components: one component normal to the plane, called *normal stress* σ_{n}, and another component parallel to this plane, called the *shearing stress* τ.

The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equivalent (equipollent) to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field $\mathbf{T}^{(\mathbf{n})}$, called the stress vector, defined on the surface S and assumed to depend continuously on the surface's unit vector $\mathbf{n}$.

To explain this principle, we consider an imaginary surface S passing through an internal material point P dividing the continuous body into two segments, as seen in Figure 2.1a or 2.1b (some authors use the cutting plane diagram and others use the diagram with the arbitrary volume inside the continuum enclosed by the surface S). The body is subjected to external surface forces $\mathbf{F}$ and body forces $\mathbf{b}$. The internal contact forces being transmitted from one segment to the other through the dividing plane, due to the action of one portion of the continuum onto the other, generate a force distribution on a small area ΔS, with a normal unit vector $\mathbf{n}$, on the dividing plane S. The force distribution is equipollent to a contact force $\Delta\mathbf{F}$ and a couple stress $\Delta\mathbf{M}$, as shown in Figure 2.1a and 2.1b. Cauchy's stress principle asserts that as ΔS becomes very small and tends to zero the ratio $\Delta\mathbf{F}/\Delta S$ becomes $\mathrm{d}\mathbf{F}/\mathrm{d}S$ and the couple stress vector $\Delta\mathbf{M}$ vanishes. In specific fields of continuum mechanics the couple stress is assumed not to vanish; however, as stated previously, in classical branches of continuum mechanics we deal with non-polar materials which do not consider couple stresses and body moments. The resultant vector $\mathrm{d}\mathbf{F}/\mathrm{d}S$ is defined as the *stress vector* or *traction vector* given by $\mathbf{T}^{(\mathbf{n})} = T_{\mathrm{i}}^{(\mathbf{n})}\,\mathbf{e}_{\mathrm{i}}$ at the point P associated with a plane with a normal vector $\mathbf{n}$:

$$T_i^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta F_i}{\Delta S} = \frac{dF_i}{dS}.$$

This equation means that the stress vector depends on its location in the body and the orientation of the plane on which it is acting.

Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to **n**, and can be resolved into two components:

- one normal to the plane, called *normal stress*

  $$\sigma_\mathrm{n} = \lim_{\Delta S \to 0} \frac{\Delta F_\mathrm{n}}{\Delta S} = \frac{dF_\mathrm{n}}{dS},$$
  where dF_n is the normal component of the force d**F** to the differential area dS

- and the other parallel to this plane, called the *shear stress*

  $$\tau = \lim_{\Delta S \to 0} \frac{\Delta F_\mathrm{s}}{\Delta S} = \frac{dF_\mathrm{s}}{dS},$$
  where dF_s is the tangential component of the force d**F** to the differential surface area dS. The shear stress can be further decomposed into two mutually perpendicular vectors.

Cauchy's postulate

According to the *Cauchy Postulate*, the stress vector $\mathbf{T}^{(\mathbf{n})}$ remains unchanged for all surfaces passing through the point P and having the same normal vector **n** at P, *i.e.* having a common tangent at P. This means that the stress vector is a function of the normal vector **n** only, and it is not influenced by the curvature of the internal surfaces.

Cauchy's fundamental lemma

A consequence of Cauchy's postulate is *Cauchy's Fundamental Lemma*, also called the *Cauchy reciprocal theorem*, which states that the stress vectors acting on opposite sides of the same surface are equal in magnitude and opposite in direction. Cauchy's fundamental lemma is equivalent to Newton's third law of motion of action and reaction, and it is expressed as

$$-\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(-\mathbf{n})}.$$

Cauchy's stress theorem – stress tensor

The state of stress at a point in the body is then defined by all the stress vectors $\mathbf{T}^{(\mathbf{n})}$ associated with all planes (infinite in number) that pass through that point. However,

according to *Cauchy's fundamental theorem*, also called *Cauchy's stress theorem*, merely by knowing the stress vectors on three mutually perpendicular planes, the stress vector on any other plane passing through that point can be found through coordinate transformation equations.

Cauchy's stress theorem states that there exists a second-order tensor field **σ**(**x**, t), called the *Cauchy stress tensor*, independent of **n**, such that **T** is a linear function of **n**:

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \mathbf{n} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i.$$

This equation implies that the stress vector $\mathbf{T}^{(\mathbf{n})}$ at any point *P* in a continuum associated with a plane with normal vector **n** can be expressed as a function of the stress vectors on the planes perpendicular to the coordinate axes, *i.e.* in terms of the components σ_{ij} of the stress tensor **σ**.

To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal area d*A* oriented in an arbitrary direction specified by a normal vector **n** (Figure 2.2). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane **n**. The stress vector on this plane is denoted by $\mathbf{T}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$, and are by definition the components σ_{ij} of the stress tensor **σ**. This tetrahedron is sometimes called the *Cauchy tetrahedron*. From equilibrium of forces, *i.e.* Euler's first law of motion (Newton's second law of motion), we have

$$\mathbf{T}^{(\mathbf{n})}\,dA - \mathbf{T}^{(\mathbf{e}_1)}\,dA_1 - \mathbf{T}^{(\mathbf{e}_2)}\,dA_2 - \mathbf{T}^{(\mathbf{e}_3)}\,dA_3 = \rho\left(\frac{h}{3}dA\right)\mathbf{a},$$

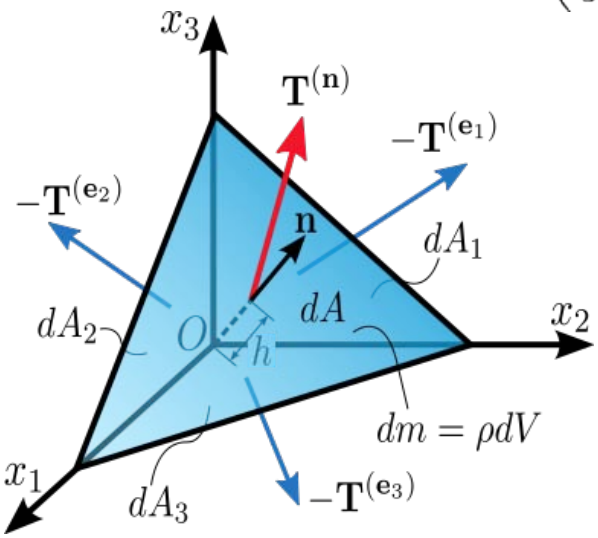


Figure 2.2. Stress vector acting on a plane with normal vector **n**.
A note on the sign convention: The tetrahedron is formed by slicing a parallelepiped along an arbitrary plane **n**. So, the force acting on the plane **n** is the reaction exerted by the other half of the parallelepiped and has an opposite sign.

where the right-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ρ is the density, **a** is the acceleration, and h is the height of the tetrahedron, considering the plane **n** as the base. The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face (using the dot product):

$$\begin{aligned} dA_1 &= (\mathbf{n} \cdot \mathbf{e}_1)\, dA = n_1\, dA, \\ dA_2 &= (\mathbf{n} \cdot \mathbf{e}_2)\, dA = n_2\, dA, \\ dA_3 &= (\mathbf{n} \cdot \mathbf{e}_3)\, dA = n_3\, dA, \end{aligned}$$

and then substituting into the equation to cancel out dA:

$$\mathbf{T}^{(\mathbf{n})} - \mathbf{T}^{(\mathbf{e}_1)} n_1 - \mathbf{T}^{(\mathbf{e}_2)} n_2 - \mathbf{T}^{(\mathbf{e}_3)} n_3 = \rho \left(\frac{h}{3} \right) \mathbf{a}.$$

To consider the limiting case as the tetrahedron shrinks to a point, h must go to 0 (intuitively, the plane **n** is translated along **n** toward O). As a result, the right-hand-side of the equation approaches 0, so

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(\mathbf{e}_1)} n_1 + \mathbf{T}^{(\mathbf{e}_2)} n_2 + \mathbf{T}^{(\mathbf{e}_3)} n_3.$$

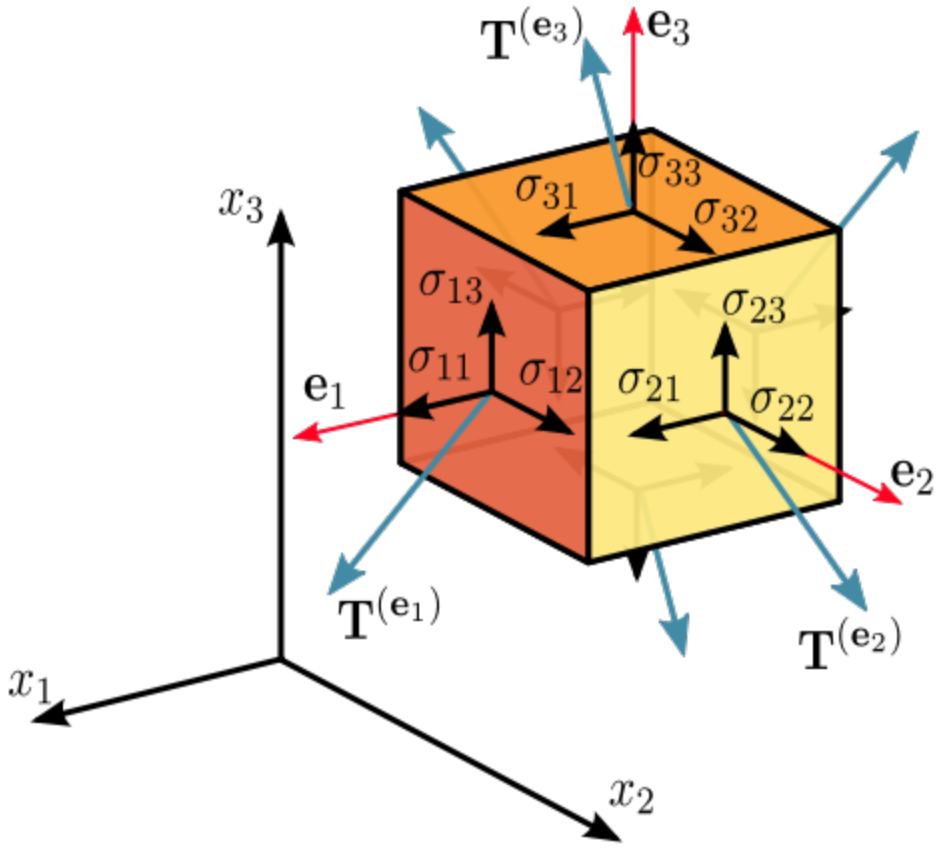


Figure 2.3 Components of stress in three dimensions

Assuming a material element (Figure 2.3) with planes perpendicular to the coordinate axes of a Cartesian coordinate system, the stress vectors associated with each of the element planes, *i.e.* $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$ can be decomposed into a normal component and two shear components, *i.e.* components in the direction of the three coordinate axes. For the particular case of a surface with normal unit vector oriented in the direction of the x_1-axis, the normal stress is denoted by σ_{11}, and the two shear stresses are denoted as σ_{12} and σ_{13}:

$$\begin{aligned}
\mathbf{T}^{(\mathbf{e}_1)} &= T_1^{(\mathbf{e}_1)}\mathbf{e}_1 + T_2^{(\mathbf{e}_1)}\mathbf{e}_2 + T_3^{(\mathbf{e}_1)}\mathbf{e}_3 = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3, \\
\mathbf{T}^{(\mathbf{e}_2)} &= T_1^{(\mathbf{e}_2)}\mathbf{e}_1 + T_2^{(\mathbf{e}_2)}\mathbf{e}_2 + T_3^{(\mathbf{e}_2)}\mathbf{e}_3 = \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3, \\
\mathbf{T}^{(\mathbf{e}_3)} &= T_1^{(\mathbf{e}_3)}\mathbf{e}_1 + T_2^{(\mathbf{e}_3)}\mathbf{e}_2 + T_3^{(\mathbf{e}_3)}\mathbf{e}_3 = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3,
\end{aligned}$$

In index notation this is

$$\mathbf{T}^{(\mathbf{e}_i)} = T_j^{(\mathbf{e}_i)}\mathbf{e}_j = \sigma_{ij}\mathbf{e}_j .$$

The nine components σ_{ij} of the stress vectors are the components of a second-order Cartesian tensor called the *Cauchy stress tensor*, which completely defines the state of stress at a point and is given by

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \mathbf{T}^{(\mathbf{e}_1)} \\ \mathbf{T}^{(\mathbf{e}_2)} \\ \mathbf{T}^{(\mathbf{e}_3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix},$$

where σ_{11}, σ_{22}, and σ_{33} are normal stresses, and σ_{12}, σ_{13}, σ_{21}, σ_{23}, σ_{31}, and σ_{32} are shear stresses. The first index i indicates that the stress acts on a plane normal to the x_i-axis, and the second index j denotes the direction in which the stress acts. A stress component is positive if it acts in the positive direction of the coordinate axes, and if the plane where it acts has an outward normal vector pointing in the positive coordinate direction.

Thus, using the components of the stress tensor

$$\begin{aligned}
\mathbf{T}^{(\mathbf{n})} &= \mathbf{T}^{(\mathbf{e}_1)}n_1 + \mathbf{T}^{(\mathbf{e}_2)}n_2 + \mathbf{T}^{(\mathbf{e}_3)}n_3 \\
&= \sum_{i=1}^{3} \mathbf{T}^{(\mathbf{e}_i)}n_i \\
&= \left(\sigma_{ij}\mathbf{e}_j\right) n_i \\
&= \sigma_{ij}n_i\mathbf{e}_j
\end{aligned}$$

or, equivalently,

$$T_j^{(\mathbf{n})} = \sigma_{ij} n_i.$$

Alternatively, in matrix form we have

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

The Voigt notation representation of the Cauchy stress tensor takes advantage of the symmetry of the stress tensor to express the stress as a six-dimensional vector of the form:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \end{bmatrix}^T \equiv \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{31} & \sigma_{12} \end{bmatrix}^T.$$

The Voigt notation is used extensively in representing stress-strain relations in solid mechanics and for computational efficiency in numerical structural mechanics software.

Transformation rule of the stress tensor

It can be shown that the stress tensor is a contravariant second order tensor, which is a statement of how it transforms under a change of the coordinate system. From an x_i-system to an x_i'-system, the components σ_{ij} in the initial system are transformed into the components σ_{ij}' in the new system according to the tensor transformation rule (Figure 2.4):

$$\sigma'_{ij} = a_{im} a_{jn} \sigma_{mn} \quad \text{or} \quad \boldsymbol{\sigma}' = \mathbf{A} \boldsymbol{\sigma} \mathbf{A}^T,$$

where **A** is a rotation matrix with components a_{ij}. In matrix form this is

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} & \sigma'_{13} \\ \sigma'_{21} & \sigma'_{22} & \sigma'_{23} \\ \sigma'_{31} & \sigma'_{32} & \sigma'_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

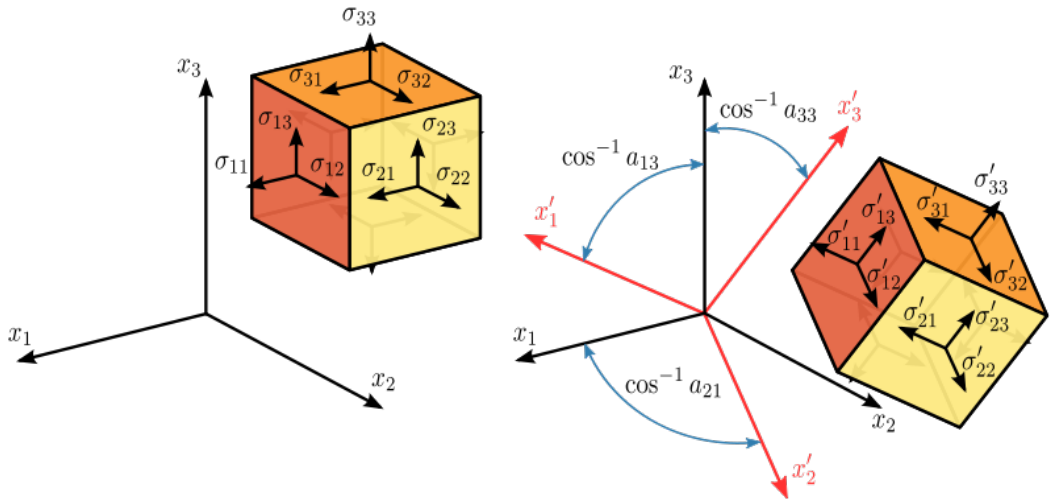


Figure 2.4 Transformation of the stress tensor

Expanding the matrix operation, and simplifying some terms by taking advantage of the symmetry of the stress tensor, gives

$$\begin{aligned}
\sigma'_{11} &= a_{11}^2\sigma_{11} + a_{12}^2\sigma_{22} + a_{13}^2\sigma_{33} + 2a_{11}a_{12}\sigma_{12} + 2a_{11}a_{13}\sigma_{13} + 2a_{12}a_{13}\sigma_{23}, \\
\sigma'_{22} &= a_{21}^2\sigma_{11} + a_{22}^2\sigma_{22} + a_{23}^2\sigma_{33} + 2a_{21}a_{22}\sigma_{12} + 2a_{21}a_{23}\sigma_{13} + 2a_{22}a_{23}\sigma_{23}, \\
\sigma'_{33} &= a_{31}^2\sigma_{11} + a_{32}^2\sigma_{22} + a_{33}^2\sigma_{33} + 2a_{31}a_{32}\sigma_{12} + 2a_{31}a_{33}\sigma_{13} + 2a_{32}a_{33}\sigma_{23}, \\
\sigma'_{12} &= a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + a_{13}a_{23}\sigma_{33} \\
&\quad + (a_{11}a_{22} + a_{12}a_{21})\sigma_{12} + (a_{12}a_{23} + a_{13}a_{22})\sigma_{23} + (a_{11}a_{23} + a_{13}a_{21})\sigma_{13}, \\
\sigma'_{23} &= a_{21}a_{31}\sigma_{11} + a_{22}a_{32}\sigma_{22} + a_{23}a_{33}\sigma_{33} \\
&\quad + (a_{21}a_{32} + a_{22}a_{31})\sigma_{12} + (a_{22}a_{33} + a_{23}a_{32})\sigma_{23} + (a_{21}a_{33} + a_{23}a_{31})\sigma_{13}, \\
\sigma'_{13} &= a_{11}a_{31}\sigma_{11} + a_{12}a_{32}\sigma_{22} + a_{13}a_{33}\sigma_{33} \\
&\quad + (a_{11}a_{32} + a_{12}a_{31})\sigma_{12} + (a_{12}a_{33} + a_{13}a_{32})\sigma_{23} + (a_{11}a_{33} + a_{13}a_{31})\sigma_{13}.
\end{aligned}$$

The Mohr circle for stress is a graphical representation of this transformation of stresses.

Normal and shear stresses

The magnitude of the normal stress component σ_n of any stress vector $\mathbf{T}^{(\mathbf{n})}$ acting on an arbitrary plane with normal vector $\mathbf{n}$ at a given point, in terms of the components σ_{ij} of the stress tensor $\boldsymbol{\sigma}$, is the dot product of the stress vector and the normal vector:

$$\begin{aligned}\sigma_{\mathrm{n}} &= \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n} \\ &= T_i^{(\mathbf{n})} n_i \\ &= \sigma_{ij} n_i n_j.\end{aligned}$$

The magnitude of the shear stress component τ_{n}, acting in the plane spanned by the two vectors $\mathbf{T}^{(\mathbf{n})}$ and $\mathbf{n}$, can then be found using the Pythagorean theorem:

$$\begin{aligned}\tau_{\mathrm{n}} &= \sqrt{\left(T^{(\mathbf{n})}\right)^2 - \sigma_{\mathrm{n}}^2} \\ &= \sqrt{T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} - \sigma_{\mathrm{n}}^2},\end{aligned}$$

where

$$\left(T^{(\mathbf{n})}\right)^2 = T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} = \left(\sigma_{ij} n_j\right)\left(\sigma_{ik} n_k\right) = \sigma_{ij}\sigma_{ik} n_j n_k.$$

Equilibrium equations and symmetry of the stress tensor

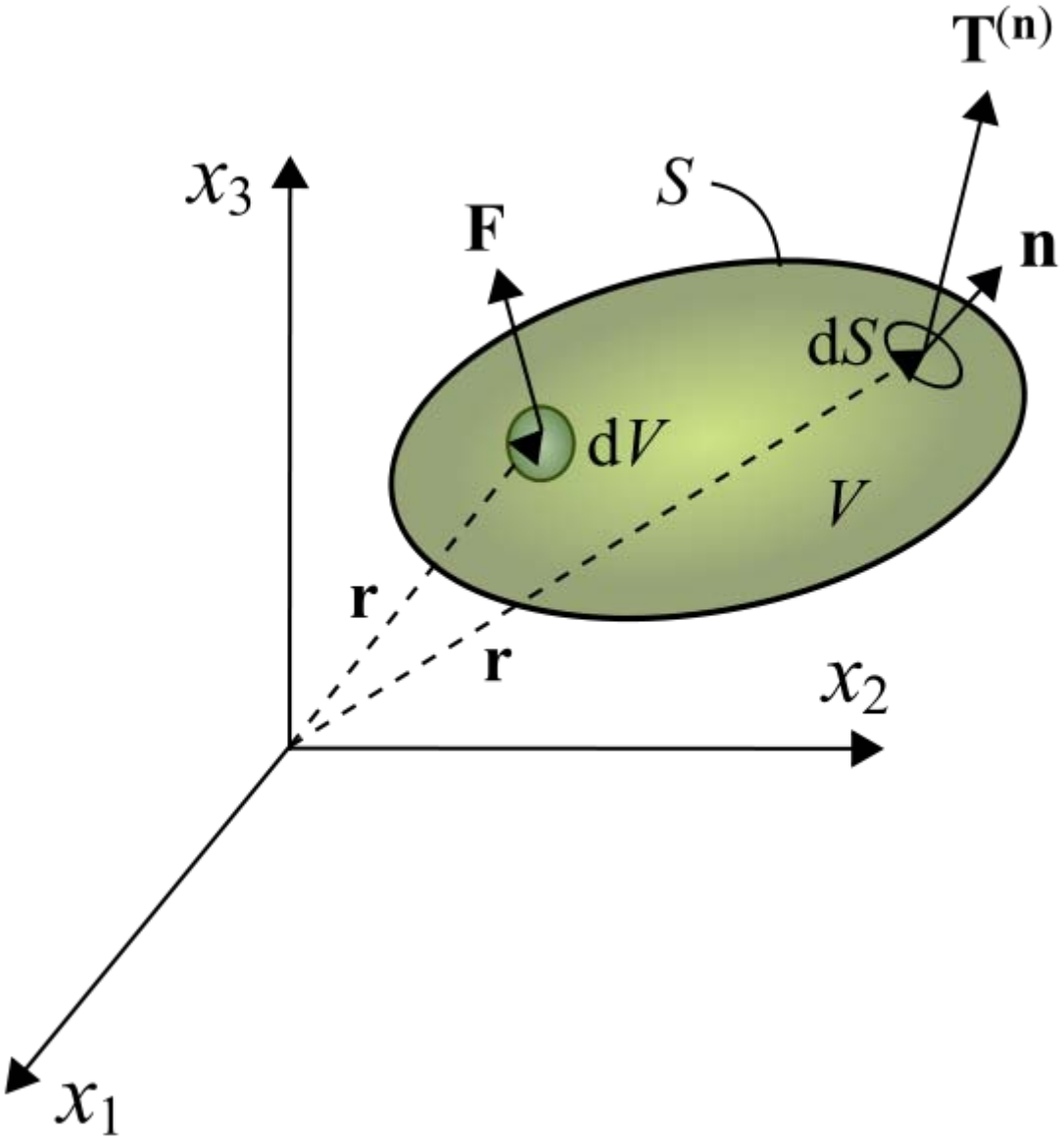


Figure 4. Continuum body in equilibrium

When a body is in equilibrium the components of the stress tensor in every point of the body satisfy the equilibrium equations,

$$\sigma_{ji,j} + F_i = 0$$

For example, for a hydrostatic fluid in equilibrium conditions, the stress tensor takes on the form:

$$\sigma_{ij} = -p\delta_{ij},$$

where *p* is the hydrostatic pressure, and δ_{ij} is the kronecker delta.

At the same time, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}$$

However, in the presence of couple-stresses, i.e. moments per unit volume, the stress tensor is non-symmetric. This also is the case when the Knudsen number is close to one, $K_n \rightarrow 1$, or the continuum is a non-Newtonian fluid, which can lead to rotationally non-invariant fluids, such as polymers.

Principal stresses and stress invariants

At every point in a stressed body there are at least three planes, called *principal planes*, with normal vectors $\mathbf{n}$, called *principal directions*, where the corresponding stress vector is perpendicular to the plane, i.e., parallel or in the same direction as the normal vector $\mathbf{n}$, and where there are no normal shear stresses $\tau_{\mathbf{n}}$. The three stresses normal to these principal planes are called *principal stresses*.

The components σ_{ij} of the stress tensor depend on the orientation of the coordinate system at the point under consideration. However, the stress tensor itself is a physical quantity and as such, it is independent of the coordinate system chosen to represent it. There are certain invariants associated with every tensor which are also independent of the coordinate system. For example, a vector is a simple tensor of rank one. In three dimensions, it has three components. The value of these components will depend on the coordinate system chosen to represent the vector, but the length of the vector is a physical quantity (a scalar) and is independent of the coordinate system chosen to represent the vector. Similarly, every second rank tensor (such as the stress and the strain tensors) has three independent invariant quantities associated with it. One set of such invariants are the principal stresses of the stress tensor, which are just the eigenvalues of the stress tensor. Their direction vectors are the principal directions or eigenvectors.

A stress vector parallel to the normal vector $\mathbf{n}$ is given by:

$$\mathbf{T}^{(\mathbf{n})} = \lambda\mathbf{n} = \sigma_{\mathbf{n}}\mathbf{n}$$

where λ is a constant of proportionality, and in this particular case corresponds to the magnitudes σ_{n} of the normal stress vectors or principal stresses.

Knowing that $T_i^{(n)} = \sigma_{ij}n_j$ and $n_i = \delta_{ij}n_j$, we have

$$
\begin{aligned}
T_i^{(n)} &= \lambda n_i \\
\sigma_{ij} n_j &= \lambda n_i \\
\sigma_{ij} n_j - \lambda n_i &= 0 \\
\left(\sigma_{ij} - \lambda \delta_{ij}\right) n_j &= 0
\end{aligned}
$$

This is a homogeneous system, i.e. equal to zero, of three linear equations where n_j are the unknowns. To obtain a nontrivial (non-zero) solution for n_j, the determinant matrix of the coefficients must be equal to zero, i.e. the system is singular. Thus,

$$
\left|\sigma_{ij} - \lambda \delta_{ij}\right| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0
$$

Expanding the determinant leads to the *characteristic equation*

$$
\left|\sigma_{ij} - \lambda \delta_{ij}\right| = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0
$$

where

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_{kk} \\
I_2 &= \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) \\
I_3 &= \det(\sigma_{ij}) \\
&= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22}
\end{aligned}
$$

The characteristic equation has three real roots λ, i.e. not imaginary due to the symmetry of the stress tensor. The three roots $\lambda_1 = \sigma_1$, $\lambda_2 = \sigma_2$, and $\lambda_3 = \sigma_3$ are the eigenvalues or principal stresses, and they are the roots of the Cayley–Hamilton theorem. The principal stresses are unique for a given stress tensor. Therefore, from the characteristic equation it is seen that the coefficients I_1, I_2 and I_3, called the first, second, and third *stress invariants*, respectively, have always the same value regardless of the orientation of the coordinate system chosen.

For each eigenvalue, there is a non-trivial solution for n_j in the equation $(\sigma_{ij} - \lambda\delta_{ij})\, n_j = 0$. These solutions are the principal directions or eigenvectors defining the plane where the principal stresses act. The principal stresses and principal directions characterize the stress at a point and are independent of the orientation of the coordinate system.

If we choose a coordinate system with axes oriented to the principal directions, then the normal stresses will be the principal stresses and the stress tensor is represented by a diagonal matrix:

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The principal stresses may be combined to form the stress invariants, I_1, I_2, and I_3.The first and third invariant are the trace and determinant respectively, of the stress tensor. Thus,

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Because of its simplicity, working and thinking in the principal coordinate system is often very useful when considering the state of the elastic medium at a particular point.

Principal stresses are often expressed in the following equation for evaluating stresses in the x and y directions or axial and bending stresses on a part. The principal normal stresses can then be used to calculate the Von Mises stress and ultimately the safety factor and margin of safety.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Using just the part of the equation under the square root is equal to the maximum and minimum shear stress for plus and minus. This is shown as:

$$\tau_{max}, \tau_{min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum and minimum shear stresses

The maximum shear stress or maximum principal shear stress is equal to one-half the difference between the largest and smallest principal stresses, and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses, i.e. the plane of the maximum shear stress is oriented 45° from the principal stress planes. The maximum shear stress is expressed as

$$\tau_{\mathbf{max}} = \frac{1}{2}\left|\sigma_{\mathbf{max}} - \sigma_{\mathbf{min}}\right|$$

Assuming $\sigma_1 \geq \sigma_2 \geq \sigma_3$ then

$$\tau_{\mathbf{max}} = \frac{1}{2}\left|\sigma_1 - \sigma_3\right|$$

The normal stress component acting on the plane for the maximum shear stress is non-zero and it is equal to

$$\sigma_{\mathbf{n}} = \frac{1}{2}\left(\sigma_1 + \sigma_3\right)$$

Stress deviator tensor

The stress tensor σ_{ij} can be expressed as the sum of two other stress tensors:

1. a *mean hydrostatic stress tensor* or *volumetric stress tensor* or *mean normal stress tensor*, $p\delta_{ij}$, which tends to change the volume of the stressed body; and
2. a deviatoric component called the *stress deviator tensor*, s_{ij}, which tends to distort it.

So:

$$\sigma_{ij} = s_{ij} + p\delta_{ij},$$

where p is the mean stress given by

$$p = \frac{\sigma_{kk}}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \tfrac{1}{3}I_1.$$

Note that convention in solid mechanics differs slightly from what is listed above. In solid mechanics, pressure is generally defined as negative one-third the trace of the stress tensor.

The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the stress tensor:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}.$$

Invariants of the stress deviator tensor

As it is a second order tensor, the stress deviator tensor also has a set of invariants, which can be obtained using the same procedure used to calculate the invariants of the stress tensor. It can be shown that the principal directions of the stress deviator tensor s_{ij} are the same as the principal directions of the stress tensor σ_{ij}. Thus, the characteristic equation is

$$|s_{ij} - \lambda\delta_{ij}| = \lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0,$$

where J_1, J_2 and J_3 are the first, second, and third *deviatoric stress invariants*, respectively. Their values are the same (invariant) regardless of the orientation of the coordinate system chosen. These deviatoric stress invariants can be expressed as a function of the components of s_{ij} or its principal values s_1, s_2, and s_3, or alternatively, as a function of σ_{ij} or its principal values σ_1, σ_2, and σ_3. Thus,

$$\begin{aligned}
J_1 &= s_{kk} = 0, \\
J_2 &= \tfrac{1}{2}s_{ij}s_{ji} \\
&= -s_1s_2 - s_2s_3 - s_3s_1 \\
&= \tfrac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \\
&= \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \\
&= \tfrac{1}{3}I_1^2 - I_2, \\
J_3 &= \det(s_{ij}) \\
&= \tfrac{1}{3}s_{ij}s_{jk}s_{ki} \\
&= s_1s_2s_3 \\
&= \tfrac{2}{27}I_1^3 - \tfrac{1}{3}I_1I_2 + I_3.
\end{aligned}$$

Because $s_{kk} = 0$, the stress deviator tensor is in a state of pure shear.

A quantity called the equivalent stress or von Mises stress is commonly used in solid mechanics. The equivalent stress is defined as

$$\sigma_{\mathrm{e}} = \sqrt{3\ J_2} = \sqrt{\tfrac{1}{2}\ [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]}\,.$$

Octahedral stresses

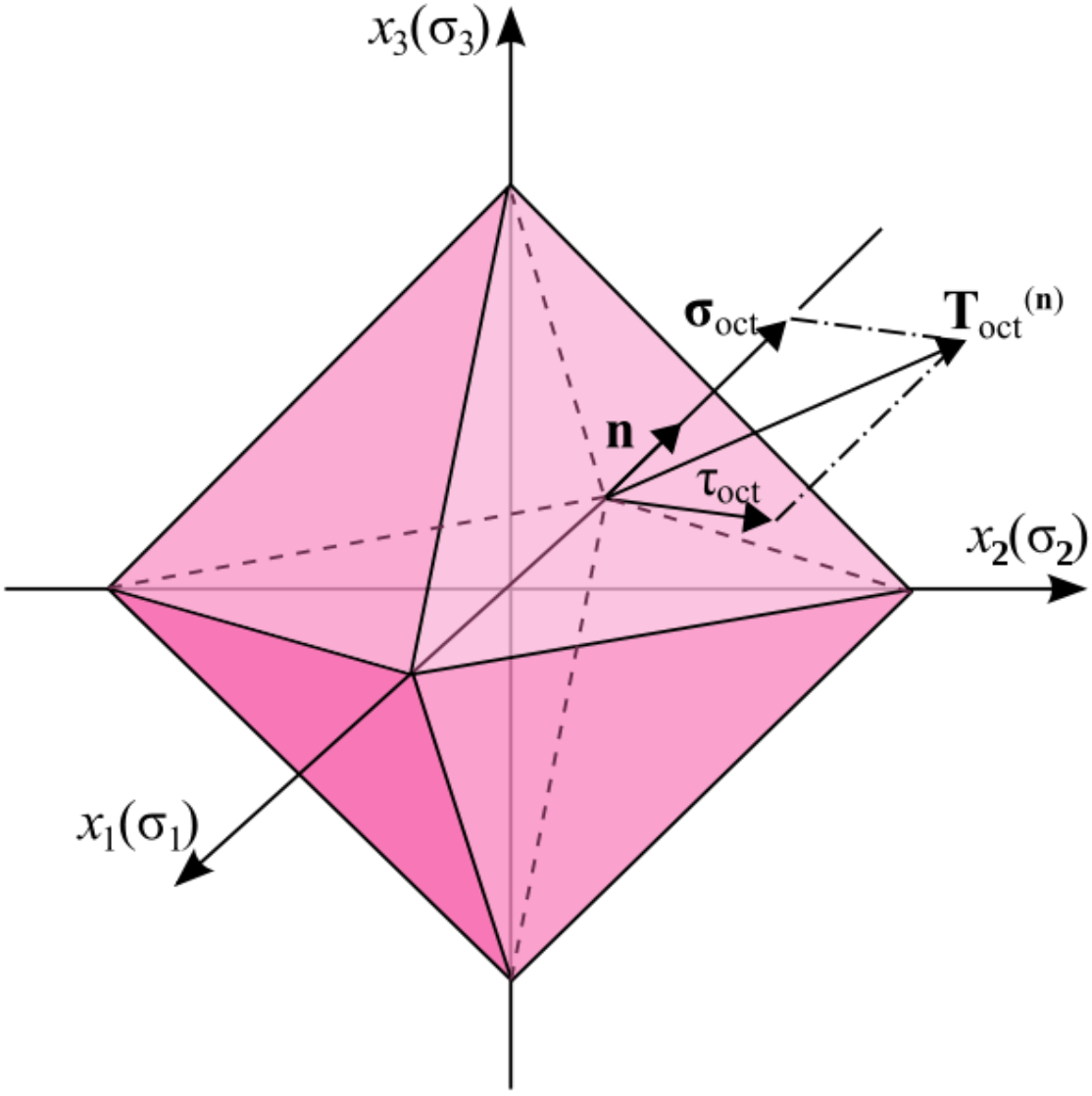


Figure 6. Octahedral stress planes

Considering the principal directions as the coordinate axes, a plane whose normal vector makes equal angles with each of the principal axes (i.e. having direction cosines equal to

$|1/\sqrt{3}|$) is called an *octahedral plane*. There are a total of eight octahedral planes (Figure 6). The normal and shear components of the stress tensor on these planes are called *octahedral normal stress* $\sigma_{\mathbf{oct}}$and *octahedral shear stress* $\tau_{\mathbf{oct}}$, respectively.

Knowing that the stress tensor of point O (Figure 6) in the principal axes is

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

the stress vector on an octahedral plane is then given by:

$$\begin{aligned} \mathbf{T}_{\mathbf{oct}}^{(\mathbf{n})} &= \sigma_{ij} n_i \mathbf{e}_j \\ &= \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \\ &= \tfrac{1}{\sqrt{3}}(\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 + \sigma_3 \mathbf{e}_3) \end{aligned}$$

The normal component of the stress vector at point O associated with the octahedral plane is

$$\begin{aligned} \sigma_{\mathbf{oct}} &= T_i^{(n)} n_i \\ &= \sigma_{ij} n_i n_j \\ &= \sigma_1 n_1 n_1 + \sigma_2 n_2 n_2 + \sigma_3 n_3 n_3 \\ &= \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3} I_1 \end{aligned}$$

which is the mean normal stress or hydrostatic stress. This value is the same in all eight octahedral planes. The shear stress on the octahedral plane is then

$$\begin{aligned} \tau_{\mathbf{oct}} &= \sqrt{T_i^{(n)} T_i^{(n)} - \sigma_{\mathbf{n}}^2} \\ &= \left[\tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2\right]^{1/2} \\ &= \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \tfrac{1}{3}\sqrt{2I_1^2 - 6I_2} = \sqrt{\tfrac{2}{3} J_2} \end{aligned}$$

Alternative measures of stress

The Cauchy stress tensor is not the only measure of stress that is used in practice. Other measures of stress include the first and second Piola–Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor.

Piola–Kirchhoff stress tensor

In the case of finite deformations, the *Piola–Kirchhoff stress tensors* are used to express the stress relative to the reference configuration. This is in contrast to the Cauchy stress tensor which expresses the stress relative to the present configuration. For infinitesimal deformations or rotations, the Cauchy and Piola–Kirchhoff tensors are identical. These tensors take their names from Gabrio Piola and Gustav Kirchhoff.

Whereas the Cauchy stress tensor, $\boldsymbol{\sigma}$relates stresses in the current configuration, the deformation gradient and strain tensors are described by relating the motion to the reference configuration; thus not all tensors describing the state of the material are in either the reference or current configuration. Having the stress, strain and deformation all described either in the reference or current configuration would make it easier to define constitutive models (for example, the Cauchy Stress tensor is variant to a pure rotation, while the deformation strain tensor is invariant; thus creating problems in defining a constitutive model that relates a varying tensor, in terms of an invariant one during pure rotation; as by definition constitutive models have to be invariant to pure rotations). The 1st Piola–Kirchhoff stress tensor, $\boldsymbol{P}$is one possible solution to this problem. It defines a family of tensors, which describe the configuration of the body in either the current or the reference state.

The 1st Piola–Kirchhoff stress tensor, $\boldsymbol{P}$relates forces in the *present* configuration with areas in the *reference* ("material") configuration.

$$\boldsymbol{P} = J\ \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$$

where $\boldsymbol{F}$is the deformation gradient and $J = \det \boldsymbol{F}$is the Jacobian determinant.

In terms of components with respect to an orthonormal basis, the first Piola–Kirchhoff stress is given by

$$P_{iL} = J\ \sigma_{ik}\ F^{-1}_{Lk} = J\ \sigma_{ik}\ \frac{\partial X_L}{\partial x_k}$$

Because it relates different coordinate systems, the 1st Piola–Kirchhoff stress is a two-point tensor. In general, it is not symmetric. The 1st Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress state (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the deformation gradient.

2nd Piola–Kirchhoff stress tensor

Whereas the 1st Piola–Kirchhoff stress relates forces in the current configuration to areas in the reference configuration, the 2nd Piola–Kirchhoff stress tensor $\boldsymbol{S}$ relates forces in the reference configuration to areas in the reference configuration. The force in the reference configuration is obtained via a mapping that preserves the relative relationship between the force direction and the area normal in the current configuration.

$$\boldsymbol{S} = J\ \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\ .$$

In index notation with respect to an orthonormal basis,

$$S_{IL} = J\ F^{-1}_{Ik}\ F^{-1}_{Lm}\ \sigma_{km} = J\ \frac{\partial X_I}{\partial x_k}\ \frac{\partial X_L}{\partial x_m}\ \sigma_{km}$$

This tensor is symmetric.

If the material rotates without a change in stress state (rigid rotation), the components of the 2nd Piola–Kirchhoff stress tensor will remain constant, irrespective of material orientation.

Chapter 4

Deformation (Mechanics)

Deformation in continuum mechanics is the transformation of a body from a *reference* configuration to a *current* configuration. A configuration is a set containing the positions of all particles of the body. Contrary to the common definition of deformation, which implies distortion or change in shape, the continuum mechanics definition includes rigid body motions where shape changes do not take place (, footnote 4, p. 48).

The cause of a deformation is not pertinent to the definition of the term. However, it is usually assumed that a deformation is caused by external loads, body forces (such as gravity or electromagnetic forces), or temperature changes within the body.

Strain is a description of deformation in terms of *relative* displacement of particles in the body.

Different equivalent choices may be made for the expression of a strain field depending on whether it is defined in the initial or in the final placement and on whether the metric tensor or its dual is considered.

In a continuous body, a deformation field results from a stress field induced by applied forces or is due to changes in the temperature field inside the body. The relation between stresses and induced strains is expressed by constitutive equations, e.g., Hooke's law for linear elastic materials. Deformations which are recovered after the stress field has been removed are called *elastic deformations*. In this case, the continuum completely recovers its original configuration. On the other hand, irreversible deformations remain even after stresses have been removed. One type of irreversible deformation is *plastic deformation*, which occurs in material bodies after stresses have attained a certain threshold value known as the *elastic limit* or yield stress, and are the result of slip, or dislocation mechanisms at the atomic level. Another type of irreversible deformation is viscous deformation, which is the irreversible part of viscoelastic deformation.

In the case of elastic deformations, the response function linking strain to the deforming stress is the compliance tensor of the material.

Strain

A strain is a normalized measure of deformation representing the displacement between particles in the body relative to a reference length.

A general deformation of a body can be expressed in the form $\mathbf{x} = \boldsymbol{F}(\mathbf{X})$where $\mathbf{X}$is the reference position of material points in the body. Such a measure does not distinguish between rigid body motions (translations and rotations) and changes in shape (and size) of the body. A deformation has units of length.

We could, for example, define strain to be

$$\boldsymbol{\varepsilon} \doteq \frac{\partial}{\partial \mathbf{X}}(\mathbf{x} - \mathbf{X}) = \frac{\partial \boldsymbol{F}}{\partial \mathbf{X}} - \mathbf{1}$$

Hence strains are dimensionless and are usually expressed as a decimal fraction, a percentage or in parts-per notation. Strains measure how much a given deformation differs locally from a rigid-body deformation.

A strain is in general a tensor quantity. Physical insight into strains can be gained by observing that a given strain can be decomposed into normal and shear components. The amount of stretch or compression along a material line elements or fibers is the *normal strain*, and the amount of distortion associated with the sliding of plane layers over each other is the *shear strain*, within a deforming body. This could be applied by elongation, shortening, or volume changes, or angular distortion.

The state of strain at a material point of a continuum body is defined as the totality of all the changes in length of material lines or fibers, the *normal strain*, which pass through that point and also the totality of all the changes in the angle between pairs of lines initially perpendicular to each other, the *shear strain*, radiating from this point. However, it is sufficient to know the normal and shear components of strain on a set of three mutually perpendicular directions.

If there is an increase in length of the material line, the normal strain is called *tensile strain*, otherwise, if there is reduction or compression in the length of the material line, it is called *compressive strain*.

Strain measures

Depending on the amount of strain, or local deformation, the analysis of deformation is subdivided into three deformation theories:

- Finite strain theory, also called *large strain theory*, *large deformation theory*, deals with deformations in which both rotations and strains are arbitrarily large. In this case, the undeformed and deformed configurations of the continuum are

significantly different and a clear distinction has to be made between them. This is commonly the case with elastomers, plastically-deforming materials and other fluids and biological soft tissue.
- Infinitesimal strain theory, also called *small strain theory*, *small deformation theory*, *small displacement theory*, or *small displacement-gradient theory* where strains and rotations are both small. In this case, the undeformed and deformed configurations of the body can be assumed identical. The infinitesimal strain theory is used in the analysis of deformations of materials exhibiting elastic behavior, such as materials found in mechanical and civil engineering applications, e.g. concrete and steel.
- *Large-displacement* or *large-rotation theory*, which assumes small strains but large rotations and displacements.

In each of these theories the strain is then defined differently. The *engineering strain* is the most common definition applied to materials used in mechanical and structural engineering, which are subjected to very small deformations. On the other hand, for some materials, e.g. elastomers and polymers, subjected to large deformations, the engineering definition of strain is not applicable, e.g. typical engineering strains greater than 1%, thus other more complex definitions of strain are required, such as *stretch*, *logarithmic strain*, *Green strain*, and *Almansi strain*.

Engineering strain

The **Cauchy strain** or **engineering strain** is expressed as the ratio of total deformation to the initial dimension of the material body in which the forces are being applied. The *engineering normal strain* or *engineering extensional strain* or *nominal strain e* of a material line element or fiber axially loaded is expressed as the change in length ΔL per unit of the original length L of the line element or fibers. The normal strain is positive if the material fibers are stretched or negative if they are compressed. Thus, we have

$$e = \frac{\Delta L}{L} = \frac{\ell - L}{L}$$

where e is the *engineering normal strain, L* is the original length of the fiber and ℓ is the final length of the fiber.

The *engineering shear strain* is defined as the change in the angle between two material line elements initially perpendicular to each other in the undeformed or initial configuration.

Stretch ratio

The **stretch ratio** or **extension ratio** is a measure of the extensional or normal strain of a differential line element, which can be defined at either the undeformed configuration or the deformed configuration. It is defined as the ratio between the final length ℓ and the initial length L of the material line.

$$\lambda = \frac{\ell}{L}$$

The extension ratio is related to the engineering strain by

$$e = \frac{\ell - L}{L} = \lambda - 1$$

This equation implies that the normal strain is zero, so that there is no deformation when the stretch is equal to unity.

The stretch ratio is used in the analysis of materials that exhibit large deformations, such as elastomers, which can sustain stretch ratios of 3 or 4 before they fail. On the other hand, traditional engineering materials, such as concrete or steel, fail at much lower stretch ratios.

True strain

The **logarithmic strain** ε, also called *natural strain*, *true strain* or *Hencky strain*. Considering an incremental strain (Ludwik)

$$\delta\varepsilon = \frac{\delta\ell}{\ell}$$

the logarithmic strain is obtained by integrating this incremental strain:

$$\begin{aligned}\int \delta\varepsilon &= \int_L^\ell \frac{\delta\ell}{\ell} \\ \varepsilon &= \ln\left(\frac{\ell}{L}\right) = \ln\lambda \\ &= \ln(1+e) \\ &= e - e^2/2 + e^3/3 - \cdots\end{aligned}$$

where *e* is the engineering strain. The logarithmic strain provides the correct measure of the final strain when deformation takes place in a series of increments, taking into account the influence of the strain path.

Green strain

The Green strain is defined as:

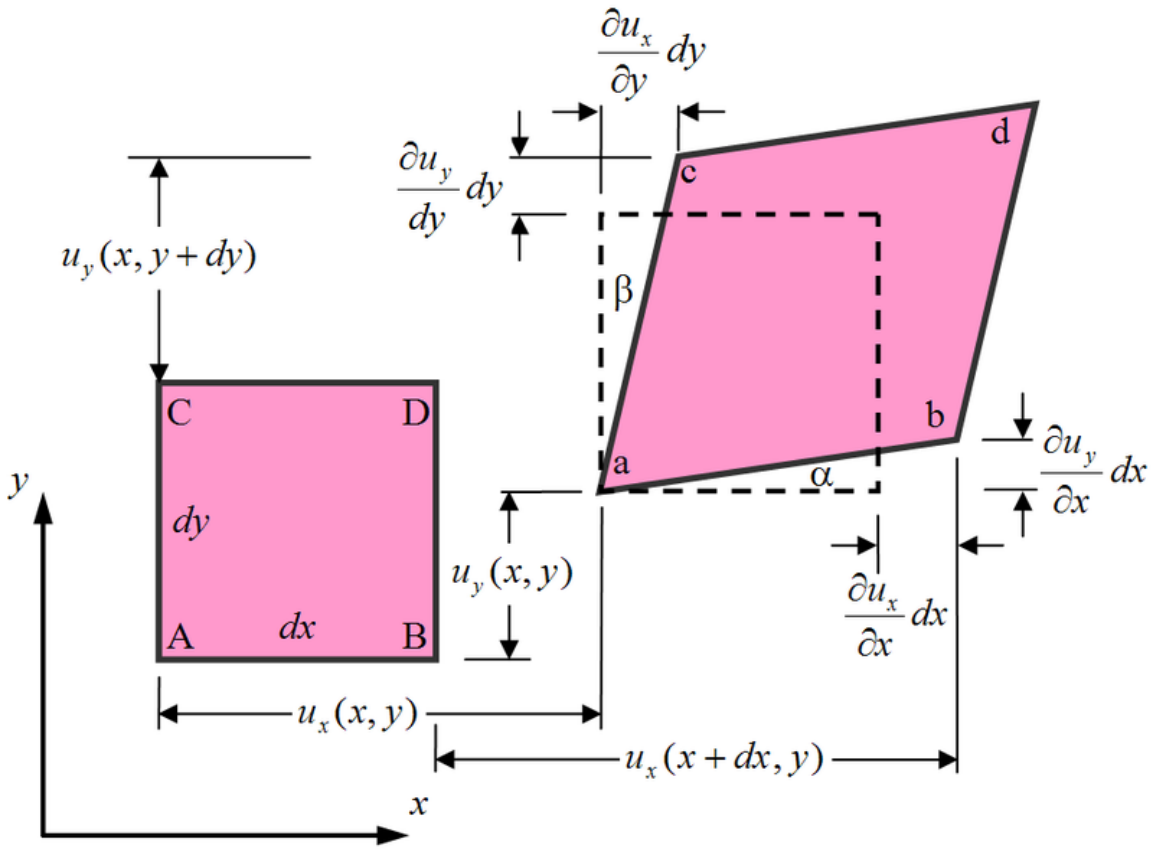
$$\varepsilon_G = \frac{1}{2}\left(\frac{\ell^2 - L^2}{L^2}\right) = \frac{1}{2}(\lambda^2 - 1)$$

Almansi strain

The Euler-Almansi strain is defined as

$$\varepsilon_E = \frac{1}{2}\left(\frac{\ell^2 - L^2}{\ell^2}\right) = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right)$$

Normal strain



Two-dimensional geometric deformation of an infinitesimal material element.

As with stresses, strains may also be classified as 'normal strain' and 'shear strain' (i.e. acting perpendicular to or along the face of an element respectively). For an isotropic material that obeys Hooke's law, a normal stress will cause a normal strain. **Normal strains** produce *dilations*.

Consider a two-dimensional infinitesimal rectangular material element with dimensions $dx \times dy$, which after deformation, takes the form of a rhombus. From the geometry of the adjacent figure we have

$$\text{length}(AB) = dx$$

and

$$\begin{aligned}\text{length}(ab) &= \sqrt{\left(dx + \frac{\partial u_x}{\partial x}dx\right)^2 + \left(\frac{\partial u_y}{\partial x}dx\right)^2} \\ &= dx\ \sqrt{1 + 2\frac{\partial u_x}{\partial x} + \left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2}\end{aligned}$$

For very small displacement gradients the squares of the derivatives are negligible and we have

$$\text{length}(ab) \approx dx + \frac{\partial u_x}{\partial x}dx$$

The normal strain in the x-direction of the rectangular element is defined by

$$\varepsilon_x = \frac{\text{extension}}{\text{original length}} = \frac{\text{length}(ab) - \text{length}(AB)}{\text{length}(AB)} = \frac{\partial u_x}{\partial x}$$

Similarly, the normal strain in the y-direction, and z-direction, becomes

$$\varepsilon_y = \frac{\partial u_y}{\partial y} \quad , \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}$$

Shear strain

Shear strain

SI symbol:	γ or ϵ
SI unit:	1, or radian
Derivations from other quantities:	γ = τ / G

The engineering shear strain is defined as (γ_{xy}) is the change in angle between lines $\overline{AC}$ and $\overline{AB}$. Therefore,

$$\gamma_{xy} = \alpha + \beta$$

From the geometry of the figure, we have

$$\tan\alpha = \frac{\frac{\partial u_y}{\partial x}dx}{dx + \frac{\partial u_x}{\partial x}dx} = \frac{\frac{\partial u_y}{\partial x}}{1 + \frac{\partial u_x}{\partial x}}$$

$$\tan\beta = \frac{\frac{\partial u_x}{\partial y}dy}{dy + \frac{\partial u_y}{\partial y}dy} = \frac{\frac{\partial u_x}{\partial y}}{1 + \frac{\partial u_y}{\partial y}}$$

For small displacement gradients we have

$$\frac{\partial u_x}{\partial x} \ll 1 \; ; \quad \frac{\partial u_y}{\partial y} \ll 1$$

For small rotations, i.e. $\alpha$and $\beta$are $\ll 1$we have $\tan\alpha \approx \alpha, \;\; \tan\beta \approx \beta$. Therefore,

$$\alpha \approx \frac{\partial u_y}{\partial x} \; ; \quad \beta \approx \frac{\partial u_x}{\partial y}$$

thus

$$\gamma_{xy} = \alpha + \beta = \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}$$

By interchanging $x$and $y$and $u_x$and u_y, it can be shown that $\gamma_{xy} = \gamma_{yx}$

Similarly, for the y-z and x-z planes, we have

$$\gamma_{yz} = \gamma_{zy} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \quad , \qquad \gamma_{zx} = \gamma_{xz} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$$

The tensorial shear strain components of the infinitesimal strain tensor can then be expressed using the engineering strain definition, γ, as

$$\underline{\underline{\boldsymbol{\varepsilon}}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \gamma_{xy}/2 & \gamma_{xz}/2 \\ \gamma_{yx}/2 & \varepsilon_{yy} & \gamma_{yz}/2 \\ \gamma_{zx}/2 & \gamma_{zy}/2 & \varepsilon_{zz} \end{bmatrix}$$

Metric tensor

A strain field associated with a displacement is defined, at any point, by the change in length of the tangent vectors representing the speeds of arbitrarily parametrized curves passing through that point.

A basic geometric result, due to Fréchet, von Neumann and Jordan, states that, if the lengths of the tangent vectors fulfill the axioms of a norm and the parallelogram law, then the length of a vector is the square root of the value of the quadratic form associated, by the polarization formula, with a positive definite bilinear map called the metric tensor.

Description of deformation

Deformation is the change in the metric properties of a continuous body, meaning that a curve drawn in the initial body placement changes its length when displaced to a curve in the final placement. If all the curves do not change length, it is said that a rigid body displacement occurred.

It is convenient to identify a reference configuration or initial geometric state of the continuum body which all subsequent configurations are referenced from. The reference configuration need not to be one the body actually will ever occupy. Often, the configuration at $t = 0$ is considered the reference configuration, $\kappa_0(\mathbf{B})$. The configuration at the current time t is the *current configuration*.

For deformation analysis, the reference configuration is identified as *undeformed configuration*, and the current configuration as *deformed configuration*. Additionally, time is not considered when analyzing deformation, thus the sequence of configurations between the undeformed and deformed configurations are of no interest.

The components X_i of the position vector $\mathbf{X}$ of a particle in the reference configuration, taken with respect to the reference coordinate system, are called the *material or reference coordinates*. On the other hand, the components x_i of the position vector $\mathbf{x}$ of a particle in the deformed configuration, taken with respect to the spatial coordinate system of reference, are called the *spatial coordinates*

There are two methods for analysing the deformation of a continuum. One description is made in terms of the material or referential coordinates, called material description or Lagrangian description. A second description is of deformation is made in terms of the spatial coordinates it is called the spatial description or Eulerian description.

There is continuity during deformation of a continuum body in the sense that:

- The material points forming a closed curve at any instant will always form a closed curve at any subsequent time.

- The material points forming a closed surface at any instant will always form a closed surface at any subsequent time and the matter within the closed surface will always remain within.

Affine deformation

A deformation is called an affine deformation, if it can be described by an affine transformation. Such a transformation is composed of a linear transformation (such as rotation, shear, extension and compression) and a rigid body translation. Affine deformations are also called homogeneous deformations.

Therefore an affine deformation has the form

$$\mathbf{x}(\mathbf{X},t) = \boldsymbol{F}(t)\cdot\mathbf{X} + \mathbf{c}(t)$$

where $\mathbf{x}$is the position of a point in the deformed configuration, $\mathbf{X}$is the position in a reference configuration, *t* is a time-like parameter, $\boldsymbol{F}$is the linear transformer and $\mathbf{c}$is the translation. In matrix form, where the components are with respect to an orthonormal basis,

$$\begin{bmatrix} x_1(X_1,X_2,X_3,t) \\ x_2(X_1,X_2,X_3,t) \\ x_3(X_1,X_2,X_3,t) \end{bmatrix} = \begin{bmatrix} F_{11}(t) & F_{12}(t) & F_{13}(t) \\ F_{21}(t) & F_{22}(t) & F_{23}(t) \\ F_{31}(t) & F_{32}(t) & F_{33}(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}$$

The above deformation becomes *non-affine* or *inhomogeneous* if $\boldsymbol{F} = \boldsymbol{F}(\mathbf{X},t)$or $\mathbf{c} = \mathbf{c}(\mathbf{X},t)$.

Rigid body motion

A rigid body motion is a special affine deformation that does not involve any shear, extension or compression. The transformation matrix $\boldsymbol{F}$is proper orthogonal in order to allow rotations but no reflections.

A rigid body motion can be described by

$$\mathbf{x}(\mathbf{X},t) = \boldsymbol{Q}(t)\cdot\mathbf{X} + \mathbf{c}(t)$$

where

$$\boldsymbol{Q}\cdot\boldsymbol{Q}^T = \boldsymbol{Q}^T\cdot\boldsymbol{Q} = \boldsymbol{1}$$

In matrix form,

$$\begin{bmatrix} x_1(X_1, X_2, X_3, t) \\ x_2(X_1, X_2, X_3, t) \\ x_3(X_1, X_2, X_3, t) \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) & Q_{13}(t) \\ Q_{21}(t) & Q_{22}(t) & Q_{23}(t) \\ Q_{31}(t) & Q_{32}(t) & Q_{33}(t) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ c_3(t) \end{bmatrix}$$

Displacement

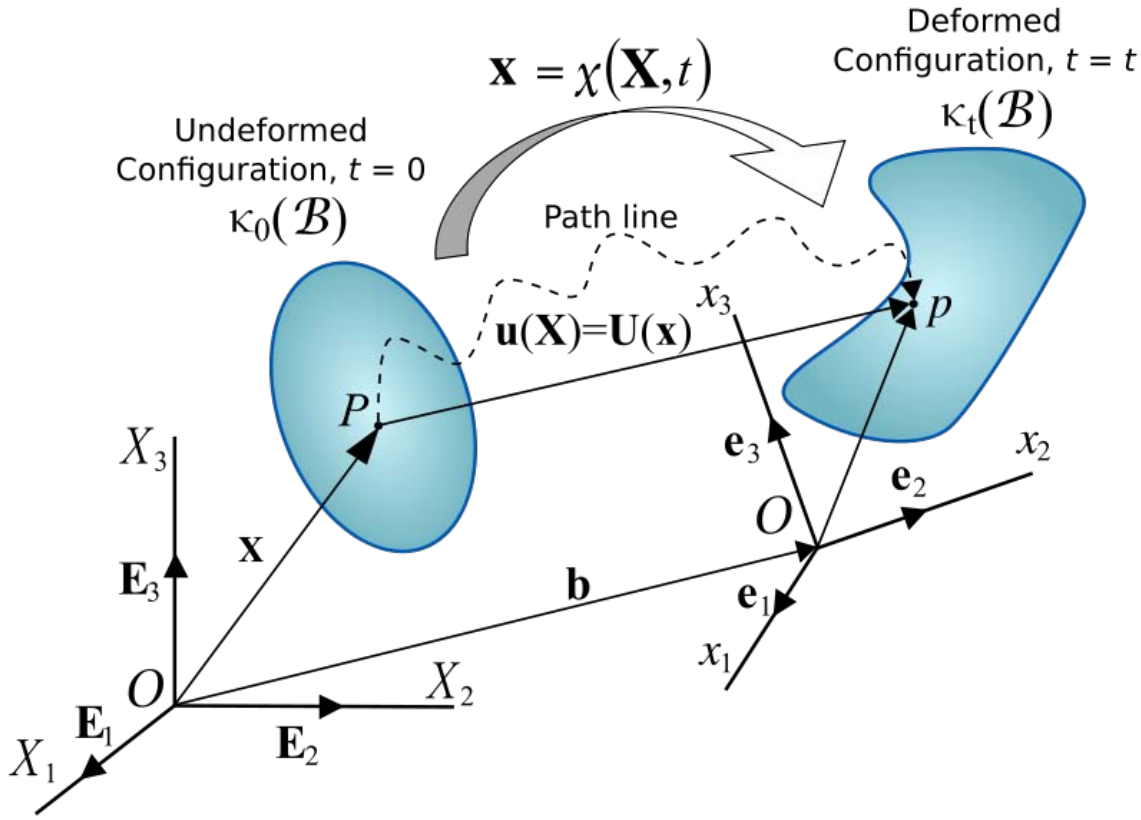


Figure 1. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consist of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 1).

If after a displacement of the continuum there is a relative displacement between particles, a deformation has occurred. On the other hand, if after displacement of the continuum the relative displacement between particles in the current configuration is zero, then there is no deformation and a rigid-body displacement is said to have occurred.

The vector joining the positions of a particle *P* in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i$ in the Lagrangian description, or $\mathbf{U}(\mathbf{x},t) = U_J\mathbf{E}_J$ in the Eulerian description.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field, In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X},t) = \mathbf{b}(\mathbf{X},t) + \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ}b_J + x_i - \alpha_{iJ}X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{b}(\mathbf{x},t) + \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji}x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ}U_J \qquad \text{or} \qquad U_J = \alpha_{Ji}u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ}\mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X},t) = u_i\mathbf{e}_i = u_i(\alpha_{iJ}\mathbf{E}_J) = U_J\mathbf{E}_J = \mathbf{U}(\mathbf{x},t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in $\mathbf{b} = 0$, and the direction cosines become Kronecker deltas:

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}.$$

Thus, we have

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ}X_J = x_i - X_i$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x},t) = \mathbf{x} - \mathbf{X}(\mathbf{x},t) \qquad \text{or} \qquad U_J = \delta_{Ji}x_i - X_J = x_J - X_J$$

Displacement gradient tensor

The partial differentiation of the displacement vector with respect to the material coordinates yields the *material displacement gradient tensor* $\nabla_{\mathbf{X}}\mathbf{u}$. Thus we have:

$$\begin{aligned}\mathbf{u}(\mathbf{X},t) &= \mathbf{x}(\mathbf{X},t) - \mathbf{X} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \nabla_{\mathbf{X}}\mathbf{x} - \mathbf{I} \\ \nabla_{\mathbf{X}}\mathbf{u} &= \mathbf{F} - \mathbf{I}\end{aligned} \qquad \text{or} \qquad \begin{aligned} u_i &= x_i - \delta_{iJ}X_J = x_i - X_i \\ \frac{\partial u_i}{\partial X_K} &= \frac{\partial x_i}{\partial X_K} - \delta_{iK}\end{aligned}$$

where $\mathbf{F}$ is the *deformation gradient tensor*.

Similarly, the partial differentiation of the displacement vector with respect to the spatial coordinates yields the *spatial displacement gradient tensor* $\nabla_{\mathbf{x}}\mathbf{U}$. Thus we have,

$$\begin{aligned}\mathbf{U}(\mathbf{x},t) &= \mathbf{x} - \mathbf{X}(\mathbf{x},t) \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \nabla_{\mathbf{x}}\mathbf{X} \\ \nabla_{\mathbf{x}}\mathbf{U} &= \mathbf{I} - \mathbf{F}^{-1}\end{aligned} \qquad \text{or} \qquad \begin{aligned} U_J &= \delta_{Ji}x_i - X_J = x_J - X_J \\ \frac{\partial U_J}{\partial x_k} &= \delta_{Jk} - \frac{\partial X_J}{\partial x_k}\end{aligned}$$

Examples of deformations

Homogeneous (or affine) deformations are useful in elucidating the behavior of materials. Some homogeneous deformations of interest are

- uniform extension
- pure dilation
- simple shear
- pure shear

Plane deformations are also of interest, particularly in the experimental context.

Plane deformation

A plane deformation, also called *plane strain*, is one where the deformation is restricted to one of the planes in the reference configuration. If the deformation is restricted to the plane described by the basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, the deformation gradient has the form

$$\boldsymbol{F} = F_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + F_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + F_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + F_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3$$

In matrix form,

$$\boldsymbol{F} = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the polar decomposition theorem, the deformation gradient can be decomposed into a stretch and a rotation. Since all the deformation is in a plane, we can write

$$\boldsymbol{F} = \boldsymbol{R} \cdot \boldsymbol{U} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where θ is the angle of rotation and λ_1,λ_2 are the principal stretches.

Isochoric plane deformation

If the deformation is isochoric (volume preserving) then $\det(\boldsymbol{F}) = 1$ and we have

$F_{11}F_{22} - F_{12}F_{21} = 1$

Alternatively,

$\lambda_1\lambda_2 = 1$

Simple shear

A simple shear deformation is defined as an isochoric plane deformation in which there are a set of line elements with a given reference orientation that do not change length and orientation during the deformation.

If $\mathbf{e}_1$ is the fixed reference orientation in which line elements do not deform during the deformation then $\lambda_1 = 1$ and $\boldsymbol{F} \cdot \mathbf{e}_1 = \mathbf{e}_1$. Therefore,

$$F_{11}\mathbf{e}_1 + F_{21}\mathbf{e}_2 = \mathbf{e}_1 \quad \Longrightarrow \quad F_{11} = 1 \; ; \quad F_{21} = 0$$

Since the deformation is isochoric,

$$F_{11}F_{22} - F_{12}F_{21} = 1 \quad \Longrightarrow \quad F_{22} = 1$$

Define $\gamma := F_{12}$. Then, the deformation gradient in simple shear can be expressed as

$$\boldsymbol{F} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now,

$$\boldsymbol{F} \cdot \mathbf{e}_2 = F_{12}\mathbf{e}_1 + F_{22}\mathbf{e}_2 = \gamma \mathbf{e}_1 + \mathbf{e}_2 \quad \Longrightarrow \quad \boldsymbol{F} \cdot (\mathbf{e}_2 \otimes \mathbf{e}_2) = \gamma \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_2$$

Since $\mathbf{e}_i \otimes \mathbf{e}_i = \boldsymbol{1}$ we can also write the deformation gradient as

$$\boldsymbol{F} = \boldsymbol{1} + \gamma \mathbf{e}_1 \otimes \mathbf{e}_2$$

Chapter 5

Finite Strain Theory

In continuum mechanics, the **finite strain theory**—also called **large strain theory**, or **large deformation theory**—deals with deformations in which both rotations and strains are arbitrarily large, i.e. invalidates the assumptions inherent in infinitesimal strain theory. In this case, the undeformed and deformed configurations of the continuum are significantly different and a clear distinction has to be made between them. This is commonly the case with elastomers, plastically-deforming materials and other fluids and biological soft tissue.

Displacement

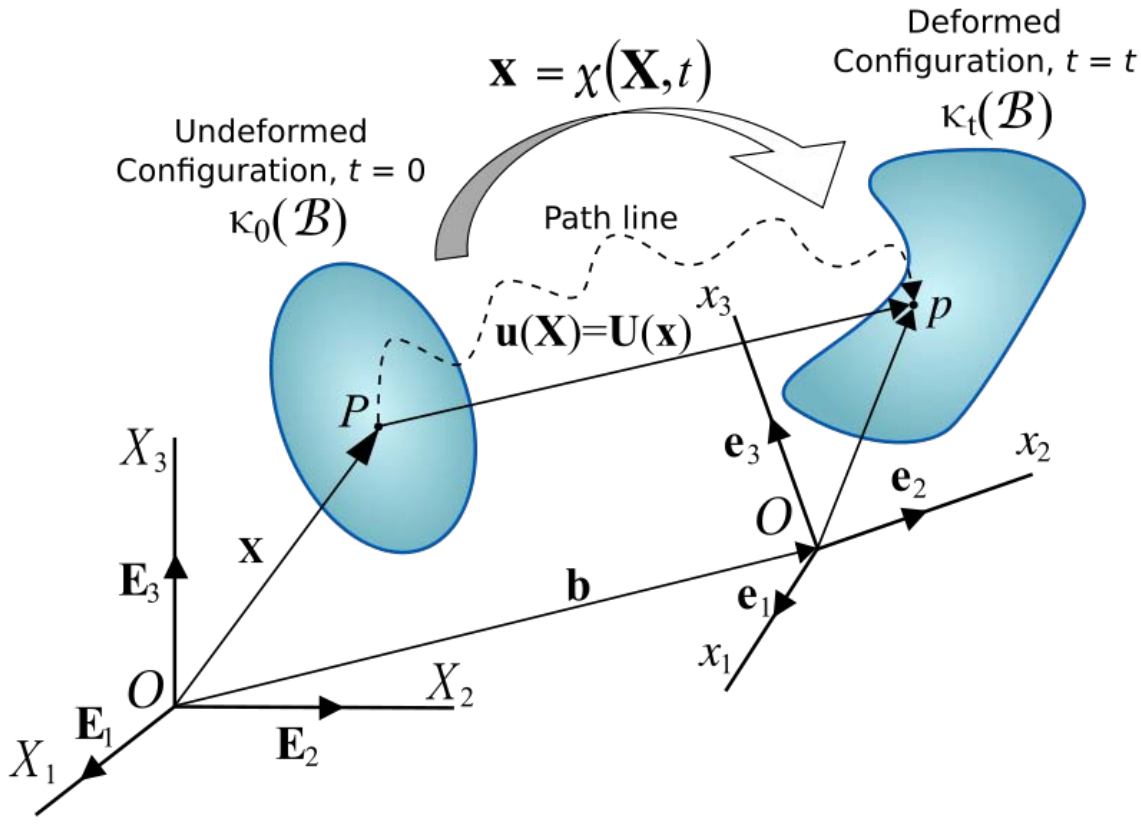


Figure 1. Motion of a continuum body.

A change in the configuration of a continuum body results in a displacement. The displacement of a body has two components: a rigid-body displacement and a deformation. A rigid-body displacement consists of a simultaneous translation and rotation of the body without changing its shape or size. Deformation implies the change in shape and/or size of the body from an initial or undeformed configuration $\kappa_0(\mathcal{B})$ to a current or deformed configuration $\kappa_t(\mathcal{B})$ (Figure 1).

If after a displacement of the continuum there is a relative displacement between particles, a deformation has occurred. On the other hand, if after displacement of the continuum the relative displacement between particles in the current configuration is zero i.e. the distance between particles remains unchanged, then there is no deformation and a rigid-body displacement is said to have occurred.

The vector joining the positions of a particle P in the undeformed configuration and deformed configuration is called the displacement vector $\mathbf{u}(\mathbf{X}, t) = u_i \mathbf{e}_i$ in the Lagrangian description, or $\mathbf{U}(\mathbf{x}, t) = U_J \mathbf{E}_J$ in the Eulerian description.

A *displacement field* is a vector field of all displacement vectors for all particles in the body, which relates the deformed configuration with the undeformed configuration. It is convenient to do the analysis of deformation or motion of a continuum body in terms of the displacement field. In general, the displacement field is expressed in terms of the material coordinates as

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{b}(\mathbf{X}, t) + \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \qquad \text{or} \qquad u_i = \alpha_{iJ} b_J + x_i - \alpha_{iJ} X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{b}(\mathbf{x}, t) + \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \qquad \text{or} \qquad U_J = b_J + \alpha_{Ji} x_i - X_J$$

where α_{Ji} are the direction cosines between the material and spatial coordinate systems with unit vectors $\mathbf{E}_J$ and $\mathbf{e}_i$, respectively. Thus

$$\mathbf{E}_J \cdot \mathbf{e}_i = \alpha_{Ji} = \alpha_{iJ}$$

and the relationship between u_i and U_J is then given by

$$u_i = \alpha_{iJ} U_J \qquad \text{or} \qquad U_J = \alpha_{Ji} u_i$$

Knowing that

$$\mathbf{e}_i = \alpha_{iJ} \mathbf{E}_J$$

then

$$\mathbf{u}(\mathbf{X}, t) = u_i \mathbf{e}_i = u_i (\alpha_{iJ} \mathbf{E}_J) = U_J \mathbf{E}_J = \mathbf{U}(\mathbf{x}, t)$$

It is common to superimpose the coordinate systems for the undeformed and deformed configurations, which results in $\mathbf{b} = 0$, and the direction cosines become Kronecker deltas, i.e.

$$\mathbf{E}_J \cdot \mathbf{e}_i = \delta_{Ji} = \delta_{iJ}$$

Thus, we have

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \qquad \text{or} \qquad u_i = x_i - \delta_{iJ} X_J$$

or in terms of the spatial coordinates as

$$\mathbf{U}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \qquad \text{or} \qquad U_J = \delta_{Ji} x_i - X_J$$

Displacement gradient tensor

The partial differentiation of the displacement vector with respect to the material coordinates yields the *material displacement gradient tensor* $\nabla_{\mathbf{X}} \mathbf{u}$. Thus we have,

$$\begin{aligned} \mathbf{u}(\mathbf{X}, t) &= \mathbf{x}(\mathbf{X}, t) - \mathbf{X} & u_i &= x_i - \delta_{iJ} X_J = x_i - X_i \\ \nabla_{\mathbf{X}} \mathbf{u} &= \nabla_{\mathbf{X}} \mathbf{x} - \mathbf{I} \qquad \text{or} & \frac{\partial u_i}{\partial X_K} &= \frac{\partial x_i}{\partial X_K} - \delta_{iK} \\ \nabla_{\mathbf{X}} \mathbf{u} &= \mathbf{F} - \mathbf{I} \end{aligned}$$

where $\mathbf{F}$ is the *deformation gradient tensor*.

Similarly, the partial differentiation of the displacement vector with respect to the spatial coordinates yields the *spatial displacement gradient tensor* $\nabla_{\mathbf{x}} \mathbf{U}$. Thus we have,

$$\begin{aligned} \mathbf{U}(\mathbf{x}, t) &= \mathbf{x} - \mathbf{X}(\mathbf{x}, t) & U_J &= \delta_{Ji} x_i - X_J = x_J - X_J \\ \nabla_{\mathbf{x}} \mathbf{U} &= \mathbf{I} - \nabla_{\mathbf{x}} \mathbf{X} \qquad \text{or} & \frac{\partial U_J}{\partial x_k} &= \delta_{Jk} - \frac{\partial X_J}{\partial x_k} \\ \nabla_{\mathbf{x}} \mathbf{U} &= \mathbf{I} - \mathbf{F}^{-1} \end{aligned}$$

Deformation gradient tensor

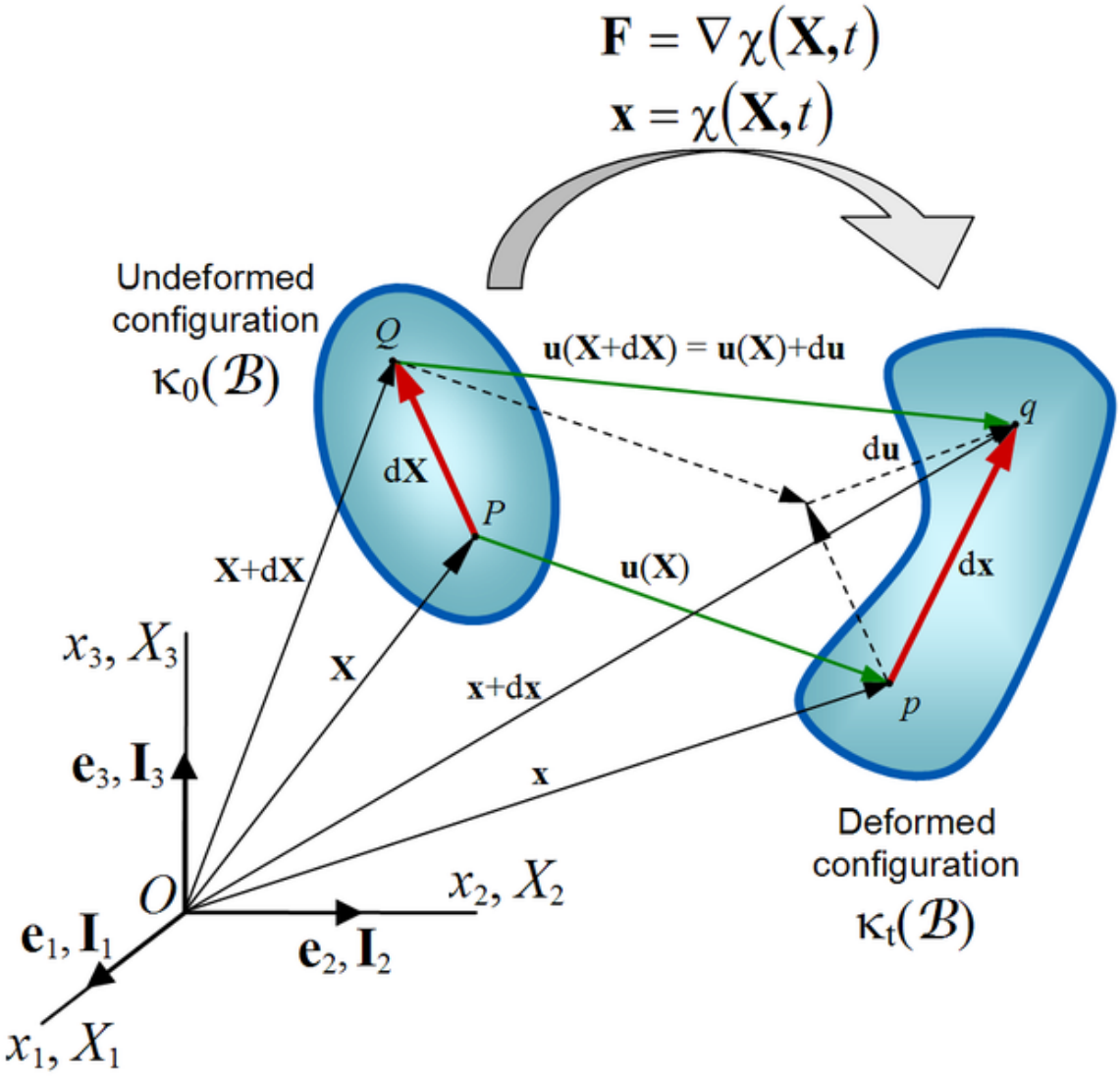


Figure 2. Deformation of a continuum body.

Consider a particle or material point P with position vector $\mathbf{X} = X_I \mathbf{I}_J$ in the undeformed configuration (Figure 2). After a displacement of the body, the new position of the particle indicated by p in the new configuration is given by the vector position $\mathbf{x} = x_i \mathbf{e}_j$. The coordinate systems for the undeformed and deformed configuration can be superimposed for convenience.

Consider now a material point Q neighboring P, with position vector $\mathbf{X} + \Delta\mathbf{X} = (X_I + \Delta X_I)\mathbf{I}_j$. In the deformed configuration this particle has a new position q given by the position vector $\mathbf{x} + \Delta\mathbf{x}$. Assuming that the line segments

ΔX and $\Delta \mathbf{x}$ joining the particles P and Q in both the undeformed and deformed configuration, respectively, to be very small, then we can express them as $d\mathbf{X}$ and $d\mathbf{x}$. Thus from Figure 2 we have

$$\begin{aligned}\mathbf{x} + d\mathbf{x} &= \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\ d\mathbf{x} &= \mathbf{X} - \mathbf{x} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) \\ &= d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}) - \mathbf{u}(\mathbf{X}) \\ &= d\mathbf{X} + d\mathbf{u}\end{aligned}$$

where $\mathbf{du}$ is the **relative displacement vector**, which represents the relative displacement of Q with respect to P in the deformed configuration.

For an infinitesimal element $d\mathbf{X}$, and assuming continuity on the displacement field, it is possible to use a Taylor series expansion around point P, neglecting higher-order terms, to approximate the components of the relative displacement vector for the neighboring particle Q as

$$\begin{aligned}\mathbf{u}(\mathbf{X} + d\mathbf{X}) &= \mathbf{u}(\mathbf{X}) + d\mathbf{u} \\ &\approx \mathbf{u}(\mathbf{X}) + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X}\end{aligned} \quad \text{or} \quad \begin{aligned}u_i^* &= u_i + du_i \\ &\approx u_i + \frac{\partial u_i}{\partial X_J} dX_J\end{aligned}$$

Thus, the previous equation $d\mathbf{x} = d\mathbf{X} + d\mathbf{u}$ can be written as

$$\begin{aligned}d\mathbf{x} &= d\mathbf{X} + d\mathbf{u} \\ &= d\mathbf{X} + \nabla_{\mathbf{X}}\mathbf{u} \cdot d\mathbf{X} \\ &= (\mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u})\, d\mathbf{X} \\ &= \mathbf{F} d\mathbf{X}\end{aligned}$$

The *material deformation gradient tensor* $\mathbf{F}(\mathbf{X}, t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$ is a second-order tensor that represents the gradient of the mapping function or functional relation $\chi(\mathbf{X}, t)$, which describes the motion of a continuum. The material deformation gradient tensor characterizes the local deformation at a material point with position vector $\mathbf{X}$, i.e. deformation at neighbouring points, by transforming (linear transformation) a material line element emanating from that point from the reference configuration to the current or deformed configuration, assuming continuity in the mapping function $\chi(\mathbf{X}, t)$, i.e. differentiable function of $\mathbf{X}$ and time t, which implies that cracks and voids do not open or close during the deformation. Thus we have,

$$
\begin{aligned}
d\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}}\, d\mathbf{X} \\
&= \nabla \chi(\mathbf{X}, t)\, d\mathbf{X} \\
&= \mathbf{F}(\mathbf{X}, t)\, d\mathbf{X}
\end{aligned}
\qquad \text{or} \qquad
\begin{aligned}
dx_j &= \frac{\partial x_j}{\partial X_K}\, dX_K \\
dx_j &= F_{jK}\, dX_K
\end{aligned}
$$

The deformation gradient tensor $\mathbf{F}(\mathbf{X}, t) = F_{jK}\mathbf{e}_j \otimes \mathbf{I}_K$ is related to both the reference and current configuration, as seen by the unit vectors $\mathbf{e}_j$ and $\mathbf{I}_K$, therefore it is a *two-point tensor*.

Due to the assumption of continuity of $\chi(\mathbf{X}, t)$, $\mathbf{F}$ has the inverse $\mathbf{H} = \mathbf{F}^{-1}$, where $\mathbf{H}$ is the *spatial deformation gradient tensor*. Then, by the implicit function theorem (Lubliner), the Jacobian determinant $J(\mathbf{X}, t)$ must be nonsingular, i.e.
$J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t) \neq 0$

Transformation of a surface and volume element

To transform quantities that are defined with respect to areas in a deformed configuration to those relative to areas in a reference configuration, and vice versa, we use the Nanson's relation, expressed as

$$da\, \mathbf{n} = J\, dA\, \mathbf{F}^{-T} \cdot \mathbf{N}$$

where da is an area of a region in the deformed configuration, dA is the same area in the reference configuration, and $\mathbf{n}$ is the outward normal to the area element in the current configuration while $\mathbf{N}$ is the outward normal in the reference configuration, $\mathbf{F}$ is the deformation gradient, and $J = \det \mathbf{F}$.

Polar decomposition of the deformation gradient tensor

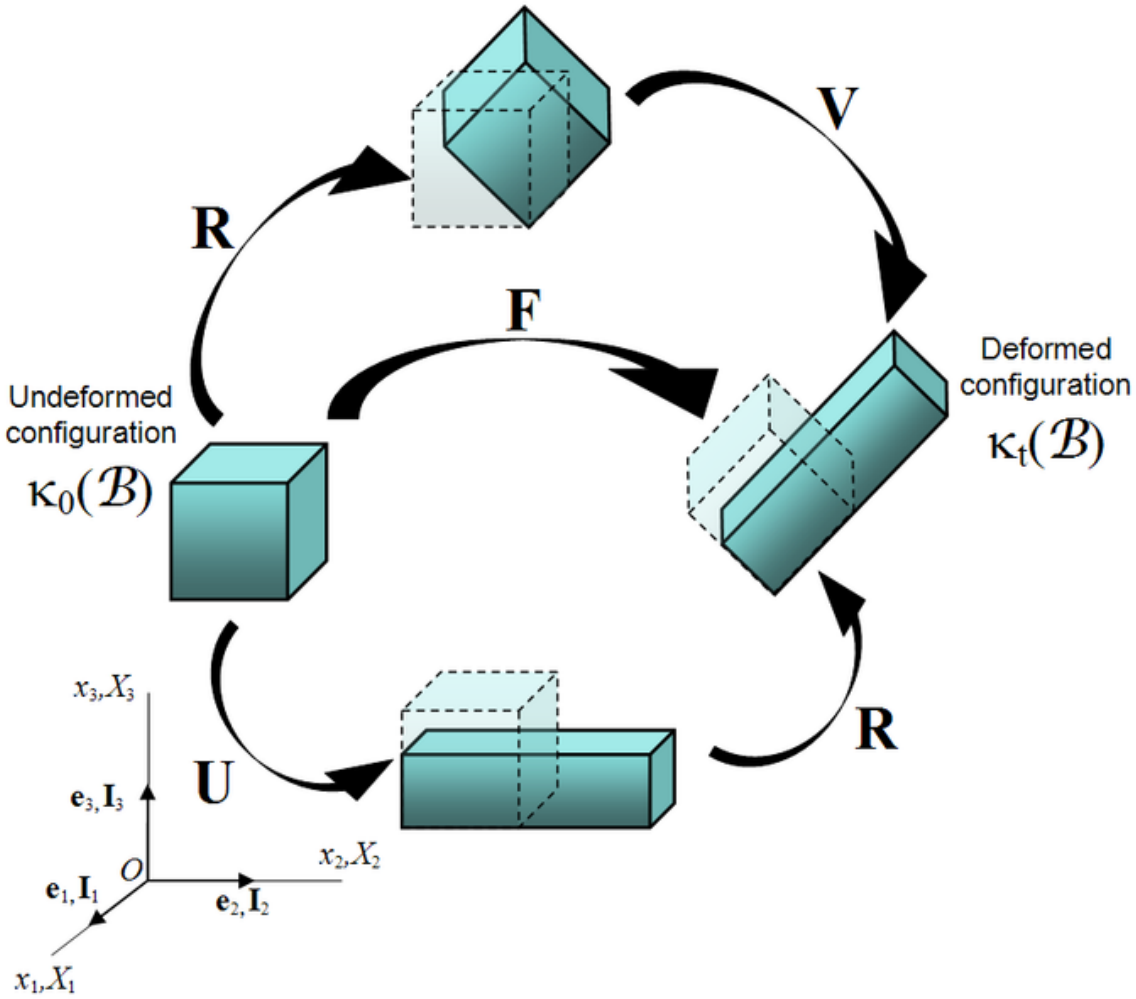


Figure 3. Representation of the polar decomposition of the deformation gradient

The deformation gradient $\mathbf{F}$, like any second-order tensor, can be decomposed, using the polar decomposition theorem, into a product of two second-order tensors (Truesdell and Noll, 1965): an orthogonal tensor and a positive definite symmetric tensor, i.e.

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}$$

where the tensor $\mathbf{R}$ is a proper orthogonal tensor, i.e. $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\det \mathbf{R} = +1$, representing a rotation; the tensor $\mathbf{U}$ is the *right stretch tensor*; and $\mathbf{V}$ the *left stretch tensor*. The terms *right* and *left* means that they are to the right and left of the rotation tensor $\mathbf{R}$, respectively. $\mathbf{U}$ and $\mathbf{V}$ are both positive definite, i.e. $\mathbf{x} \cdot \mathbf{U} \cdot \mathbf{x} \geq 0$ and $\mathbf{x} \cdot \mathbf{V} \cdot \mathbf{x} \geq 0$, and symmetric tensors, i.e. $\mathbf{U} = \mathbf{U}^T$ and $\mathbf{V} = \mathbf{V}^T$, of second order.

This decomposition implies that the deformation of a line element $d\mathbf{X}$in the undeformed configuration onto $d\mathbf{x}$in the deformed configuration, i.e. $d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$, may be obtained either by first stretching the element by $\mathbf{U}$, i.e. $d\mathbf{x}' = \mathbf{U}\, d\mathbf{X}$, followed by a rotation $\mathbf{R}$, i.e. $d\mathbf{x} = \mathbf{R}\, d\mathbf{x}'$; or equivalently, by applying a rigid rotation $\mathbf{R}$first, i.e. $d\mathbf{x}' = \mathbf{R}\, d\mathbf{X}$, followed later by a stretching $\mathbf{V}$, i.e. $d\mathbf{x} = \mathbf{V}\, d\mathbf{x}'$.

It can be shown that,

$$\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^T$$

so that $\mathbf{U}$and $\mathbf{V}$have the same eigenvalues or principal stretches, but different eigenvectors or *principal directions* $\mathbf{N}_i$and $\mathbf{n}_i$, respectively. The principal directions are related by

$$\mathbf{n}_i = \mathbf{R}\mathbf{N}_i.$$

This polar decomposition is unique as $\mathbf{F}$is non-symmetric.

Deformation tensors

Several rotation-independent deformation tensors are used in mechanics. In solid mechanics, the most popular of these are the right and left Cauchy-Green deformation tensors.

Since a pure rotation should not induce any stresses in a deformable body, it is often convenient to use rotation-independent measures of deformation in continuum mechanics. As a rotation followed by its inverse rotation leads to no change ($\mathbf{R}\mathbf{R}^T = \mathbf{R}^T\mathbf{R} = \mathbf{1}$) we can exclude the rotation by multiplying $\mathbf{F}$by its transpose.

The Right Cauchy-Green deformation tensor

In 1839, George Green introduced a deformation tensor known as the *right Cauchy-Green deformation tensor* or *Green's deformation tensor*, defined as:

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2 \qquad \text{or} \qquad C_{IJ} = F_{kI}\, F_{kJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J}.$$

Physically, the Cauchy-Green tensor gives us the square of local change in distances due to deformation, i.e. $d\mathbf{x}^2 = d\mathbf{X} \cdot \mathbf{C}d\mathbf{X}$

Invariants of $\mathbf{C}$are often used in the expressions for strain energy density functions. The most commonly used invariants are

$$I_1^C := \mathrm{tr}(\mathbf{C}) = C_{II} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2^C := \tfrac{1}{2}\left[(\mathrm{tr}\ \mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2)\right] = \tfrac{1}{2}\left[(C_{JJ})^2 - C_{IK}C_{KI}\right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3^C := \det(\mathbf{C}) = \lambda_1^2\lambda_2^2\lambda_3^2.$$

The Finger deformation tensor

The IUPAC recommends that the inverse of the right Cauchy-Green deformation tensor, i. e., $\mathbf{C}^{-1}$, be called the **Finger tensor**. However, that nomenclature is not universally accepted in applied mechanics.

$$\mathbf{f} = \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{F}^{-T} \qquad \text{or} \qquad f_{IJ} = \frac{\partial X_I}{\partial x_k}\frac{\partial X_J}{\partial x_k}$$

The Left Cauchy-Green or Finger deformation tensor

Reversing the order of multiplication in the formula for the right Green-Cauchy deformation tensor leads to the *left Cauchy-Green deformation tensor* which is defined as:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2 \qquad \text{or} \qquad B_{ij} = \frac{\partial x_i}{\partial X_K}\frac{\partial x_j}{\partial X_K}$$

The left Cauchy-Green deformation tensor *is often called the* Finger deformation tensor*, named after Josef Finger (1894).*

Invariants of $\mathbf{B}$are also used in the expressions for strain energy density functions. The conventional invariants are defined as

$$I_1 := \mathrm{tr}(\mathbf{B}) = B_{ii} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$$
$$I_2 := \tfrac{1}{2}\left[(\mathrm{tr}\ \mathbf{B})^2 - \mathrm{tr}(\mathbf{B}^2)\right] = \tfrac{1}{2}\left(B_{ii}^2 - B_{jk}B_{kj}\right) = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$$
$$I_3 := \det \mathbf{B} = J^2 = \lambda_1^2\lambda_2^2\lambda_3^2$$

where $J := \det \mathbf{F}$is the determinant of the deformation gradient.

For nearly incompressible materials, a slightly different set of invariants is used:

$$(\bar{I}_1 := J^{-2/3}I_1\ ;\quad \bar{I}_2 := J^{-4/3}I_2\ ;\quad J = 1)\ .$$

The Cauchy deformation tensor

Earlier in 1828 , Augustin Louis Cauchy introduced a deformation tensor defined as the inverse of the left Cauchy-Green deformation tensor, $\mathbf{B}^{-1}$. This tensor has also been

called the **Piola tensor** and the **Finger tensor** in the rheology and fluid dynamics literature.

$$\mathbf{c} = \mathbf{B}^{-1} = \mathbf{F}^{-T}\mathbf{F}^{-1} \qquad \text{or} \qquad c_{ij} = \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j}$$

Spectral representation

If there are three distinct *principal stretches* λ_i, the spectral decompositions of $\mathbf{C}$ and $\mathbf{B}$ is given by

$$\mathbf{C} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{N}_i \otimes \mathbf{N}_i \qquad \text{and} \qquad \mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{n}_i \otimes \mathbf{n}_i$$

Furthermore,

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{N}_i \otimes \mathbf{N}_i \ ; \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i$$
$$\mathbf{R} = \sum_{i=1}^{3} \mathbf{n}_i \otimes \mathbf{N}_i \ ; \quad \mathbf{F} = \sum_{i=1}^{3} \lambda_i \mathbf{n}_i \otimes \mathbf{N}_i$$

Observe that

$$\mathbf{V} = \mathbf{R}\ \mathbf{U}\ \mathbf{R}^T = \sum_{i=1}^{3} \lambda_i\ \mathbf{R}\ (\mathbf{N}_i \otimes \mathbf{N}_i)\ \mathbf{R}^T = \sum_{i=1}^{3} \lambda_i\ (\mathbf{R}\ \mathbf{N}_i) \otimes (\mathbf{R}\ \mathbf{N}_i)$$

Therefore the uniqueness of the spectral decomposition also implies that $\mathbf{n}_i = \mathbf{R}\ \mathbf{N}_i$. The left stretch ($\mathbf{V}$) is also called the *spatial stretch tensor* while the right stretch ($\mathbf{U}$) is called the *material stretch tensor*.

The effect of $\mathbf{F}$ acting on $\mathbf{N}_i$ is to stretch the vector by λ_i and to rotate it to the new orientation $\mathbf{n}_i$, i.e.,

$$\mathbf{F}\ \mathbf{N}_i = \lambda_i\ (\mathbf{R}\ \mathbf{N}_i) = \lambda_i\ \mathbf{n}_i$$

In a similar vein,

$$\mathbf{F}^{-T}\ \mathbf{N}_i = \frac{1}{\lambda_i}\ \mathbf{n}_i \ ; \quad \mathbf{F}^T\ \mathbf{n}_i = \lambda_i\ \mathbf{N}_i \ ; \quad \mathbf{F}^{-1}\ \mathbf{n}_i = \frac{1}{\lambda_i}\ \mathbf{N}_i \ .$$

Derivatives of stretch

Derivatives of the stretch with respect to the right Cauchy-Green deformation tensor are used to derive the stress-strain relations of many solids, particularly hyperelastic materials. These derivatives are

$$\frac{\partial \lambda_i}{\partial \mathbf{C}} = \frac{1}{2\lambda_i}\,\mathbf{N}_i \otimes \mathbf{N}_i = \frac{1}{2\lambda_i}\,\mathbf{R}^T\,(\mathbf{n}_i \otimes \mathbf{n}_i)\,\mathbf{R}\;;\quad i = 1,2,3$$

and follow from the observations that

$$\mathbf{C} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \lambda_i^2\;;\qquad \frac{\partial \mathbf{C}}{\partial \mathbf{C}} = \mathsf{I}^{(s)}\;;\qquad \mathsf{I}^{(s)} : (\mathbf{N}_i \otimes \mathbf{N}_i) = \mathbf{N}_i \otimes \mathbf{N}_i.$$

Physical interpretation of deformation tensors

Let $\mathbf{X} = X^i\ \boldsymbol{E}_i$ be a Cartesian coordinate system defined on the undeformed body and let $\mathbf{x} = x^i\ \boldsymbol{E}_i$ be another system defined on the deformed body. Let a curve $\mathbf{X}(s)$ in the undeformed body be parametrized using $s \in [0,1]$. Its image in the deformed body is $\mathbf{x}(\mathbf{X}(s))$.

The undeformed length of the curve is given by

$$l_X = \int_0^1 \left|\frac{d\mathbf{X}}{ds}\cdot\frac{d\mathbf{X}}{ds}\right|\,ds = \int_0^1 \left|\frac{d\mathbf{X}}{ds}\cdot\boldsymbol{I}\cdot\frac{d\mathbf{X}}{ds}\right|\,ds$$

After deformation, the length becomes

$$\begin{aligned} l_x &= \int_0^1 \left|\frac{d\mathbf{x}}{ds}\cdot\frac{d\mathbf{x}}{ds}\right|\,ds = \int_0^1 \left|\left(\frac{d\mathbf{x}}{d\mathbf{X}}\cdot\frac{d\mathbf{X}}{ds}\right)\cdot\left(\frac{d\mathbf{x}}{d\mathbf{X}}\cdot\frac{d\mathbf{X}}{ds}\right)\right|\,ds \\ &= \int_0^1 \left|\frac{d\mathbf{X}}{ds}\cdot\left[\left(\frac{d\mathbf{x}}{d\mathbf{X}}\right)^T\cdot\frac{d\mathbf{x}}{d\mathbf{X}}\right]\cdot\frac{d\mathbf{X}}{ds}\right|\,ds \end{aligned}$$

Note that the right Cauchy-Green deformation tensor is defined as

$$\boldsymbol{C} := \boldsymbol{F}^T\cdot\boldsymbol{F} = \left(\frac{d\mathbf{x}}{d\mathbf{X}}\right)^T\cdot\frac{d\mathbf{x}}{d\mathbf{X}}$$

Hence,

$$l_x = \int_0^1 \left| \frac{d\mathbf{X}}{ds} \cdot \boldsymbol{C} \cdot \frac{d\mathbf{X}}{ds} \right| \, ds$$

which indicates that changes in length are characterized by $\boldsymbol{C}$.

Finite strain tensors

The concept of *strain* is used to evaluate how much a given displacement differs locally from a rigid body displacement (Ref. Lubliner). One of such strains for large deformations is the *Lagrangian finite strain tensor*, also called the *Green-Lagrangian strain tensor* or *Green - St-Venant strain tensor*, defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \qquad \text{or} \qquad E_{KL} = \frac{1}{2}\left(\frac{\partial x_j}{\partial X_K}\frac{\partial x_j}{\partial X_L} - \delta_{KL}\right)$$

or as a function of the displacement gradient tensor

$$\mathbf{E} = \frac{1}{2}\left[(\nabla_{\mathbf{X}}\mathbf{u})^T + \nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T \cdot \nabla_{\mathbf{X}}\mathbf{u}\right]$$

or

$$E_{KL} = \frac{1}{2}\left(\frac{\partial U_K}{\partial X_L} + \frac{\partial U_L}{\partial X_K} + \frac{\partial U_M}{\partial X_K}\frac{\partial U_M}{\partial X_L}\right)$$

The Green-Lagrangian strain tensor is a measure of how much $\mathbf{C}$ differs from $\mathbf{I}$. It can be shown that this tensor is a special case of a general formula for Lagrangian strain tensors (Hill 1968):

$$\mathbf{E}_{(m)} = \frac{1}{2m}(\mathbf{U}^{2m} - \mathbf{I})$$

For different values of m we have:

$\mathbf{E}_{(1)} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$ Green-Lagrangian strain tensor

$\mathbf{E}_{(1/2)} = (\mathbf{U} - \mathbf{I})$ Biot strain tensor

$\mathbf{E}_{(0)} = \ln \mathbf{U}$ Logarithmic strain, Natural strain, True strain, or Hencky strain

The *Eulerian-Almansi finite strain tensor*, referenced to the deformed configuration, i.e. Eulerian description, is defined as

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{c}) \qquad \text{or} \qquad e_{rs} = \frac{1}{2}\left(\delta_{rs} - \frac{\partial X_M}{\partial x_r}\frac{\partial X_M}{\partial x_s}\right)$$

or as a function of the displacement gradients we have

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)$$

Stretch ratio

The **stretch ratio** is a measure of the extensional or normal strain of a differential line element, which can be defined at either the undeformed configuration or the deformed configuration.

The stretch ratio for the differential element $d\mathbf{X} = dX\mathbf{N}$(Figure) in the direction of the unit vector $\mathbf{N}$at the material point P, in the undeformed configuration, is defined as

$$\Lambda_{(\mathbf{N})} = \frac{dx}{dX}$$

where $dx$is the deformed magnitude of the differential element $d\mathbf{X}$.

Similarly, the stretch ratio for the differential element $d\mathbf{x} = dx\mathbf{n}$(Figure), in the direction of the unit vector $\mathbf{n}$at the material point p, in the deformed configuration, is defined as

$$\frac{1}{\Lambda_{(\mathbf{n})}} = \frac{dX}{dx}.$$

The normal strain $e_{\mathbf{N}}$in any direction $\mathbf{N}$can be expressed as a function of the stretch ratio,

$$e_{(\mathbf{N})} = \frac{dx - dX}{dX} = \Lambda_{(\mathbf{N})} - 1.$$

This equation implies that the normal strain is zero, i.e. no deformation, when the stretch is equal to unity. Some materials, such as elastometers can sustain stretch ratios of 3 or 4 before they fail, whereas traditional engineering materials, such as concrete or steel, fail at much lower stretch ratios, perhaps of the order of 1.001 (reference?)

Physical interpretation of the finite strain tensor

The diagonal components E_{KL} of the Lagrangian finite strain tensor are related to the normal strain, e.g.

$$E_{11} = e_{(\mathbf{I}_1)} + \frac{1}{2} e_{(\mathbf{I}_1)}^2$$

where $e_{(\mathbf{I}_1)}$ is the normal strain or engineering strain in the direction $\mathbf{I}_1$.

The off-diagonal components E_{KL} of the Lagrangian finite strain tensor are related to shear strain, e.g.

$$E_{12} = \frac{1}{2}\sqrt{2E_{11}+1}\sqrt{2E_{22}+1}\sin\phi_{12}$$

where ϕ_{12} is the change in the angle between two line elements that were originally perpendicular with directions $\mathbf{I}_1$ and $\mathbf{I}_2$, respectively.

Under certain circumstances, i.e. small displacements and small displacement rates, the components of the Lagrangian finite strain tensor may be approximated by the components of the infinitesimal strain tensor

Deformation tensors in curvilinear coordinates

A representation of deformation tensors in curvilinear coordinates is useful for many problems in continuum mechanics such as nonlinear shell theories and large plastic deformations. Let $\mathbf{x} = \mathbf{x}(\xi^1, \xi^2, \xi^3)$ be a given deformation where the space is characterized by the coordinates (ξ^1,ξ^2,ξ^3). The tangent vector to the coordinate curve ξ^i at $\mathbf{x}$ is given by

$$\mathbf{g}_i = \frac{\partial \mathbf{x}}{\partial \xi^i}$$

The three tangent vectors at $\mathbf{x}$ form a basis. These vectors are related the reciprocal basis vectors by

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

Let us define a field

$$g_{ij} := \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \mathbf{g}_i \cdot \mathbf{g}_j$$

The Christoffel symbols of the first kind can be expressed as

$$\Gamma_{ijk} = \tfrac{1}{2}[(\mathbf{g}_i \cdot \mathbf{g}_k)_{,j} + (\mathbf{g}_j \cdot \mathbf{g}_k)_{,i} - (\mathbf{g}_i \cdot \mathbf{g}_j)_{,k}]$$

To see how the Christoffel symbols are related to the Right Cauchy-Green deformation tensor let us define two sets of bases

$$\mathbf{G}_i := \frac{\partial \mathbf{X}}{\partial \xi^i} \; ; \quad \mathbf{G}_i \cdot \mathbf{G}^j = \delta_i^j \; ; \quad \mathbf{g}_i := \frac{\partial \mathbf{x}}{\partial \xi^i} \; ; \quad \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$$

The deformation gradient in curvilinear coordinates

Using the definition of the gradient of a vector field in curvilinear coordinates, the deformation gradient can be written as

$$\boldsymbol{F} = \boldsymbol{\nabla}_{\mathbf{X}} \mathbf{x} = \frac{\partial \mathbf{x}}{\partial \xi^i} \otimes \mathbf{G}^i = \mathbf{g}_i \otimes \mathbf{G}^i$$

The right Cauchy-Green tensor in curvilinear coordinates

The right Cauchy-Green deformation tensor is given by

$$\boldsymbol{C} = \boldsymbol{F}^T \cdot \boldsymbol{F} = (\mathbf{G}^i \otimes \mathbf{g}_i) \cdot (\mathbf{g}_i \otimes \mathbf{G}^i) = (\mathbf{g}_i \cdot \mathbf{g}_j)(\mathbf{G}^i \otimes \mathbf{G}^j)$$

If we express $\boldsymbol{C}$ in terms of components with respect to the basis $\{\mathbf{G}^i\}$ we have

$$\boldsymbol{C} = C_{ij} \; \mathbf{G}^i \otimes \mathbf{G}^j$$

Therefore

$$C_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = g_{ij}$$

and the Christoffel symbol of the first kind may be written in the following form.

$$\Gamma_{ijk} = \tfrac{1}{2}[C_{ik,j} + C_{jk,i} - C_{ij,k}] = \tfrac{1}{2}[(\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,j} + (\mathbf{G}_j \cdot \boldsymbol{C} \cdot \mathbf{G}_k)_{,i} - (\mathbf{G}_i \cdot \boldsymbol{C} \cdot \mathbf{G}_j)_{,k}]$$

Some relations between deformation measures and Christoffel symbols

Let us consider a one-to-one mapping from $\mathbf{X} = \{X^1, X^2, X^3\}$to $\mathbf{x} = \{x^1, x^2, x^3\}$and let us assume that there exist two positive definite, symmetric second-order tensor fields $\boldsymbol{G}$and $\boldsymbol{g}$that satisfy

$$G_{ij} = \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,g_{\alpha\beta}$$

Then,

$$\frac{\partial G_{ij}}{\partial x^k} = \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\,\frac{\partial X^\beta}{\partial x^j} + \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial^2 X^\beta}{\partial x^j \partial x^k}\right) g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial g_{\alpha\beta}}{\partial x^k}$$

Noting that

$$\frac{\partial g_{\alpha\beta}}{\partial x^k} = \frac{\partial X^\gamma}{\partial x^k}\,\frac{\partial g_{\alpha\beta}}{\partial X^\gamma}$$

and $g_{\alpha\beta} = g_{\beta\alpha}$ we have

$$\begin{aligned}
\frac{\partial G_{ij}}{\partial x^k} &= \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\,\frac{\partial X^\beta}{\partial x^j} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}\,\frac{\partial X^\beta}{\partial x^i}\right) g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} \\
\frac{\partial G_{ik}}{\partial x^j} &= \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^j \partial x^k}\,\frac{\partial X^\beta}{\partial x^i}\right) g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^k}\,\frac{\partial X^\gamma}{\partial x^j}\,\frac{\partial g_{\alpha\beta}}{\partial X^\gamma} \\
\frac{\partial G_{jk}}{\partial x^i} &= \left(\frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^k}\,\frac{\partial X^\beta}{\partial x^j}\right) g_{\alpha\beta} + \frac{\partial X^\alpha}{\partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,\frac{\partial X^\gamma}{\partial x^i}\,\frac{\partial g_{\alpha\beta}}{\partial X^\gamma}
\end{aligned}$$

Define

$$\begin{aligned}
{}_{(x)}\Gamma_{ijk} &:= \frac{1}{2}\left(\frac{\partial G_{ik}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^i} - \frac{\partial G_{ij}}{\partial x^k}\right) \\
{}_{(X)}\Gamma_{\alpha\beta\gamma} &:= \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}\right)
\end{aligned}$$

Hence

$${}_{(x)}\Gamma_{ijk} = \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,\frac{\partial X^\gamma}{\partial x^k}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}\,\frac{\partial X^\beta}{\partial x^k}\,g_{\alpha\beta}$$

Define

$$[G^{ij}] = [G_{ij}]^{-1} \; ; \quad [g^{\alpha\beta}] = [g_{\alpha\beta}]^{-1}$$

Then

$$G^{ij} = \frac{\partial x^i}{\partial X^\alpha} \frac{\partial x^j}{\partial X^\beta} g^{\alpha\beta}$$

Define the Christoffel symbols of the second kind as

$${}_{(x)}\Gamma^m_{ij} := G^{mk} \, {}_{(x)}\Gamma_{ijk} \; ; \quad {}_{(X)}\Gamma^\nu_{\alpha\beta} := g^{\nu\gamma} \, {}_{(X)}\Gamma_{\alpha\beta\gamma}$$

Then

$$\begin{aligned}
{}_{(x)}\Gamma^m_{ij} &= G^{mk} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \frac{\partial X^\gamma}{\partial x^k} \, {}_{(X)}\Gamma_{\alpha\beta\gamma} + G^{mk} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \frac{\partial X^\beta}{\partial x^k} g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu} \frac{\partial x^k}{\partial X^\rho} g^{\nu\rho} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \frac{\partial X^\gamma}{\partial x^k} \, {}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu} \frac{\partial x^k}{\partial X^\rho} g^{\nu\rho} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \frac{\partial X^\beta}{\partial x^k} g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu} \delta^\gamma_\rho \, g^{\nu\rho} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu} \delta^\beta_\rho \, g^{\nu\rho} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu} g^{\nu\gamma} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma_{\alpha\beta\gamma} + \frac{\partial x^m}{\partial X^\nu} g^{\nu\beta} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} g_{\alpha\beta} \\
&= \frac{\partial x^m}{\partial X^\nu} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\nu} \delta^\nu_\alpha \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}
\end{aligned}$$

Therefore

$${}_{(x)}\Gamma^m_{ij} = \frac{\partial x^m}{\partial X^\nu} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial x^m}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j}$$

The invertibility of the mapping implies that

$$\begin{aligned}
\frac{\partial X^\mu}{\partial x^m} \, {}_{(x)}\Gamma^m_{ij} &= \frac{\partial X^\mu}{\partial x^m} \frac{\partial x^m}{\partial X^\nu} \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma^\nu_{\alpha\beta} + \frac{\partial X^\mu}{\partial x^m} \frac{\partial x^m}{\partial X^\alpha} \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\
&= \delta^\mu_\nu \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma^\nu_{\alpha\beta} + \delta^\mu_\alpha \frac{\partial^2 X^\alpha}{\partial x^i \partial x^j} \\
&= \frac{\partial X^\alpha}{\partial x^i} \frac{\partial X^\beta}{\partial x^j} \, {}_{(X)}\Gamma^\mu_{\alpha\beta} + \frac{\partial^2 X^\mu}{\partial x^i \partial x^j}
\end{aligned}$$

We can also formulate a similar result in terms of derivatives with respect to *x*. Therefore

$$\frac{\partial^2 X^\mu}{\partial x^i \partial x^j} = \frac{\partial X^\mu}{\partial x^m}\,{}_{(x)}\Gamma^m_{ij} - \frac{\partial X^\alpha}{\partial x^i}\,\frac{\partial X^\beta}{\partial x^j}\,{}_{(X)}\Gamma^\mu_{\alpha\beta}$$
$$\frac{\partial^2 x^m}{\partial X^\alpha \partial X^\beta} = \frac{\partial x^m}{\partial X^\mu}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - \frac{\partial x^i}{\partial X^\alpha}\,\frac{\partial x^j}{\partial X^\beta}\,{}_{(x)}\Gamma^m_{ij}$$

Compatibility conditions

The problem of compatibility in continuum mechanics involves the determination of allowable single-valued continuous fields on bodies. These allowable conditions leave the body without unphysical gaps or overlaps after a deformation. Most such conditions apply to simply-connected bodies. Additional conditions are required for the internal boundaries of multiply connected bodies.

Compatibility of the deformation gradient

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{F}$ field over a simply connected body are

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$$

Compatibility of the right Cauchy-Green deformation tensor

The necessary and sufficient conditions for the existence of a compatible $\boldsymbol{C}$ field over a simply connected body are

$$R^\gamma_{\alpha\beta\rho} := \frac{\partial}{\partial X^\rho}[\,{}_{(X)}\Gamma^\gamma_{\alpha\beta}] - \frac{\partial}{\partial X^\beta}[\,{}_{(X)}\Gamma^\gamma_{\alpha\rho}] + {}_{(X)}\Gamma^\gamma_{\mu\rho}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - {}_{(X)}\Gamma^\gamma_{\mu\beta}\,{}_{(X)}\Gamma^\mu_{\alpha\rho} = 0$$

We can show these are the mixed components of the Riemann-Christoffel curvature tensor. Therefore the necessary conditions for $\boldsymbol{C}$ -compatibility are that the Riemann-Christoffel curvature of the deformation is zero.

Compatibility of the left Cauchy-Green deformation tensor

No general sufficiency conditions are known for the left Cauchy-Green deformation tensor in three-dimensions. Compatibility conditions for two-dimensional $\boldsymbol{B}$ fields have been found by Janet Blume.

Chapter 6

Peridynamics

A ductile fracture of an Al-Mg-Si alloy. A fracture is a mathematical singularity to which the classical equations of continuum mechanics cannot be applied directly –

Peridynamics offers a numerical method.

Peridynamics is a formulation of continuum mechanics that is oriented toward deformations with discontinuities, especially fractures.

Purpose of peridynamics

The peridynamic theory is based on integral equations, in contrast with the classical theory of continuum mechanics, which is based on partial differential equations. Since partial derivatives do not exist on crack surfaces and other singularities, the classical equations of continuum mechanics cannot be applied directly when such features are present in a deformation. The integral equations of the peridynamic theory can be applied directly, because they do not require partial derivatives.

The ability to apply the same equations directly at all points in a mathematical model of a deforming structure helps the peridynamic approach avoid the need for the special techniques of fracture mechanics. For example, in peridynamics, there is no need for a separate crack growth law based on a stress intensity factor.

Definition and basic terminology

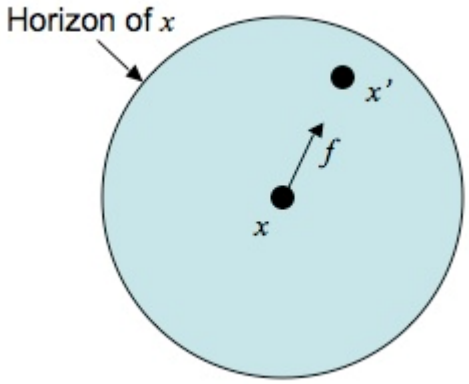
The basic equation of peridynamics is the following equation of motion:

$$\rho(x)\ddot{u}(x,t) = \int_R f(u(x',t) - u(x,t), x' - x, x)dV_{x'} + b(x,t)$$

where x is a point in a body R, t is time, u is the displacement vector field, and ρ is the mass density in the undeformed body. x' is a dummy variable of integration.

The vector valued function f is the force density that x' exerts on x. This force density depends on the relative displacement and relative position vectors between x' and x. The dimensions of f are force per volume squared. The function f is called the "pairwise force function" and contains all the constitutive (material-dependent) properties. It describes how the internal forces depend on the deformation.

The interaction between any x and x' is called a "bond." The physical mechanism in this interaction need not be specified. It is usually assumed that f vanishes whenever x' is outside a neighborhood of x (in the undeformed configuration) called the *horizon.*



The term "peridynamic," an adjective, was proposed in the year 2000 and comes from the prefix *peri,* which means *all around*, *near*, or *surrounding*; and the root *dyna*, which means *force* or *power.* The term "peridynamics," a noun, is a shortened form of the phrase *peridynamic model of solid mechanics.*

Pairwise force functions

Using the abbreviated notation $u = u(x,t)$ and $u' = u(x',t)$ Newton's third law places the following restriction on f:

$$f(u-u', x-x', x') = -f(u'-u, x'-x, x)$$

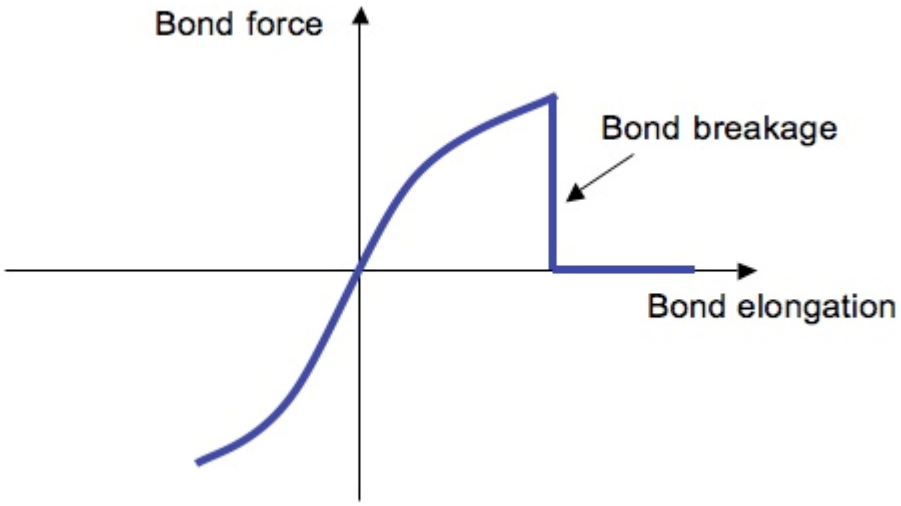
for any x,x',u,u'. This equation states that the force density vector that x exerts on x' equals minus the force density vector that x' exerts on x. Balance of angular momentum requires that f be parallel to the vector connecting the deformed position of x to the deformed position of x':

$$((x'+u')-(x+u)) \times f(u'-u, x'-x, x) = 0.$$

A pairwise force function is specified by a graph of $|f|$ versus bond elongation e, defined by

$$e = |(x'+u')-(x+u)| - |x'-x|.$$

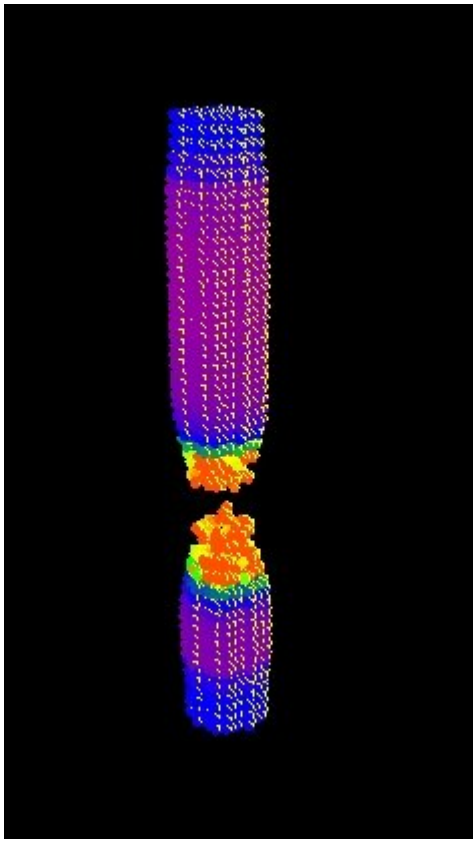
A schematic of a pairwise force function for the bond connecting two typical points is shown in the following figure:



Damage

Damage is incorporated in the pairwise force function by allowing bonds to break when their elongation exceeds some prescribed value. After a bond breaks, it no longer sustains any force, and the endpoints are effectively disconnected from each other. When a bond breaks, the force it was carrying is redistributed to other bonds that have not yet broken. This increased load makes it more likely that these other bonds will break. The process of bond breakage and load redistribution, leading to further breakage, is how cracks grow in the peridynamic model.

Peridynamic states



Computer model of the necking of an aluminum rod under tension. Colors indicate temperature increase due to plastic heating. Calculation performed with the Emu computer code using peridynamic states.

The theory described above assumes that each peridynamic bond responds independently of all the others. This is an oversimplification for most materials and leads to restrictions on the types of materials that can be modeled. In particular, this assumption implies that any isotropic linear elastic solid is restricted to a Poisson ratio of 1/4.

To address this lack of generality, the idea of "peridynamic states" was introduced. This allows the force density in each bond to depend on the stretches in all the bonds connected to its endpoints, in addition to its own stretch. For example, the force in a bond could depend on the net volume changes at the endpoints. The effect of this volume change, relative to the effect of the bond stretch, determines the Poisson ratio. With peridynamic states, any material that can be modeled within the standard theory of continuum mechanics can be modeled as a peridynamic material, while retaining the advantages of the peridynamic theory for fracture.

Chapter 7

Tensor Derivative (Continuum Mechanics)

The derivatives of scalars, vectors, and second-order tensors with respect to second-order tensors are of considerable use in continuum mechanics. These derivatives are used in the theories of nonlinear elasticity and plasticity, particularly in the design of algorithms for numerical simulations.

The directional derivative provides a systematic way of finding these derivatives.

Derivatives with respect to vectors and second-order tensors

The definitions of directional derivatives for various situations are given below. It is assumed that the functions are sufficiently smooth that derivatives can be taken.

Derivatives of scalar valued functions of vectors

Let $f(\mathbf{v})$ be a real valued function of the vector $\mathbf{v}$. Then the derivative of $f(\mathbf{v})$ with respect to $\mathbf{v}$ (or at $\mathbf{v}$) in the direction $\mathbf{u}$ is the **vector** defined as

$$\frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{u} = Df(\mathbf{v})[\mathbf{u}] = \left[\frac{d}{d\alpha}\, f(\mathbf{v} + \alpha\, \mathbf{u})\right]_{\alpha=0}$$

for all vectors $\mathbf{u}$.

Properties:

1) If $f(\mathbf{v}) = f_1(\mathbf{v}) + f_2(\mathbf{v})$ then $\frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}} + \frac{\partial f_2}{\partial \mathbf{v}}\right) \cdot \mathbf{u}$

2) If $f(\mathbf{v}) = f_1(\mathbf{v})\ f_2(\mathbf{v})$ then
$$\frac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{u} = \left(\frac{\partial f_1}{\partial \mathbf{v}} \cdot \mathbf{u}\right)\ f_2(\mathbf{v}) + f_1(\mathbf{v})\ \left(\frac{\partial f_2}{\partial \mathbf{v}} \cdot \mathbf{u}\right)$$

3) If $f(\mathbf{v}) = f_1(f_2(\mathbf{v}))$ then $\dfrac{\partial f}{\partial \mathbf{v}} \cdot \mathbf{u} = \dfrac{\partial f_1}{\partial f_2}\ \dfrac{\partial f_2}{\partial \mathbf{v}} \cdot \mathbf{u}$

Derivatives of vector valued functions of vectors

Let $\mathbf{f}(\mathbf{v})$ be a vector valued function of the vector $\mathbf{v}$. Then the derivative of $\mathbf{f}(\mathbf{v})$ with respect to $\mathbf{v}$ (or at $\mathbf{v}$) in the direction $\mathbf{u}$ is the **second order tensor** defined as

$$\frac{\partial \mathbf{f}}{\partial \mathbf{v}} \cdot \mathbf{u} = D\mathbf{f}(\mathbf{v})[\mathbf{u}] = \left[\frac{d}{d\alpha}\ \mathbf{f}(\mathbf{v} + \alpha\ \mathbf{u})\right]_{\alpha=0}$$

for all vectors $\mathbf{u}$.

Properties:

1) If $\mathbf{f}(\mathbf{v}) = \mathbf{f}_1(\mathbf{v}) + \mathbf{f}_2(\mathbf{v})$ then $\dfrac{\partial \mathbf{f}}{\partial \mathbf{v}} \cdot \mathbf{u} = \left(\dfrac{\partial \mathbf{f}_1}{\partial \mathbf{v}} + \dfrac{\partial \mathbf{f}_2}{\partial \mathbf{v}}\right) \cdot \mathbf{u}$

2) If $\mathbf{f}(\mathbf{v}) = \mathbf{f}_1(\mathbf{v}) \times \mathbf{f}_2(\mathbf{v})$ then
$$\frac{\partial \mathbf{f}}{\partial \mathbf{v}} \cdot \mathbf{u} = \left(\frac{\partial \mathbf{f}_1}{\partial \mathbf{v}} \cdot \mathbf{u}\right) \times \mathbf{f}_2(\mathbf{v}) + \mathbf{f}_1(\mathbf{v}) \times \left(\frac{\partial \mathbf{f}_2}{\partial \mathbf{v}} \cdot \mathbf{u}\right)$$

3) If $\mathbf{f}(\mathbf{v}) = \mathbf{f}_1(\mathbf{f}_2(\mathbf{v}))$ then $\dfrac{\partial \mathbf{f}}{\partial \mathbf{v}} \cdot \mathbf{u} = \dfrac{\partial \mathbf{f}_1}{\partial \mathbf{f}_2} \cdot \left(\dfrac{\partial \mathbf{f}_2}{\partial \mathbf{v}} \cdot \mathbf{u}\right)$

Derivatives of scalar valued functions of second-order tensors

Let $f(\boldsymbol{S})$ be a real valued function of the second order tensor $\boldsymbol{S}$. Then the derivative of $f(\boldsymbol{S})$ with respect to $\boldsymbol{S}$ (or at $\boldsymbol{S}$) in the direction $\boldsymbol{T}$ is the **second order tensor** defined as

$$\frac{\partial f}{\partial \boldsymbol{S}} : \boldsymbol{T} = Df(\boldsymbol{S})[\boldsymbol{T}] = \left[\frac{d}{d\alpha}\ f(\boldsymbol{S} + \alpha\ \boldsymbol{T})\right]_{\alpha=0}$$

for all second order tensors $\boldsymbol{T}$.

Properties:

1) If $f(\boldsymbol{S}) = f_1(\boldsymbol{S}) + f_2(\boldsymbol{S})$ then $\dfrac{\partial f}{\partial \boldsymbol{S}} : \boldsymbol{T} = \left(\dfrac{\partial f_1}{\partial \boldsymbol{S}} + \dfrac{\partial f_2}{\partial \boldsymbol{S}}\right) : \boldsymbol{T}$

2) If $f(\boldsymbol{S}) = f_1(\boldsymbol{S})\ f_2(\boldsymbol{S})$ then

$$\frac{\partial f}{\partial \boldsymbol{S}} : \boldsymbol{T} = \left(\frac{\partial f_1}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)\ f_2(\boldsymbol{S}) + f_1(\boldsymbol{S})\ \left(\frac{\partial f_2}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)$$

3) If $f(\boldsymbol{S}) = f_1(f_2(\boldsymbol{S}))$ then $\dfrac{\partial f}{\partial \boldsymbol{S}} : \boldsymbol{T} = \dfrac{\partial f_1}{\partial f_2}\ \left(\dfrac{\partial f_2}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)$

Derivatives of tensor valued functions of second-order tensors

Let $\boldsymbol{F}(\boldsymbol{S})$ be a second order tensor valued function of the second order tensor $\boldsymbol{S}$. Then the derivative of $\boldsymbol{F}(\boldsymbol{S})$ with respect to $\boldsymbol{S}$ (or at $\boldsymbol{S}$) in the direction $\boldsymbol{T}$ is the **fourth order tensor** defined as

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{S}} : \boldsymbol{T} = D\boldsymbol{F}(\boldsymbol{S})[\boldsymbol{T}] = \left[\frac{d}{d\alpha}\ \boldsymbol{F}(\boldsymbol{S} + \alpha\ \boldsymbol{T})\right]_{\alpha=0}$$

for all second order tensors $\boldsymbol{T}$.

Properties:

1) If $\boldsymbol{F}(\boldsymbol{S}) = \boldsymbol{F}_1(\boldsymbol{S}) + \boldsymbol{F}_2(\boldsymbol{S})$ then $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{S}} : \boldsymbol{T} = \left(\dfrac{\partial \boldsymbol{F}_1}{\partial \boldsymbol{S}} + \dfrac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{S}}\right) : \boldsymbol{T}$

2) If $\boldsymbol{F}(\boldsymbol{S}) = \boldsymbol{F}_1(\boldsymbol{S}) \cdot \boldsymbol{F}_2(\boldsymbol{S})$ then

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{S}} : \boldsymbol{T} = \left(\frac{\partial \boldsymbol{F}_1}{\partial \boldsymbol{S}} : \boldsymbol{T}\right) \cdot \boldsymbol{F}_2(\boldsymbol{S}) + \boldsymbol{F}_1(\boldsymbol{S}) \cdot \left(\frac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)$$

3) If $\boldsymbol{F}(\boldsymbol{S}) = \boldsymbol{F}_1(\boldsymbol{F}_2(\boldsymbol{S}))$ then $\dfrac{\partial \boldsymbol{F}}{\partial \boldsymbol{S}} : \boldsymbol{T} = \dfrac{\partial \boldsymbol{F}_1}{\partial \boldsymbol{F}_2} : \left(\dfrac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)$

4) If $f(\boldsymbol{S}) = f_1(\boldsymbol{F}_2(\boldsymbol{S}))$ then $\dfrac{\partial f}{\partial \boldsymbol{S}} : \boldsymbol{T} = \dfrac{\partial f_1}{\partial \boldsymbol{F}_2} : \left(\dfrac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{S}} : \boldsymbol{T}\right)$

Gradient of a tensor field

The gradient, $\boldsymbol{\nabla T}$, of a tensor field $\boldsymbol{T}(\mathbf{x})$ in the direction of an arbitrary constant vector $\mathbf{c}$ is defined as:

$$\boldsymbol{\nabla T} \cdot \mathbf{c} = \left.\frac{d}{d\alpha}\, \boldsymbol{T}(\mathbf{x} + \alpha \mathbf{c})\right|_{\alpha=0}$$

The gradient of a tensor field of order n is a tensor field of order $n + 1$.

Cartesian coordinates

> *Note: the Einstein summation convention of summing on repeated indices is used below.*

If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the basis vectors in a Cartesian coordinate system, with coordinates of points denoted by (x_1,x_2,x_3), then the gradient of the tensor field $\boldsymbol{T}$ is given by

$$\boldsymbol{\nabla T} = \frac{\partial \boldsymbol{T}}{\partial x_i} \otimes \mathbf{e}_i$$

Since the basis vectors do not vary in a Cartesian coordinate system we have the following relations for the gradients of a scalar field φ, a vector field $\mathbf{v}$, and a second-order tensor field $\boldsymbol{S}$.

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial x_i}\, \mathbf{e}_i$$

$$\boldsymbol{\nabla}\mathbf{v} = \frac{\partial (v_j \mathbf{e}_j)}{\partial x_i} \otimes \mathbf{e}_i = \frac{\partial v_j}{\partial x_i}\, \mathbf{e}_j \otimes \mathbf{e}_i$$

$$\boldsymbol{\nabla S} = \frac{\partial (S_{jk} \mathbf{e}_j \otimes \mathbf{e}_k)}{\partial x_i} \otimes \mathbf{e}_i = \frac{\partial S_{jk}}{\partial x_i}\, \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_i$$

Curvilinear coordinates

If $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$ are the contravariant basis vectors in a curvilinear coordinate system, with coordinates of points denoted by (ξ^1,ξ^2,ξ^3), then the gradient of the tensor field $\boldsymbol{T}$ is given by.

$$\boldsymbol{\nabla T} = \frac{\partial \boldsymbol{T}}{\partial \xi^i} \otimes \mathbf{g}^i$$

From this definition we have the following relations for the gradients of a scalar field φ, a vector field $\mathbf{v}$, and a second-order tensor field $\boldsymbol{S}$.

$$\boldsymbol{\nabla}\phi = \frac{\partial \phi}{\partial \xi^i}\,\mathbf{g}^i$$

$$\boldsymbol{\nabla}\mathbf{v} = \frac{\partial (v^j \mathbf{g}_j)}{\partial \xi^i} \otimes \mathbf{g}^i = \left(\frac{\partial v^j}{\partial \xi^i} + v^k\ \Gamma^j_{ik}\right)\ \mathbf{g}_j \otimes \mathbf{g}^i = \left(\frac{\partial v_j}{\partial \xi^i} - v_k\ \Gamma^k_{ij}\right)\ \mathbf{g}^j \otimes \mathbf{g}^i$$

$$\boldsymbol{\nabla}\boldsymbol{S} = \frac{\partial (S_{jk}\ \mathbf{g}^j \otimes \mathbf{g}^k)}{\partial \xi^i} \otimes \mathbf{g}^i = \left(\frac{\partial S_{jk}}{\partial \xi_i} - S_{lk}\ \Gamma^l_{ij} - S_{jl}\ \Gamma^l_{ik}\right)\ \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^i$$

where the Christoffel symbol $\Gamma^k_{ij}$is defined using

$$\Gamma^k_{ij}\ \mathbf{g}_k = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \quad \Longrightarrow \quad \Gamma^k_{ij} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}_k = -\mathbf{g}_i \cdot \frac{\partial \mathbf{g}^k}{\partial \xi^j}$$

Cylindrical polar coordinates

In cylindrical coordinates, the gradient is given by

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial\phi}{\partial\theta}\,\mathbf{e}_\theta + \frac{\partial\phi}{\partial z}\,\mathbf{e}_z$$

$$\begin{aligned}\boldsymbol{\nabla}\mathbf{v} = {} & \frac{\partial v_r}{\partial r}\,\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right)\,\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial v_r}{\partial z}\,\mathbf{e}_r\otimes\mathbf{e}_z \\ & + \frac{\partial v_\theta}{\partial r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right)\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial v_\theta}{\partial z}\,\mathbf{e}_\theta\otimes\mathbf{e}_z \\ & + \frac{\partial v_z}{\partial r}\,\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial v_z}{\partial\theta}\,\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial v_z}{\partial z}\,\mathbf{e}_z\otimes\mathbf{e}_z\end{aligned}$$

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{S} = {} & \frac{\partial S_{rr}}{\partial r}\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{rr}}{\partial\theta} - (S_{\theta r} + S_{r\theta})\right]\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial S_{rr}}{\partial z}\,\mathbf{e}_r\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\ & + \frac{\partial S_{r\theta}}{\partial r}\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{r\theta}}{\partial\theta} + (S_{rr} - S_{\theta\theta})\right]\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial S_{r\theta}}{\partial z}\,\mathbf{e}_r\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\ & + \frac{\partial S_{rz}}{\partial r}\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{rz}}{\partial\theta} - S_{\theta z}\right]\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial S_{rz}}{\partial z}\,\mathbf{e}_r\otimes\mathbf{e}_z\otimes\mathbf{e}_z \\ & + \frac{\partial S_{\theta r}}{\partial r}\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{\theta r}}{\partial\theta} + (S_{rr} - S_{\theta\theta})\right]\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial S_{\theta r}}{\partial z}\,\mathbf{e}_\theta\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\ & + \frac{\partial S_{\theta\theta}}{\partial r}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{\theta\theta}}{\partial\theta} + (S_{r\theta} + S_{\theta r})\right]\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial S_{\theta\theta}}{\partial z}\,\mathbf{e}_\theta\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\ & + \frac{\partial S_{\theta z}}{\partial r}\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{\theta z}}{\partial\theta} + S_{rz}\right]\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial S_{\theta z}}{\partial z}\,\mathbf{e}_\theta\otimes\mathbf{e}_z\otimes\mathbf{e}_z \\ & + \frac{\partial S_{zr}}{\partial r}\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{zr}}{\partial\theta} - S_{z\theta}\right]\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_\theta + \frac{\partial S_{zr}}{\partial z}\,\mathbf{e}_z\otimes\mathbf{e}_r\otimes\mathbf{e}_z \\ & + \frac{\partial S_{z\theta}}{\partial r}\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{z\theta}}{\partial\theta} + S_{zr}\right]\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \frac{\partial S_{z\theta}}{\partial z}\,\mathbf{e}_z\otimes\mathbf{e}_\theta\otimes\mathbf{e}_z \\ & + \frac{\partial S_{zz}}{\partial r}\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\,\frac{\partial S_{zz}}{\partial\theta}\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_\theta + \frac{\partial S_{zz}}{\partial z}\,\mathbf{e}_z\otimes\mathbf{e}_z\otimes\mathbf{e}_z\end{aligned}$$

Divergence of a tensor field

The divergence of a tensor field $\boldsymbol{T}(\mathbf{x})$ is defined using the recursive relation

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})\cdot\mathbf{c} = \boldsymbol{\nabla}\cdot(\mathbf{c}\cdot\boldsymbol{T})\ ; \qquad \boldsymbol{\nabla}\cdot\mathbf{v} = \mathrm{tr}(\boldsymbol{\nabla}\mathbf{v})$$

where $\mathbf{c}$ is an arbitrary constant vector and $\mathbf{v}$ is a vector field. If $\boldsymbol{T}$ is a tensor field of order $n > 1$ then the divergence of the field is a tensor of order $n - 1$.

Cartesian coordinates

Note: the Einstein summation convention of summing on repeated indices is used below.

In a Cartesian coordinate system we have the following relations for the divergences of a vector field $\mathbf{v}$and a second-order tensor field $\boldsymbol{S}$.

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{S} = \frac{\partial S_{ik}}{\partial x_i}\, \mathbf{e}_k$$

Curvilinear coordinates

In curvilinear coordinates, the divergences of a vector field $\mathbf{v}$and a second-order tensor field $\boldsymbol{S}$are

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \left(\frac{\partial v_i}{\partial \xi^i} - v_k\ \Gamma^k_{ii} \right)$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{S} = \left(\frac{\partial S_{ik}}{\partial \xi_i} - S_{lk}\ \Gamma^l_{ii} - S_{il}\ \Gamma^l_{ik} \right)\ \mathbf{g}^k$$

Cylindrical polar coordinates

In cylindrical polar coordinates

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) + \frac{\partial v_z}{\partial z}$$

$$\begin{aligned}\boldsymbol{\nabla} \cdot \boldsymbol{S} = {} & \frac{\partial S_{rr}}{\partial r}\ \mathbf{e}_r + \frac{\partial S_{r\theta}}{\partial r}\ \mathbf{e}_\theta + \frac{\partial S_{rz}}{\partial r}\ \mathbf{e}_z \\ & + \frac{1}{r}\left[\frac{\partial S_{\theta r}}{\partial \theta} + (S_{rr} - S_{\theta\theta}) \right]\ \mathbf{e}_r + \frac{1}{r}\left[\frac{\partial S_{\theta\theta}}{\partial \theta} + (S_{r\theta} + S_{\theta r}) \right]\ \mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial S_{\theta z}}{\partial \theta} + S_{rz} \right]\ \mathbf{e}_z \\ & + \frac{\partial S_{zr}}{\partial z}\ \mathbf{e}_r + \frac{\partial S_{z\theta}}{\partial z}\ \mathbf{e}_\theta + \frac{\partial S_{zz}}{\partial z}\ \mathbf{e}_z\end{aligned}$$

Curl of a tensor field

The curl of an order-*n* > 1 tensor field $\boldsymbol{T}(\mathbf{x})$is also defined using the recursive relation

$$(\boldsymbol{\nabla} \times \boldsymbol{T}) \cdot \mathbf{c} = \boldsymbol{\nabla} \times (\mathbf{c} \cdot \boldsymbol{T}) \; ; \qquad (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{c} = \boldsymbol{\nabla} \cdot (\mathbf{v} \times \mathbf{c})$$

where $\mathbf{c}$is an arbitrary constant vector and $\mathbf{v}$is a vector field.

Curl of a first-order tensor (vector) field

Consider a vector field $\mathbf{v}$and an arbitrary constant vector $\mathbf{c}$. In index notation, the cross product is given by

$$\mathbf{v} \times \mathbf{c} = e_{ijk} \; v_j \; a_k \; \mathbf{e}_i$$

where e_{ijk} is the permutation symbol. Then,

$$\boldsymbol{\nabla} \cdot (\mathbf{v} \times \mathbf{c}) = e_{ijk} \; v_{j,i} \; c_k = (e_{ijk} \; v_{j,i} \; \mathbf{e}_k) \cdot \mathbf{c} = (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{c}$$

Therefore

$$\boldsymbol{\nabla} \times \mathbf{v} = e_{ijk} \; v_{j,i} \; \mathbf{e}_k$$

Curl of a second-order tensor field

For a second-order tensor $\boldsymbol{S}$

$$\mathbf{c} \cdot \boldsymbol{S} = c_m \; S_{mj} \; \mathbf{e}_j$$

Hence, using the definition of the curl of a first-order tensor field,

$$\boldsymbol{\nabla} \times (\mathbf{c} \cdot \boldsymbol{S}) = e_{ijk} \; c_m \; S_{mj,i} \; \mathbf{e}_k = (e_{ijk} \; S_{mj,i} \; \mathbf{e}_k \otimes \mathbf{e}_m) \cdot \mathbf{c} = (\boldsymbol{\nabla} \times \boldsymbol{S}) \cdot \mathbf{c}$$

Therefore, we have

$$\boldsymbol{\nabla} \times \boldsymbol{S} = e_{ijk} \; S_{mj,i} \; \mathbf{e}_k \otimes \mathbf{e}_m$$

Identities involving the curl of a tensor field

The most commonly used identity involving the curl of a tensor field, $\boldsymbol{T}$, is

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \boldsymbol{T}) = \mathbf{0}$$

This identity hold for tensor fields of all orders. For the important case of a second-order tensor, $\boldsymbol{S}$, this identity implies that

$$\boldsymbol{\nabla} \times \boldsymbol{S} = \mathbf{0} \quad \Longrightarrow \quad S_{mi,j} - S_{mj,i} = 0$$

Derivative of the determinant of a second-order tensor

The derivative of the determinant of a second order tensor $\boldsymbol{A}$ is given by

$$\frac{\partial}{\partial \boldsymbol{A}} \det(\boldsymbol{A}) = \det(\boldsymbol{A})\ [\boldsymbol{A}^{-1}]^T\ .$$

In an orthonormal basis, the components of $\boldsymbol{A}$ can be written as a matrix A. In that case, the right hand side corresponds the cofactors of the matrix.

Derivatives of the invariants of a second-order tensor

The principal invariants of a second order tensor are

$$I_1(\boldsymbol{A}) = \mathrm{tr}\boldsymbol{A}$$
$$I_2(\boldsymbol{A}) = \frac{1}{2}\left[(\mathrm{tr}\boldsymbol{A})^2 - \mathrm{tr}\boldsymbol{A}^2\right]$$
$$I_3(\boldsymbol{A}) = \det(\boldsymbol{A})$$

The derivatives of these three invariants with respect to $\boldsymbol{A}$ are

$$\frac{\partial I_1}{\partial \boldsymbol{A}} = \boldsymbol{1}$$
$$\frac{\partial I_2}{\partial \boldsymbol{A}} = I_1\ \boldsymbol{1} - \boldsymbol{A}^T$$
$$\frac{\partial I_3}{\partial \boldsymbol{A}} = \det(\boldsymbol{A})\ [\boldsymbol{A}^{-1}]^T = I_2\ \boldsymbol{1} - \boldsymbol{A}^T\ (I_1\ \boldsymbol{1} - \boldsymbol{A}^T) = (\boldsymbol{A}^2 - I_1\ \boldsymbol{A} + I_2\ \boldsymbol{1})^T$$

Derivative of the second-order identity tensor

Let $\boldsymbol{1}$ be the second order identity tensor. Then the derivative of this tensor with respect to a second order tensor $\boldsymbol{A}$ is given by

$$\frac{\partial \boldsymbol{1}}{\partial \boldsymbol{A}} : \boldsymbol{T} = \boldsymbol{0} : \boldsymbol{T} = \boldsymbol{0}$$

This is because $\boldsymbol{1}$ is independent of $\boldsymbol{A}$.

Derivative of a second-order tensor with respect to itself

Let $\boldsymbol{A}$ be a second order tensor. Then

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} : \boldsymbol{T} = \left[\frac{\partial}{\partial \alpha}(\boldsymbol{A} + \alpha\ \boldsymbol{T})\right]_{\alpha=0} = \boldsymbol{T} = \mathbf{I} : \boldsymbol{T}$$

Therefore,

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \mathbf{I}$$

Here $\mathbf{I}$ is the fourth order identity tensor. In index notation with respect to an orthonormal basis

$$\mathbf{I} = \delta_{ik}\ \delta_{jl}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

This result implies that

$$\frac{\partial \boldsymbol{A}^T}{\partial \boldsymbol{A}} : \boldsymbol{T} = \mathbf{I}^T : \boldsymbol{T} = \boldsymbol{T}^T$$

where

$$\mathbf{I}^T = \delta_{jk}\ \delta_{il}\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

Therefore, if the tensor $\boldsymbol{A}$ is symmetric, then the derivative is also symmetric and we get

$$\frac{\partial \boldsymbol{A}}{\partial \boldsymbol{A}} = \mathbf{I}^{(s)} = \frac{1}{2}\ (\mathbf{I} + \mathbf{I}^T)$$

where the symmetric fourth order identity tensor is

$$\mathbf{I}^{(s)} = \frac{1}{2}\ (\delta_{ik}\ \delta_{jl} + \delta_{il}\ \delta_{jk})\ \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

Derivative of the inverse of a second-order tensor

Let $\boldsymbol{A}$ and $\boldsymbol{T}$ be two second order tensors, then

$$\frac{\partial}{\partial \boldsymbol{A}}\left(\boldsymbol{A}^{-1}\right) : \boldsymbol{T} = -\boldsymbol{A}^{-1} \cdot \boldsymbol{T} \cdot \boldsymbol{A}^{-1}$$

In index notation with respect to an orthonormal basis

$$\frac{\partial A^{-1}_{ij}}{\partial A_{kl}}\ T_{kl} = -A^{-1}_{ik}\ T_{kl}\ A^{-1}_{lj} \implies \frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -A^{-1}_{ik}\ A^{-1}_{lj}$$

We also have

$$\frac{\partial}{\partial \boldsymbol{A}}\left(\boldsymbol{A}^{-T}\right) : \boldsymbol{T} = -\boldsymbol{A}^{-T} \cdot \boldsymbol{T} \cdot \boldsymbol{A}^{-T}$$

In index notation

$$\frac{\partial A^{-1}_{ji}}{\partial A_{kl}}\ T_{kl} = -A^{-1}_{jk}\ T_{kl}\ A^{-1}_{li} \implies \frac{\partial A^{-1}_{ji}}{\partial A_{kl}} = -A^{-1}_{li}\ A^{-1}_{jk}$$

If the tensor $\boldsymbol{A}$is symmetric then

$$\frac{\partial A^{-1}_{ij}}{\partial A_{kl}} = -\frac{1}{2}\left(A^{-1}_{ik}\ A^{-1}_{jl} + A^{-1}_{il}\ A^{-1}_{jk}\right)$$

Chapter 8

Compatibility (Mechanics)

In continuum mechanics, a **compatible** deformation (or strain) tensor field in a body is that *unique* field that is obtained when the body is subjected to a continuous, single-valued, displacement field. **Compatibility** is the study of the conditions under which such a displacement field can be guaranteed. Compatibility conditions are particular cases of integrability conditions and were first derived for linear elasticity by Barré de Saint-Venant in 1864 and proved rigorously by Beltrami in 1886.

In the continuum description of a solid body we imagine the body to be composed of a set of infinitesimal volumes or material points. Each volume is assumed to be connected to its neighbors without any gaps or overlaps. Certain mathematical conditions have to be satisfied to ensure that gaps/overlaps do not develop when a continuum body is deformed. A body that deforms without developing any gaps/overlaps is called a **compatible** body. **Compatibility conditions** are mathematical conditions that determine whether a particular deformation will leave a body in a compatible state.

In the context of infinitesimal strain theory, these conditions are equivalent to stating that the displacements in a body can be obtained by integrating the strains. Such an integration is possible if the Saint-Venant's tensor (or incompatibility tensor) $\boldsymbol{R}(\boldsymbol{\varepsilon})$ vanishes in a simply-connected body where $\boldsymbol{\varepsilon}$ is the infinitesimal strain tensor and

$$\boldsymbol{R} := \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}) \ .$$

For finite deformations the compatibility conditions take the form

$$\boldsymbol{R} := \boldsymbol{\nabla} \times \boldsymbol{F} = \boldsymbol{0}$$

where $\boldsymbol{F}$ is the deformation gradient.

Compatibility conditions for infinitesimal strains

The compatibility conditions in linear elasticity are obtained by observing that there are six strain-displacement relations that are functions of only three unknown displacements. This suggests that the three displacements may be removed from the system of equations without loss of information. The resulting expressions in terms of only the strains provide constraints on the possible forms of a strain field.

2-dimensions

For two-dimensional, plane strain problems the strain-displacement relations are

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}\,; \quad \varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] \quad ; \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

Combining these relations gives us the two-dimensional compatibility condition for strains

$$\frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} - 2\frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_1^2} = 0$$

The only displacement field that is allowed by a compatible plane strain field is a **plane displacement** field, i.e., $\mathbf{u} = \mathbf{u}(x_1, x_2)$.

3-dimensions

In three dimensions, in addition to two more equations of the form seen for two dimensions, there are three more equations of the form

$$\frac{\partial^2 \varepsilon_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left[\frac{\partial \varepsilon_{23}}{\partial x_1} + \frac{\partial \varepsilon_{31}}{\partial x_2} - \frac{\partial \varepsilon_{12}}{\partial x_3}\right]$$

Therefore there are **six** different compatibility conditions. We can write these conditions in index notation as

$$e_{ikr}\ e_{jls}\ \varepsilon_{ij,kl} = 0$$

where e_{ijk} is the permutation symbol. In direct tensor notation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}) = \mathbf{0}$$

where the curl operator can be expressed in a orthonormal coordinate system as $\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} = e_{ijk}\varepsilon_{rj,i}\mathbf{e}_k \otimes \mathbf{e}_r$.

The second-order tensor

$$\boldsymbol{R} := \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\varepsilon}) \; ; \quad R_{rs} := e_{ikr} \; e_{jls} \; \varepsilon_{ij,kl}$$

is known as the **incompatibility tensor**.

Compatibility conditions for finite strains

For solids in which the deformations are not required to be small, the compatibility conditions take the form

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$$

where $\boldsymbol{F}$ is the deformation gradient. In terms of components with respect to a Cartesian coordinate system we can write these compatibility relations as

$$e_{ABC} \; \frac{\partial F_{iB}}{\partial X_A} = 0$$

This condition is **necessary** if the deformation is to be continuous and derived from the mapping $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The same condition is also **sufficient** to ensure compatibility in a **simply connected** body.

Compatibility condition for the right Cauchy-Green deformation tensor

The compatibility condition for the right Cauchy-Green deformation tensor can be expressed as

$$R^{\gamma}_{\alpha\beta\rho} := \frac{\partial}{\partial X^{\rho}}[\Gamma^{\gamma}_{\alpha\beta}] - \frac{\partial}{\partial X^{\beta}}[\Gamma^{\gamma}_{\alpha\rho}] + \Gamma^{\gamma}_{\mu\rho} \; \Gamma^{\mu}_{\alpha\beta} - \Gamma^{\gamma}_{\mu\beta} \; \Gamma^{\mu}_{\alpha\rho} = 0$$

where Γ^{k}_{ij} is the Christoffel symbol of the second kind. The quantity R^{m}_{ijk} represents the mixed components of the Riemann-Christoffel curvature tensor.

The general compatibility problem

The problem of compatibility in continuum mechanics involves the determination of allowable single-valued continuous fields on simply connected bodies. More precisely, the problem may be stated in the following manner .

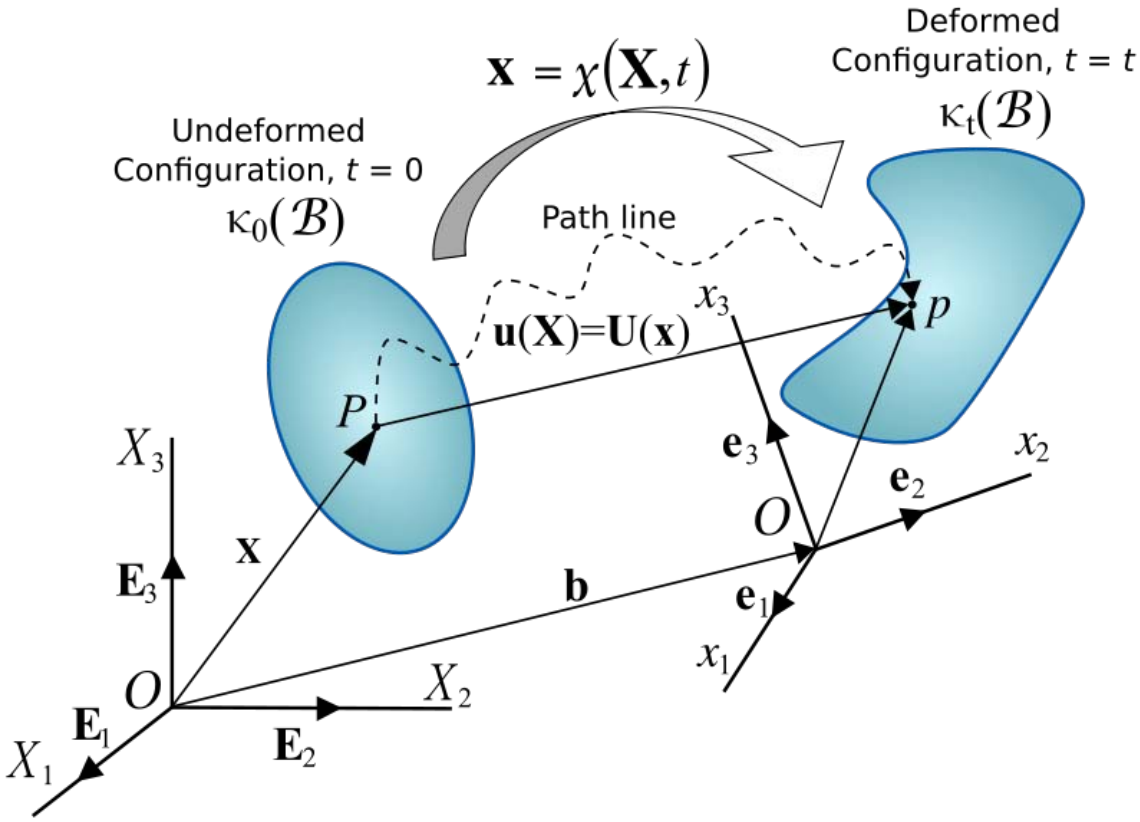


Figure 1. Motion of a continuum body.

Consider the deformation of a body shown in Figure 1. If we express all vectors in terms of the reference coordinate system $\{(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3), O\}$, the displacement of a point in the body is given by

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \; ; \quad u_i = x_i - X_i$$

Also

$$\boldsymbol{\nabla}\mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \; ; \quad \boldsymbol{\nabla}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

What conditions on a given second-order tensor field $\boldsymbol{A}(\mathbf{X})$ on a body are necessary and sufficient so that there exists a unique vector field $\mathbf{v}(\mathbf{X})$ that satisfies

$$\boldsymbol{\nabla}\mathbf{v} = \boldsymbol{A} \quad \equiv \quad v_{i,j} = A_{ij}$$

Necessary conditions

For the necessary conditions we assume that the field $\mathbf{v}$ exists and satisfies $v_{i,j} = A_{ij}$. Then

$$v_{i,jk} = A_{ij,k} \; ; \quad v_{i,kj} = A_{ik,j}$$

Since changing the order of differentiation does not affect the result we have

$$v_{i,jk} = v_{i,kj}$$

Hence

$$A_{ij,k} = A_{ik,j}$$

From the well known identify for the curl of a tensor we get the necessary condition

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \mathbf{0}$$

Sufficient conditions

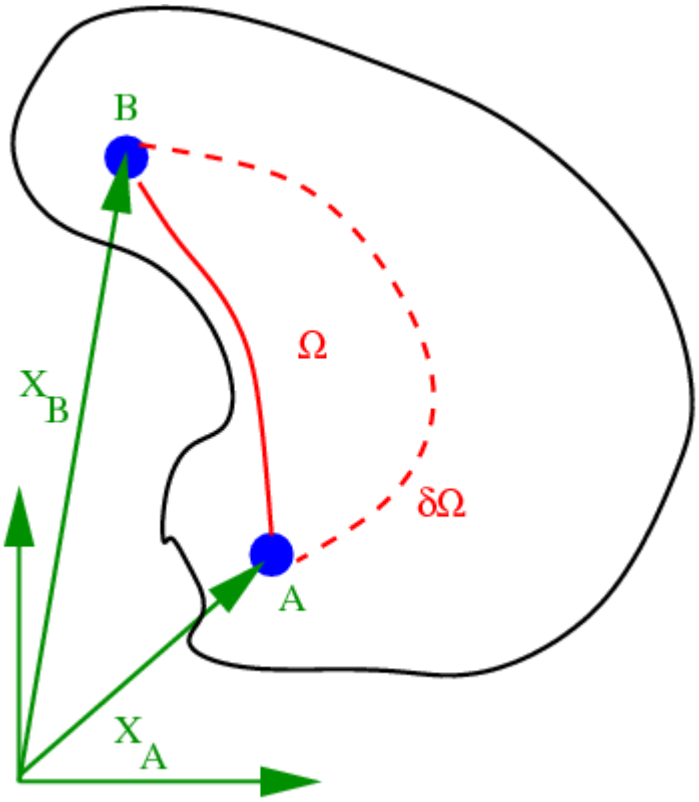


Figure 2. Integration paths used in proving the sufficiency conditions for compatibility.

To prove that this condition is sufficient to guarantee existence of a compatible second-order tensor field, we start with the assumption that a field $\boldsymbol{A}$exists such that $\boldsymbol{\nabla} \times \boldsymbol{A} = \mathbf{0}$. We will integrate this field to find the vector field $\mathbf{v}$along a line between points *A* and *B*, i.e.,

$$\mathbf{v}(\mathbf{X}_B) - \mathbf{v}(\mathbf{X}_A) = \int_{\mathbf{X}_A}^{\mathbf{X}_B} \boldsymbol{\nabla}\mathbf{v} \cdot \ d\mathbf{X} = \int_{\mathbf{X}_A}^{\mathbf{X}_B} \boldsymbol{A}(\mathbf{X}) \cdot d\mathbf{X}$$

If the vector field $\mathbf{v}$ is to be single-valued then the value of the integral should be independent of the path taken to go from *A* to *B*.

From Stokes theorem, the integral of a second order tensor along a closed path is given by

$$\oint_{\partial\Omega} \boldsymbol{A} \ ds = \int_{\Omega} \mathbf{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) \ da$$

Using the assumption that the curl of $\boldsymbol{A}$ is zero, we get

$$\oint_{\partial\Omega} \boldsymbol{A} \ ds = 0 \quad \Longrightarrow \quad \int_{AB} \boldsymbol{A} \cdot d\mathbf{X} + \int_{BA} \boldsymbol{A} \cdot d\mathbf{X} = 0$$

Hence the integral is path independent and the compatibility condition is sufficient to ensure a unique $\mathbf{v}$ field, provided that the body is simply connected.

Compatibility of the deformation gradient

The compatibility condition for the deformation gradient is obtained directly from the above proof by observing that

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \boldsymbol{\nabla}\mathbf{x}$$

Then the necessary and sufficient conditions for the existence of a compatible $\boldsymbol{F}$ field over a simply connected body are

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \mathbf{0}$$

Compatibility of infinitesimal strains

The compatibility problem for small strains can be stated as follows.

Given a symmetric second order tensor field $\boldsymbol{\epsilon}$ when is it possible to construct a vector field $\mathbf{u}$ such that

$$\boldsymbol{\epsilon} = \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T]$$

Necessary conditions

Suppose that there exists $\mathbf{u}$ such that the expression for $\boldsymbol{\epsilon}$ holds. Now

$$\boldsymbol{\nabla}\mathbf{u} = \boldsymbol{\epsilon} + \boldsymbol{\omega}$$

where

$$\boldsymbol{\omega} := \frac{1}{2}[\boldsymbol{\nabla}\mathbf{u} - (\boldsymbol{\nabla}\mathbf{u})^T]$$

Therefore, in index notation,

$$\boldsymbol{\nabla}\boldsymbol{\omega} \equiv \omega_{ij,k} = \frac{1}{2}(u_{i,jk} - u_{j,ik}) = \frac{1}{2}(u_{i,jk} + u_{k,ji} - u_{j,ik} - u_{k,ji}) = \varepsilon_{ik,j} - \varepsilon_{jk,i}$$

If $\boldsymbol{\omega}$ is continuously differentiable we have $\omega_{ij,kl} = \omega_{ij,lk}$. Hence,

$$\varepsilon_{ik,jl} - \varepsilon_{jk,il} - \varepsilon_{il,jk} + \varepsilon_{jl,ik} = 0$$

In direct tensor notation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) = \mathbf{0}$$

The above are necessary conditions. If $\mathbf{w}$ is the infinitesimal rotation vector then $\boldsymbol{\nabla} \times \boldsymbol{\epsilon} = \boldsymbol{\nabla}\mathbf{w}$. Hence the necessary condition may also be written as $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{w}) = \mathbf{0}$.

Sufficient conditions

Let us now assume that the condition $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) = \mathbf{0}$ is satisfied in a portion of a body. Is this condition sufficient to guarantee the existence of a continuous, single-valued displacement field $\mathbf{u}$?

The first step in the process is to show that this condition implies that the infinitesimal rotation tensor $\boldsymbol{\omega}$ is uniquely defined. To do that we integrate $\boldsymbol{\nabla}\mathbf{w}$ along the path $\mathbf{X}_A$ to $\mathbf{X}_B$, i.e.,

$$\mathbf{w}(\mathbf{X}_B) - \mathbf{w}(\mathbf{X}_A) = \int_{\mathbf{X}_A}^{\mathbf{X}_B} \boldsymbol{\nabla}\mathbf{w} \cdot d\mathbf{X} = \int_{\mathbf{X}_A}^{\mathbf{X}_B} (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) \cdot d\mathbf{X}$$

Note that we need to know a reference $\mathbf{w}(\mathbf{X}_A)$ to fix the rigid body rotation. The field $\mathbf{w}(\mathbf{X})$ is uniquely determined only if the contour integral along a closed contour between $\mathbf{X}_A$ and $\mathbf{X}_b$ is zero, i.e.,

$$\oint_{\mathbf{X}_A}^{\mathbf{X}_B} (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) \cdot d\mathbf{X} = \mathbf{0}$$

But from Stokes' theorem for a simply-connected body and the necessary condition for compatibility

$$\oint_{\mathbf{X}_A}^{\mathbf{X}_B} (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) \cdot d\mathbf{X} = \int_{\Omega_{AB}} \mathbf{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{\epsilon})\, da = \mathbf{0}$$

Therefore the field $\mathbf{w}$ is uniquely defined which implies that the infinitesimal rotation tensor $\boldsymbol{\omega}$ is also uniquely defined, provided the body is simply connected.

In the next step of the process we will consider the uniqueness of the displacement field $\mathbf{u}$. As before we integrate the displacement gradient

$$\mathbf{u}(\mathbf{X}_B) - \mathbf{u}(\mathbf{X}_A) = \int_{\mathbf{X}_A}^{\mathbf{X}_B} \boldsymbol{\nabla}\mathbf{u} \cdot d\mathbf{X} = \int_{\mathbf{X}_A}^{\mathbf{X}_B} (\boldsymbol{\epsilon} + \boldsymbol{\omega}) \cdot d\mathbf{X}$$

From Stokes' theorem and using the relations $\boldsymbol{\nabla} \times \boldsymbol{\epsilon} = \boldsymbol{\nabla}\mathbf{w} = -\boldsymbol{\nabla} \times \Omega$ we have

$$\oint_{\mathbf{X}_A}^{\mathbf{X}_B} (\boldsymbol{\epsilon} + \boldsymbol{\omega}) \cdot d\mathbf{X} = \int_{\Omega_{AB}} \mathbf{n} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\epsilon} + \boldsymbol{\nabla} \times \boldsymbol{\omega})\, da = \mathbf{0}$$

Hence the displacement field $\mathbf{u}$ is also determined uniquely. Hence the compatibility conditions are sufficient to guarantee the existence of a unique displacement field $\mathbf{u}$ in a simply-connected body.

Compatibility for Right Cauchy-Green Deformation field

The compatibility problem for the Right Cauchy-Green deformation field can be posed as follows.

Problem: Let $\boldsymbol{C}(\mathbf{X})$ be a positive definite symmetric tensor field defined on the reference configuration. Under what conditions on $\boldsymbol{C}$ does there exist a deformed configuration marked by the position field $\mathbf{x}(\mathbf{X})$ such that

$$(1) \quad \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right)^T \cdot \left(\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right) = \boldsymbol{C}$$

Necessary conditions

Suppose that a field $\mathbf{x}(\mathbf{X})$ exists that satisfies condition (1). In terms of components with respect to a rectangular Cartesian basis

$$\frac{\partial x^i}{\partial X^\alpha}\frac{\partial x^i}{\partial X^\beta} = C_{\alpha\beta}$$

From finite strain theory we know that $C_{\alpha\beta} = g_{\alpha\beta}$. Hence we can write

$$\delta_{ij}\ \frac{\partial x^i}{\partial X^\alpha}\ \frac{\partial x^j}{\partial X^\beta} = g_{\alpha\beta}$$

For two symmetric second-order tensor field that are mapped one-to-one we also have the relation

$$G_{ij} = \frac{\partial X^\alpha}{\partial x^i}\ \frac{\partial X^\beta}{\partial x^j}\ g_{\alpha\beta}$$

From the relation between of G_{ij} and $g_{\alpha\beta}$ that $\delta_{ij} = G_{ij}$, we have

$${}_{(x)}\Gamma^k_{ij} = 0$$

Then, from the relation

$$\frac{\partial^2 x^m}{\partial X^\alpha \partial X^\beta} = \frac{\partial x^m}{\partial X^\mu}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} - \frac{\partial x^i}{\partial X^\alpha}\ \frac{\partial x^j}{\partial X^\beta}\,{}_{(x)}\Gamma^m_{ij}$$

we have

$$\frac{\partial F^m_{\ \alpha}}{\partial X^\beta} = F^m_{\ \mu}\,{}_{(X)}\Gamma^\mu_{\alpha\beta} \qquad ; \quad F^i_{\ \alpha} := \frac{\partial x^i}{\partial X^\alpha}$$

From finite strain theory we also have

$${}_{(X)}\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial X^\beta} + \frac{\partial g_{\beta\gamma}}{\partial X^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial X^\gamma}\right) ; \ {}_{(X)}\Gamma^\nu_{\alpha\beta} = g^{\nu\gamma}\,{}_{(X)}\Gamma_{\alpha\beta\gamma} ; \ g_{\alpha\beta} = C_{\alpha\beta} ; \ g^{\alpha\beta} = C^{\alpha\beta}$$

Therefore

$$ {}_{(X)}\Gamma^{\mu}_{\alpha\beta} = \frac{C^{\mu\gamma}}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^{\beta}} + \frac{\partial C_{\beta\gamma}}{\partial X^{\alpha}} - \frac{\partial C_{\alpha\beta}}{\partial X^{\gamma}}\right) $$

and we have

$$ \frac{\partial F^{m}_{\alpha}}{\partial X^{\beta}} = F^{m}_{\mu}\,\frac{C^{\mu\gamma}}{2}\left(\frac{\partial C_{\alpha\gamma}}{\partial X^{\beta}} + \frac{\partial C_{\beta\gamma}}{\partial X^{\alpha}} - \frac{\partial C_{\alpha\beta}}{\partial X^{\gamma}}\right) $$

Again, using the commutative nature of the order of differentiation, we have

$$ \frac{\partial^2 F^{m}_{\alpha}}{\partial X^{\beta}\partial X^{\rho}} = \frac{\partial^2 F^{m}_{\alpha}}{\partial X^{\rho}\partial X^{\beta}} \implies \frac{\partial F^{m}_{\mu}}{\partial X^{\rho}}\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta} + F^{m}_{\mu}\,\frac{\partial}{\partial X^{\rho}}[\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta}] = \frac{\partial F^{m}_{\mu}}{\partial X^{\beta}}\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho} + F^{m}_{\mu}\,\frac{\partial}{\partial X^{\beta}}[\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho}] $$

or

$$ F^{m}_{\gamma}\,{}_{(X)}\Gamma^{\gamma}_{\mu\rho}\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta} + F^{m}_{\mu}\,\frac{\partial}{\partial X^{\rho}}[\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta}] = F^{m}_{\gamma}\,{}_{(X)}\Gamma^{\gamma}_{\mu\beta}\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho} + F^{m}_{\mu}\,\frac{\partial}{\partial X^{\beta}}[\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho}] $$

After collecting terms we get

$$ F^{m}_{\gamma}\left({}_{(X)}\Gamma^{\gamma}_{\mu\rho}\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta} + \frac{\partial}{\partial X^{\rho}}[\,{}_{(X)}\Gamma^{\gamma}_{\alpha\beta}] - {}_{(X)}\Gamma^{\gamma}_{\mu\beta}\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho} - \frac{\partial}{\partial X^{\beta}}[\,{}_{(X)}\Gamma^{\gamma}_{\alpha\rho}]\right) = 0 $$

From the definition of F^{m}_{γ} we observe that it is invertible and hence cannot be zero. Therefore,

$$ R^{\gamma}_{\alpha\beta\rho} := \frac{\partial}{\partial X^{\rho}}[\,{}_{(X)}\Gamma^{\gamma}_{\alpha\beta}] - \frac{\partial}{\partial X^{\beta}}[\,{}_{(X)}\Gamma^{\gamma}_{\alpha\rho}] + {}_{(X)}\Gamma^{\gamma}_{\mu\rho}\,{}_{(X)}\Gamma^{\mu}_{\alpha\beta} - {}_{(X)}\Gamma^{\gamma}_{\mu\beta}\,{}_{(X)}\Gamma^{\mu}_{\alpha\rho} = 0 $$

We can show these are the mixed components of the Riemann-Christoffel curvature tensor. Therefore the necessary conditions for $\boldsymbol{C}$ -compatibility are that the Riemann-Christoffel curvature of the deformation is zero.

Sufficient conditions

The proof of sufficiency is a bit more involved. We start with the assumption that

$$ R^{\gamma}_{\alpha\beta\rho} = 0\ ; \quad g_{\alpha\beta} = C_{\alpha\beta} $$

We have to show that there exist $\mathbf{x}$ and $\mathbf{X}$ such that

$$ \frac{\partial x^{i}}{\partial X^{\alpha}}\frac{\partial x^{i}}{\partial X^{\beta}} = C_{\alpha\beta} $$

From a theorem by T.Y.Thomas we know that the system of equations

$$\frac{\partial F^i{}_\alpha}{\partial X^\beta} = F^i{}_\gamma \ {}_{(X)}\Gamma^\gamma_{\alpha\beta}$$

has unique solutions $F^i{}_\alpha$over simply connected domains if

$${}_{(X)}\Gamma^\gamma_{\alpha\beta} = {}_{(X)}\ \Gamma^\gamma_{\beta\alpha}\ ; \quad R^\gamma_{\alpha\beta\rho} = 0$$

The first of these is true from the defining of $\Gamma^i_{jk}$and the second is assumed. Hence the assumed condition gives us a unique $F^i{}_\alpha$that is C^2 continuous.

Next consider the system of equations

$$\frac{\partial x^i}{\partial X^\alpha} = F^i{}_\alpha$$

Since $F^i{}_\alpha$is C^2 and the body is simply connected there exists some solution $x^i(X^\alpha)$ to the above equations. We can show that the x^i also satisfy the property that

$$\det\left|\frac{\partial x^i}{\partial X^\alpha}\right| \neq 0$$

We can also show that the relation

$$\frac{\partial x^i}{\partial X^\alpha}\ g^{\alpha\beta}\ \frac{\partial x^j}{\partial X^\beta} = \delta^{ij}$$

implies that

$$g_{\alpha\beta} = C_{\alpha\beta} = \frac{\partial x^k}{\partial X^\alpha}\ \frac{\partial x^k}{\partial X^\beta}$$

If we associate these quantities with tensor fields we can show that $\frac{\partial \mathbf{x}}{\partial \mathbf{X}}$is invertible and the constructed tensor field satisfies the expression for $\boldsymbol{C}$.