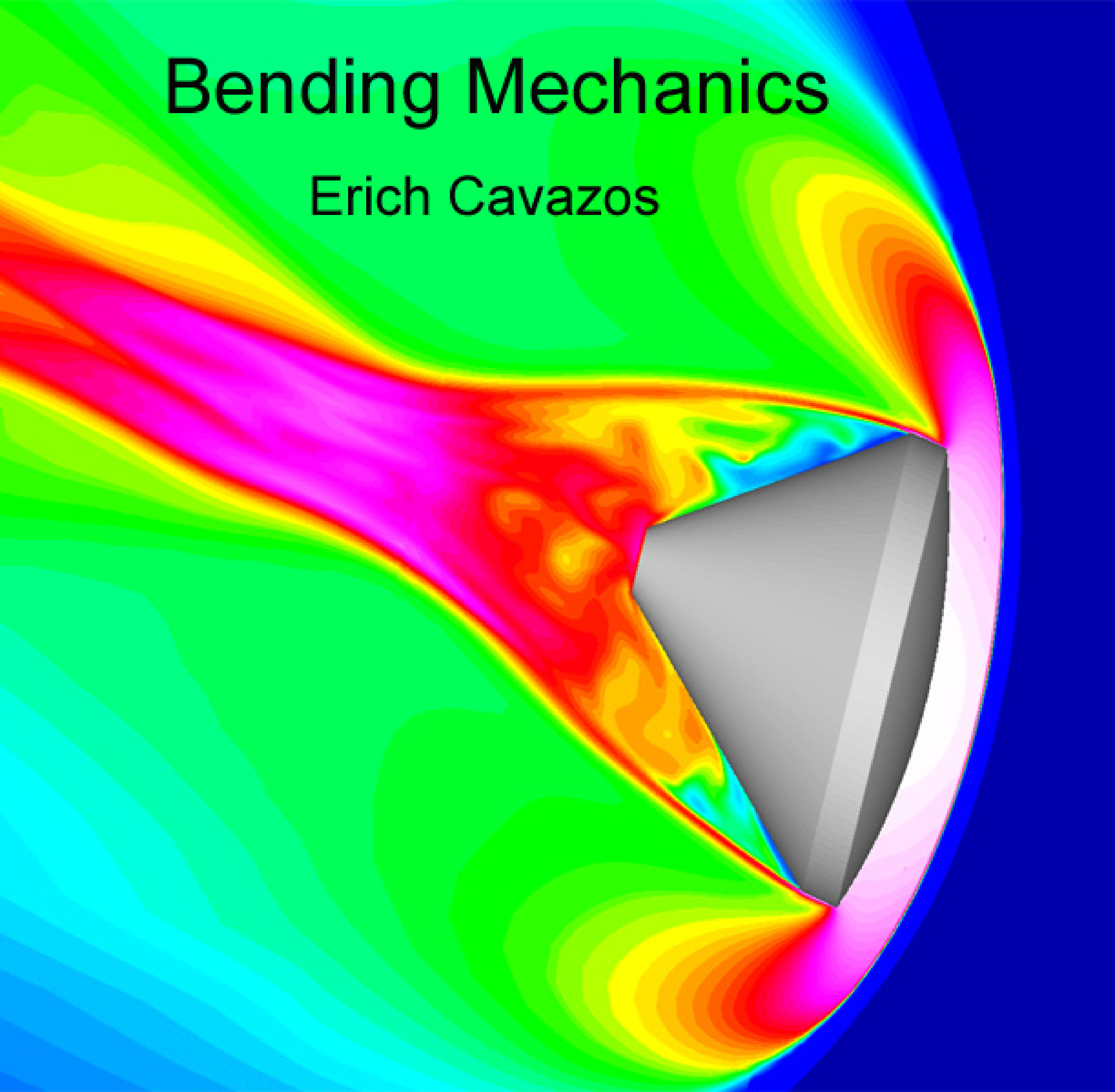
# Bending Mechanics

Erich Cavazos



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## Chapter 1

# Bending

Bending of an I-beam

In engineering mechanics, **bending** (also known as **flexure**) characterizes the behavior of a slender structural element subjected to an external load applied perpendicularly to a longitudinal axis of the element. The structural element is assumed to be such that at least one of its dimensions is a small fraction, typically 1/10 or less, of the other two. When the length is considerably longer than the width and the thickness, the element is called a beam. A closet rod sagging under the weight of clothes on clothes hangers is an example of a beam experiencing bending. On the other hand, a shell is a structure of any geometric form where the length and the width are of the same order of magnitude but the thickness of the structure (known as the 'wall') is considerably smaller. A large diameter, but thin-walled, short tube supported at its ends and loaded laterally is an example of a shell experiencing bending.

In the absence of a qualifier, the term *bending* is ambiguous because bending can occur locally in all objects. To make the usage of the term more precise, engineers refer to the *bending of rods*, the *bending of beams*, the *bending of plates*, the *bending of shells* and so on.

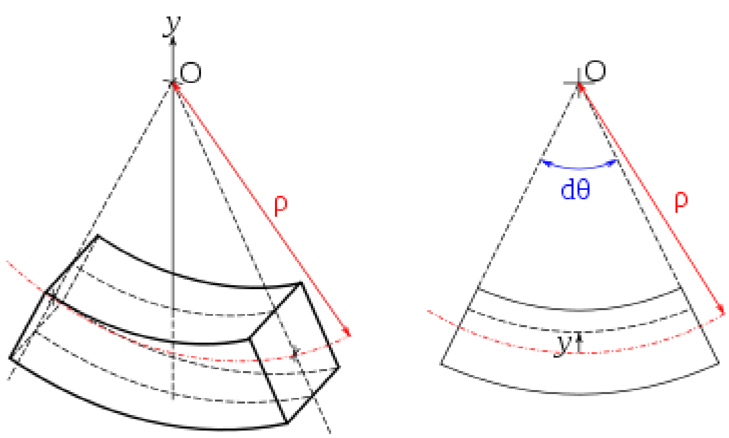
# Quasistatic bending of beams

A beam deforms and stresses develop inside it when a transverse load is applied on it. In the quasistatic case, the amount of bending deflection and the stresses that develop are assumed not to change over time. In a horizontal beam supported at the ends and loaded downwards in the middle, the material at the over-side of the beam is compressed while the material at the underside is stretched. There are two forms of internal stresses caused by lateral loads:

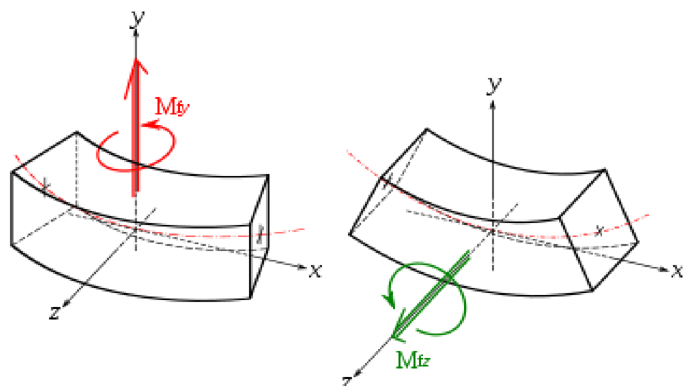
- Shear stress parallel to the lateral loading plus complementary shear stress on planes perpendicular to the load direction;
- Direct compressive stress in the upper region of the beam, and direct tensile stress in the lower region of the beam.

These last two forces form a couple or moment as they are equal in magnitude and opposite in direction. This bending moment resists the sagging deformation characteristic of a beam experiencing bending. The stress distribution in a beam can be predicted quite accurately even when some simplifying assumptions are used.

## Euler-Bernoulli bending theory




Element of a bent beam: the fibers form concentric arcs, the top fibers are compressed and bottom fibers stretched.




Bending moments in a beam

In the Euler-Bernoulli theory of slender beams, a major assumption is that 'plane sections remain plane'. In other words, any deformation due to shear across the section is not accounted for (no shear deformation). Also, this linear distribution is only applicable if the maximum stress is less than the yield stress of the material. For stresses that exceed yield, refer to article plastic bending. At yield, the maximum stress experienced in the section (at the furthest points from the neutral axis of the beam) is defined as the flexural strength.

The Euler-Bernoulli equation for the quasistatic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load $q(x)$ is

$$EI \frac{\mathrm{d}^4 w(x)}{\mathrm{d}x^4} = q(x)$$

where $E$ is the Young's modulus, $I$ is the area moment of inertia of the cross-section, and $w(x)$ is the deflection of the neutral axis of the beam.

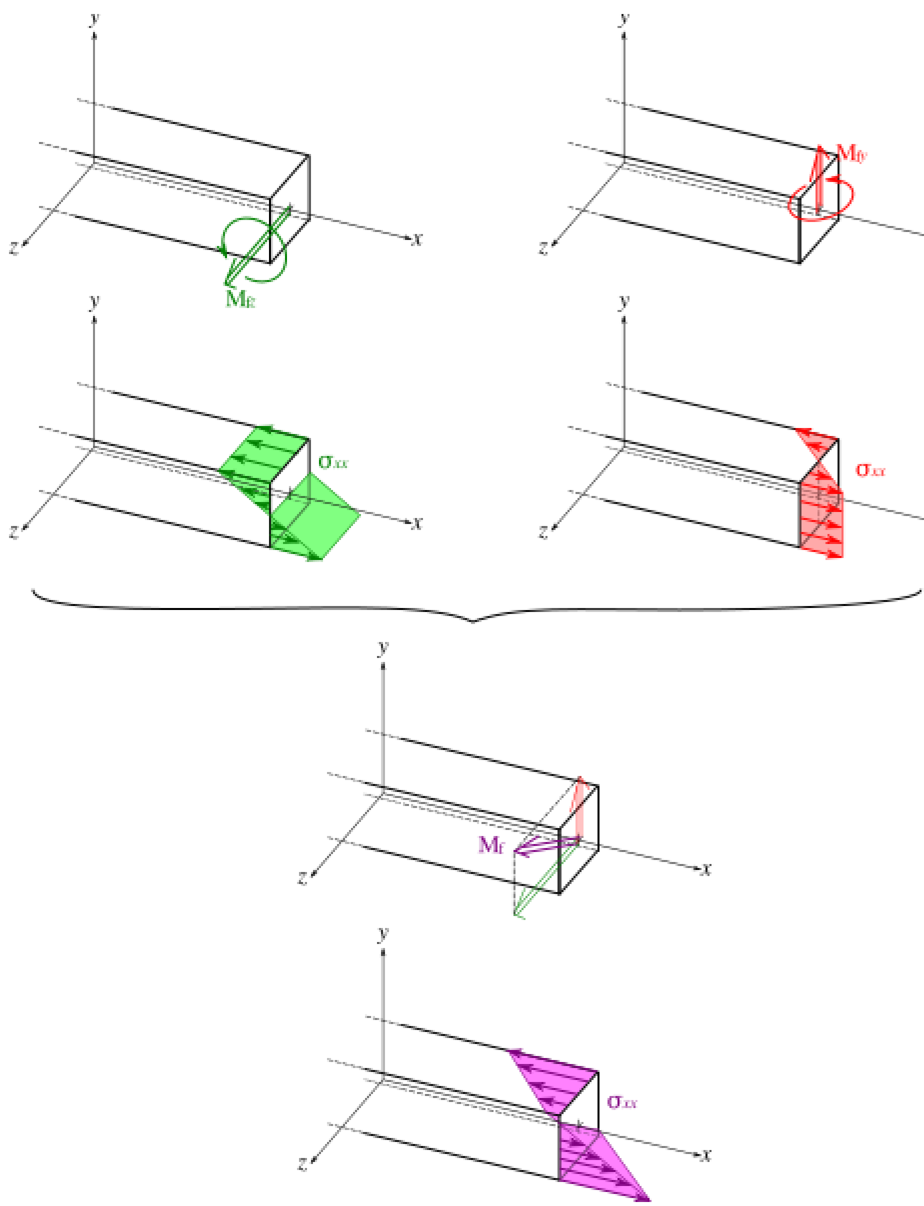
After a solution for the displacement of the beam has been obtained, the bending moment ($M$) and shear force ($Q$) in the beam can be calculated using the relations

$$M(x) = -EI \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}~; \quad Q(x) = \frac{\mathrm{d}M}{\mathrm{d}x}$$

Simple beam bending is often analyzed with the Euler-Bernoulli beam equation. The conditions for using simple bending theory are :

1. The beam is subject to pure bending. This means that the shear force is zero, and that no torsional or axial loads are present.
2. The material is isotropic and homogeneous.
3. The material obeys Hooke's law (it is linearly elastic and will not deform plastically).

4. The beam is initially straight with a cross section that is constant throughout the beam length.
5. The beam has an axis of symmetry in the plane of bending.
6. The proportions of the beam are such that it would fail by bending rather than by crushing, wrinkling or sideways buckling.
7. Cross-sections of the beam remain plane during bending.



Deflection of a beam deflected symmetrically and principle of superposition

Compressive and tensile forces develop in the direction of the beam axis under bending loads. These forces induce stresses on the beam. The maximum compressive stress is found at the uppermost edge of the beam while the maximum tensile stress is located at the lower edge of the beam. Since the stresses between these two opposing maxima vary linearly, there therefore exists a point on the linear path between them where there is no bending stress. The locus of these points is the neutral axis. Because of this area with no stress and the adjacent areas with low stress, using uniform cross section beams in bending is not a particularly efficient means of supporting a load as it does not use the full capacity of the beam until it is on the brink of collapse. Wide-flange beams (I-beams) and truss girders effectively address this inefficiency as they minimize the amount of material in this under-stressed region.

The classic formula for determining the bending stress in a beam under simple bending is:

$$\sigma = \frac{My}{I_x}$$

where

- σ is the bending stress
- *M* - the moment about the neutral axis
- *y* - the perpendicular distance to the neutral axis
- $I_x$ - the second moment of area about the neutral axis *x*

## Extensions of Euler-Bernoulli beam bending theory

### Plastic bending

The equation $\sigma = \frac{My}{I_x}$ is valid only when the stress at the extreme fiber (i.e. the portion of the beam farthest from the neutral axis) is below the yield stress of the material from which it is constructed. At higher loadings the stress distribution becomes non-linear, and ductile materials will eventually enter a *plastic hinge* state where the magnitude of the stress is equal to the yield stress everywhere in the beam, with a discontinuity at the neutral axis where the stress changes from tensile to compressive. This plastic hinge state is typically used as a limit state in the design of steel structures.

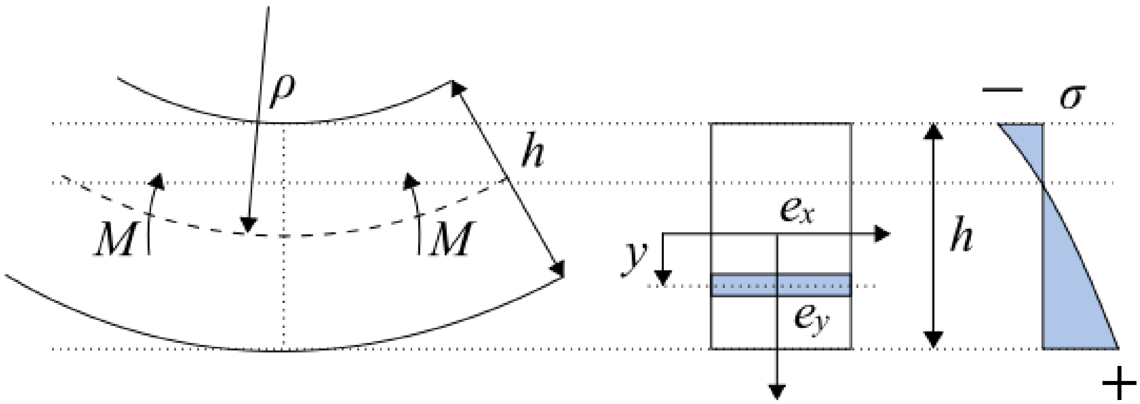
### Complex or asymmetrical bending

The equation above is only valid if the cross-section is symmetrical. For homogeneous beams with asymmetrical sections, the axial stress in the beam is given by

$$\sigma_x(y,z) = -\frac{(M_z\ I_y + M_y\ I_{yz})}{I_y\ I_z - I_{yz}^2}y + \frac{(M_y\ I_z + M_z\ I_{yz})}{I_y\ I_z - I_{yz}^2}z$$

where *y*,*z* are the coordinates of a point on the cross section at which the stress is to be determined as shown to the right, $M_y$ and $M_z$ are the bending moments about the y and z centroid axes, $I_y$ and $I_z$ are the second moments of area (distinct from moments of inertia) about the y and z axes, and $I_{yz}$ is the product of moments of area. Using this equation it is possible to calculate the bending stress at any point on the beam cross section regardless of moment orientation or cross-sectional shape. Note that $M_y, M_z, I_y, I_z, I_{yz}$ do not change from one point to another on the cross section.

**Large bending deformation**




For large deformations of the body, the stress in the cross-section is calculated using an extended version of this formula. First the following assumptions must be made:

1. Assumption of flat sections - before and after deformation the considered section of body remains flat (i.e. is not swirled).
2. Shear and normal stresses in this section that are perpendicular to the normal vector of cross section have no influence on normal stresses that are parallel to this section.

Large bending considerations should be implemented when the bending radius ρ is smaller than ten section heights h:

$\rho < 10h$

With those assumptions the stress in large bending is calculated as:

$$\sigma = \frac{F}{A} + \frac{M}{\rho A} + \frac{M}{I_x{}'} y \frac{\rho}{\rho + y}$$

where

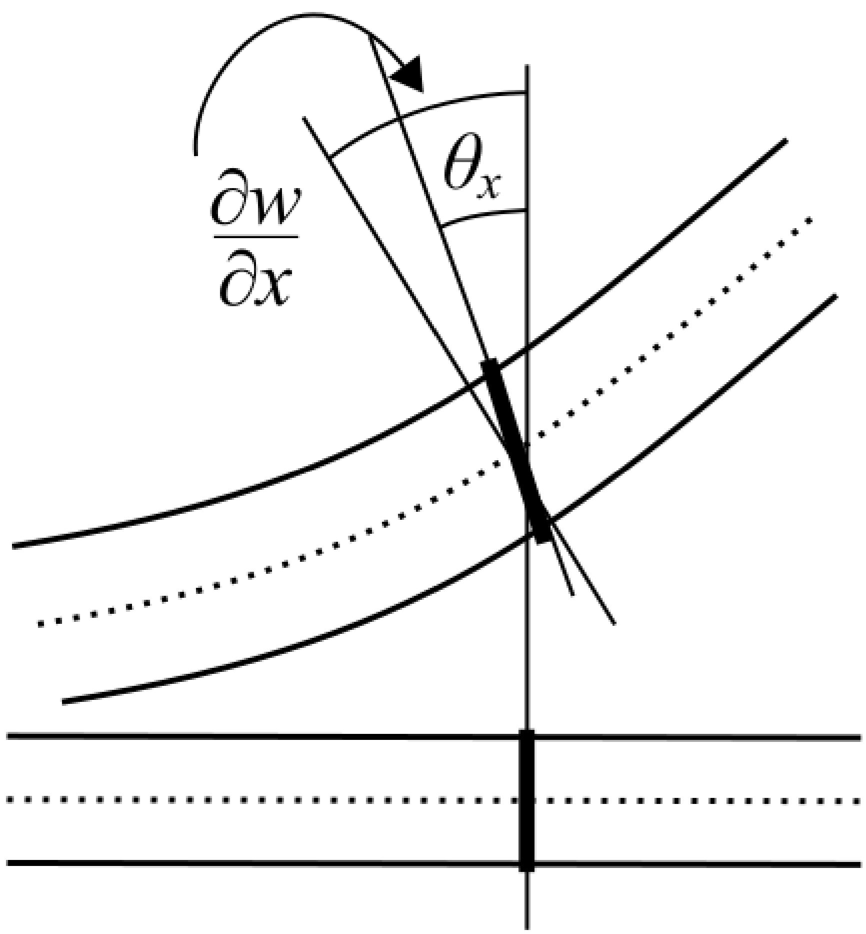
*F* is the normal force
*A* is the section area
*M* is the bending moment
ρ is the local bending radius (the radius of bending at the current section)
$I_x'$ is the area moment of inertia along the **x** axis, at the *y* place

*y* is the position along **y** axis on the section area in which the stress σ is calculated

When bending radius ρ approaches infinity and *y* is near zero, the original formula is back:

$$\sigma = \frac{F}{A} \pm \frac{My}{I}.$$

**Timoshenko bending theory**




Deformation of a Timoshenko beam. The normal rotates by an amount θ which is not equal to *dw* / *dx*.

In 1921, Timoshenko improved upon the Euler-Bernoulli theory of beams by adding the effect of shear into the beam equation. The kinematic assumptions of the Timoshenko theory are

- normals to the axis of the beam remain straight after deformation
- there is no change in beam thickness after deformation

However, normals to the axis are not required to remain perpendicular to the axis after deformation.

The equation for the quasistatic bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\ \frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - \frac{EI}{kAG}\ \frac{\mathrm{d}^2 q}{\mathrm{d}x^2}$$

where *I* is the area moment of inertia of the cross-section, *A* is the cross-sectional area, *G* is the shear modulus, and *k* is a **shear correction factor**. For materials with Poisson's ratios (ν) close to 0.3, the shear correction factor for a rectangular cross-section is approximately

$$k = \frac{5 + 5\nu}{6 + 5\nu}$$

The rotation ($\varphi(x)$) of the normal is described by the equation

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} - \frac{q(x)}{kAG}$$

The bending moment (*M*) and the shear force (*Q*) are given by

$$M(x) = -EI\ \frac{\mathrm{d}\varphi}{\mathrm{d}x}\ ;\quad Q(x) = kAG\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \varphi\right) = -EI\ \frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} = \frac{\mathrm{d}M}{\mathrm{d}x}$$

# Dynamic bending of beams

The dynamic bending of beams, also known as flexural vibrations of beams, was first investigated by Daniel Bernoulli in the late 18th century. Bernoulli's equation of motion of a vibrating beam tended to overestimate the natural frequencies of beams and was improved marginally by Rayleigh in 1877 by the addition of a mid-plane rotation. In 1921 Stephen Timoshenko improved the theory further by incorporating the effect of shear on the dynamic response of bending beams. This allowed the theory to be used for problems involving high frequencies of vibration where the dynamic Euler-Bernoulli theory is inadequate. The Euler-Bernoulli and Timoshenko theories for the dynamic bending of beams continue to be used widely by engineers.

### Euler-Bernoulli theory

The Euler-Bernoulli equation for the dynamic bending of slender, isotropic, homogeneous beams of constant cross-section under an applied transverse load $q(x,t)$ is

$$EI\ \frac{\partial^4 w}{\partial x^4} + m\ \frac{\partial^2 w}{\partial t^2} = q(x,t)$$

where $E$ is the Young's modulus, $I$ is the area moment of inertia of the cross-section, $w(x,t)$ is the deflection of the neutral axis of the beam, and $m$ is mass per unit length of the beam.

**Free vibrations**

For the situation where there is no transverse load on the beam, the bending equation takes the form

$$EI\ \frac{\partial^4 w}{\partial x^4} + m\ \frac{\partial^2 w}{\partial t^2} = 0$$

Free, harmonic vibrations of the beam can then be expressed as

$$w(x,t) = \text{Re}[\hat{w}(x)\ e^{-i\omega t}] \quad \implies \quad \frac{\partial^2 w}{\partial t^2} = -\omega^2\ w(x,t)$$

and the bending equation can be written as

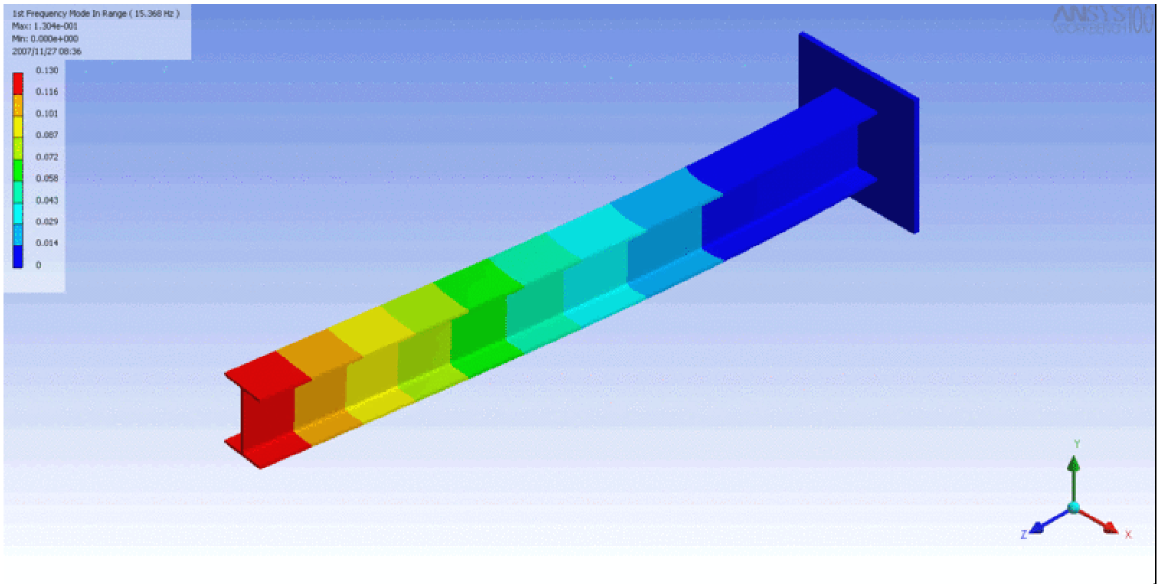
$$EI\ \frac{\mathrm{d}^4 \hat{w}}{\mathrm{d}x^4} - m\omega^2 \hat{w} = 0$$

The general solution of the above equation is

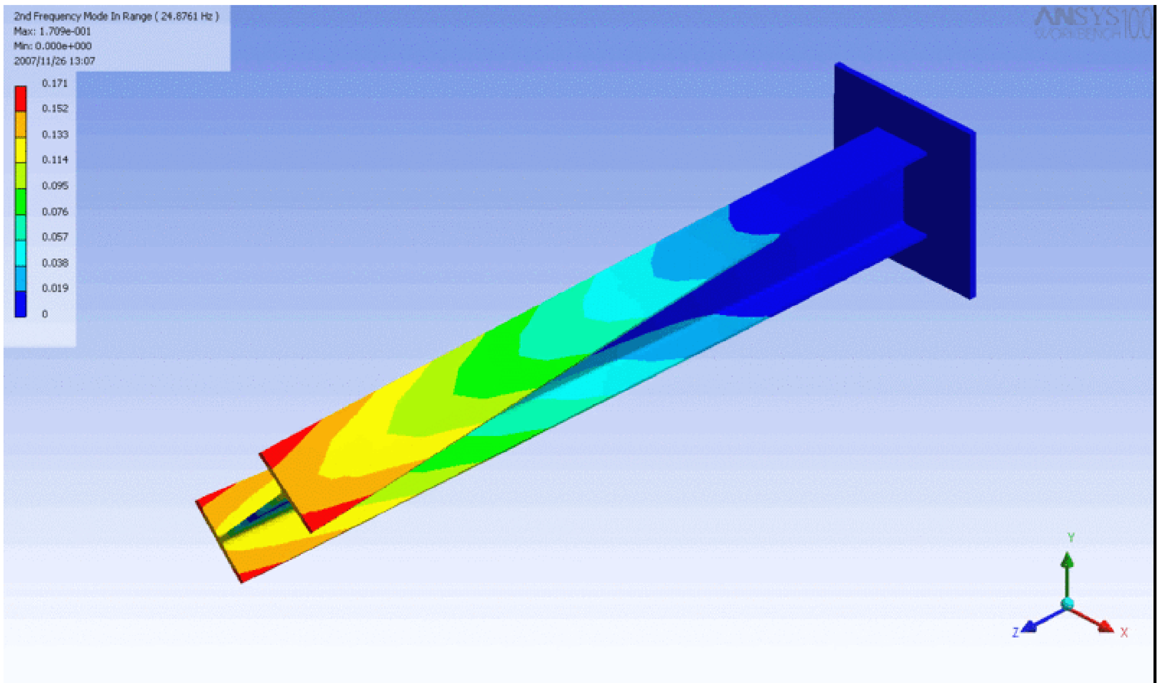
$$\hat{w} = A_1 \cosh(\beta x) + A_2 \sinh(\beta x) + A_3 \cos(\beta x) + A_4 \sin(\beta x)$$

where $A_1, A_2, A_3, A_4$ are constants and $\beta := \left(\frac{m}{EI}\,\omega^2\right)^{1/4}$

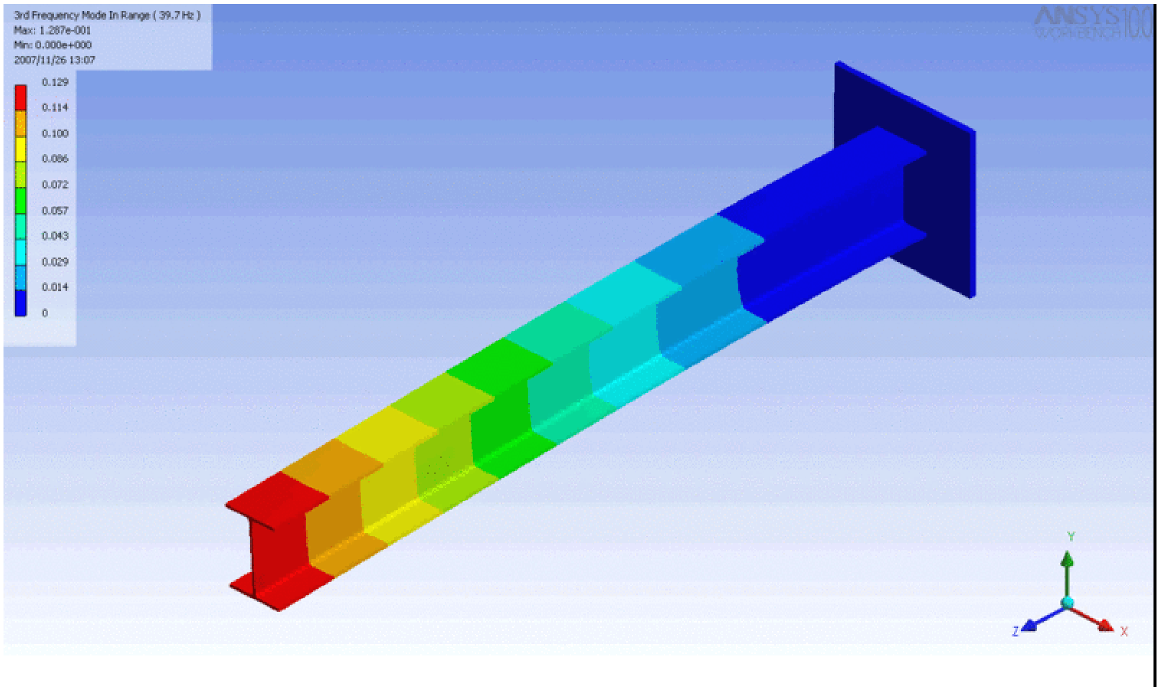
**The mode shapes of a cantilevered I-beam**



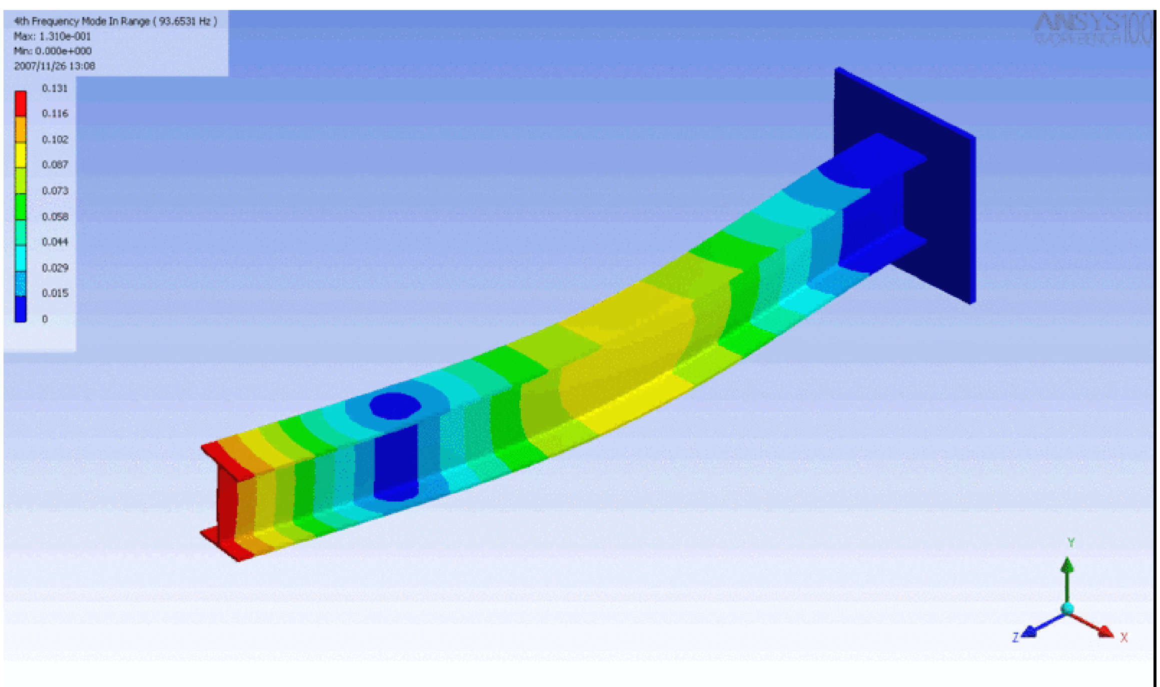

1st lateral bending



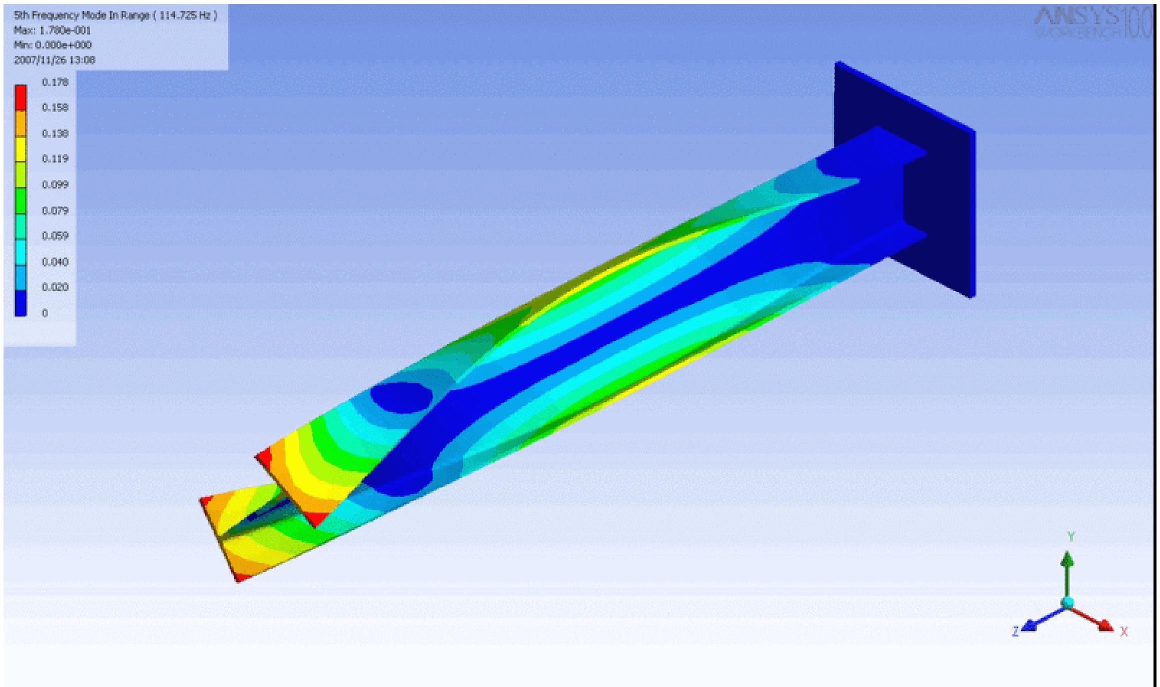

1st torsional



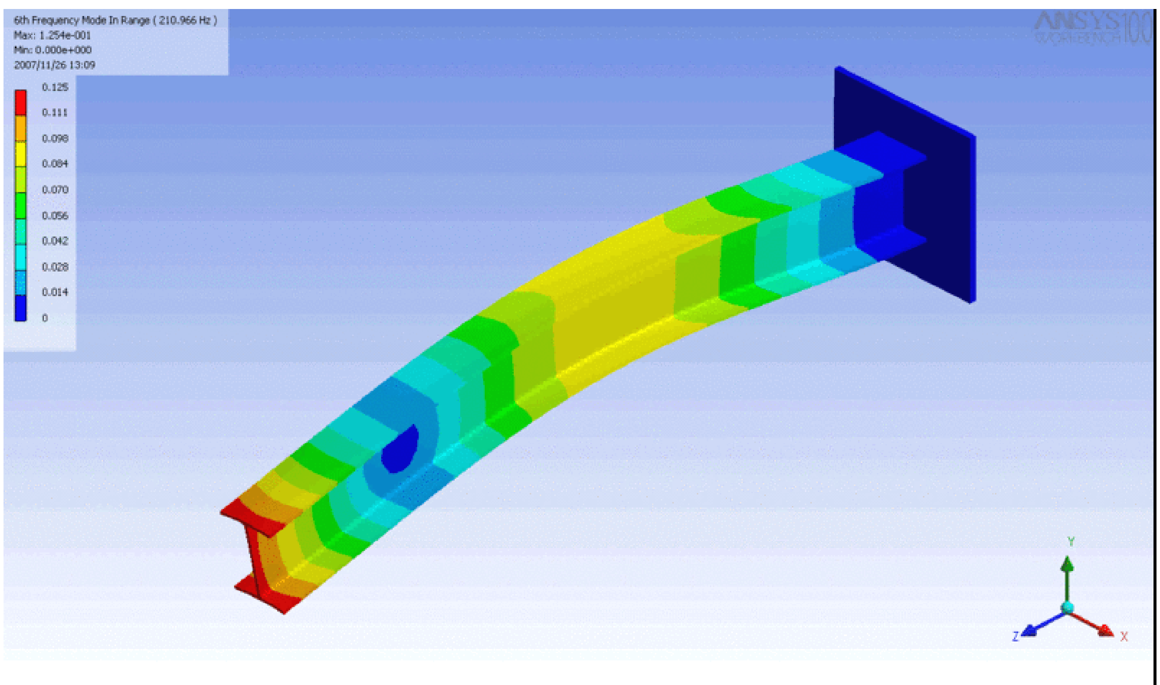

1st vertical bending




2nd lateral bending




2nd torsional




2nd vertical bending

## Timoshenko-Rayleigh theory

In 1877, Rayleigh proposed an improvement to the dynamic Euler-Bernoulli beam theory by including the effect of rotational inertia of the cross-section of the beam. Timoshenko improved upon that theory in 1922 by adding the effect of shear into the beam equation. Shear deformations of the normal to the mid-surface of the beam are allowed in the Timoshenko-Rayleigh theory.

The equation for the bending of a linear elastic, isotropic, homogeneous beam of constant cross-section beam under these assumptions is

$$EI\,\frac{\partial^4 w}{\partial x^4}+m\,\frac{\partial^2 w}{\partial t^2}-\left(J+\frac{EIm}{kAG}\right)\frac{\partial^4 w}{\partial x^2\,\partial t^2}+\frac{Jm}{kAG}\,\frac{\partial^4 w}{\partial t^4}=q(x,t)+\frac{J}{kAG}\,\frac{\partial^2 q}{\partial t^2}-\frac{EI}{kAG}\,\frac{\partial^2 q}{\partial x^2}$$

where $J = \frac{mI}{A}$ is the polar moment of inertia of the cross-section, $m = \rho A$ is the mass per unit length of the beam, $\rho$ is the density of the beam, $A$ is the cross-sectional area, $G$ is the shear modulus, and $k$ is a **shear correction factor**. For materials with Poisson's ratios ($\nu$) close to 0.3, the shear correction factor are approximately

$$\begin{aligned} k &= \tfrac{5+5\nu}{6+5\nu} \quad \text{rectangular cross-section} \\ &= \tfrac{6+12\nu+6\nu^2}{7+12\nu+4\nu^2} \quad \text{circular cross-section} \end{aligned}$$

### Free vibrations

For free, harmonic vibrations the Timoshenko-Rayleigh equations take the form

$$EI\,\frac{\mathrm{d}^4\hat{w}}{\mathrm{d}x^4}+m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\frac{\mathrm{d}^2\hat{w}}{\mathrm{d}x^2}+m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)\,\hat{w}=0$$

This equation can be solved by noting that all the derivatives of $w$ must have the same form to cancel out and hence as solution of the form $e^{kx}$ may be expected. This observation leads to the characteristic equation

$$\alpha\,k^4+\beta\,k^2+\gamma=0\,;\;\;\alpha:=EI\,,\;\;\beta:=m\omega^2\left(\frac{J}{m}+\frac{EI}{kAG}\right)\,,\;\;\gamma:=m\omega^2\left(\frac{\omega^2 J}{kAG}-1\right)$$

The solutions of this quartic equation are

$$k_1=+\sqrt{z_+}\;,\quad k_2=-\sqrt{z_+}\;,\quad k_3=+\sqrt{z_-}\;,\quad k_4=-\sqrt{z_-}$$

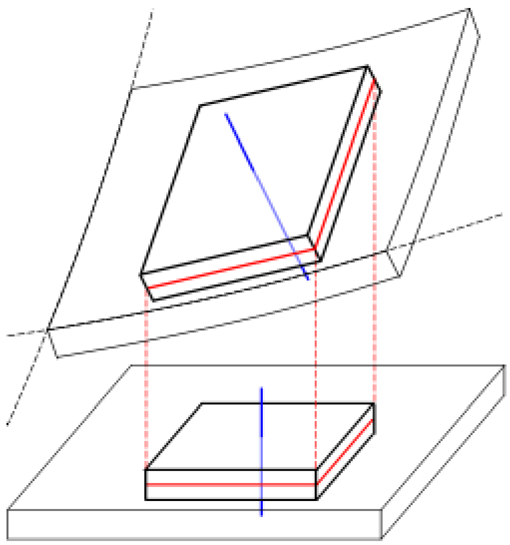
where

$$z_+ := \frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha} \ , \quad z_- := \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

The general solution of the Timoshenko-Rayleigh beam equation for free vibrations can then be written as

$$\hat{w} = A_1 \ e^{k_1 x} + A_2 \ e^{-k_1 x} + A_3 \ e^{k_3 x} + A_4 \ e^{-k_3 x}$$

# Quasistatic bending of plates



Deformation of a thin plate highlighting the displacement, the mid-surface (red) and the normal to the mid-surface (blue)

The defining feature of beams is that one of the dimensions is much *larger* than the other two. A structure is called a plate when it is flat and one of its dimensions is much *smaller* than the other two. There several theories that attempt to describe the deformation and stress in a plate under applied loads two of which have been used widely. These are

- the Kirchhoff-Love theory of plates (also called classical plate theory)
- the Mindlin-Reissner plate theory (also called the first-order shear theory of plates)

## Kirchhoff-Love theory of plates

The assumptions of Kirchhoff-Love theory are

- straight lines normal to the mid-surface remain straight after deformation

- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

These assumptions imply that

$$u_\alpha(\mathbf{x}) = -x_3 \; \frac{\partial w^0}{\partial x_\alpha} = -x_3 \; w^0_{,\alpha} \; ; \quad \alpha = 1,2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

where $\mathbf{u}$is the displacement of a point in the plate and $w^0$ is the displacement of the mid-surface.

The strain-displacement relations are

$$\varepsilon_{\alpha\beta} = -x_3 \; w^0_{,\alpha\beta}$$
$$\varepsilon_{\alpha 3} = 0$$
$$\varepsilon_{33} = 0$$

The equilibrium equations are

$$M_{\alpha\beta,\alpha\beta} + q(x) = 0 \; ; \quad M_{\alpha\beta} := \int_{-h}^{h} x_3 \; \sigma_{\alpha\beta} \; dx_3$$

where $q(x)$ is an applied load normal to the surface of the plate.

In terms of displacements, the equilibrium equations for an isotropic, linear elastic plate in the absence of external load can be written as

$$w^0_{,1111} + 2 \; w^0_{,1212} + w^0_{,2222} = 0$$

In direct tensor notation,

$$\nabla^2 \nabla^2 w = 0$$

## Mindlin-Reissner theory of plates

The special assumption of this theory is that normals to the mid-surface remain straight and inextensible but not necessarily normal to the mid-surface after deformation. The displacements of the plate are given by

$$u_\alpha(\mathbf{x}) = -x_3 \; \varphi_\alpha \; ; \quad \alpha = 1,2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

where $\varphi_\alpha$ are the rotations of the normal.

The strain-displacement relations that result from these assumptions are

$$\varepsilon_{\alpha\beta} = -x_3 \; \varphi_{\alpha,\beta}$$
$$\varepsilon_{\alpha 3} = \frac{1}{2} \, \kappa \left(w^0_{,\alpha} - \varphi_\alpha\right)$$
$$\varepsilon_{33} = 0$$

where κ is a shear correction factor.

The equilibrium equations are

$$M_{\alpha\beta,\beta} - Q_\alpha = 0$$
$$Q_{\alpha,\alpha} + q = 0$$

where

$$Q_\alpha := \kappa \; \int_{-h}^{h} \sigma_{\alpha 3} \; dx_3$$

# Dynamic bending of plates

## Dynamics of thin Kirchhoff plates

The dynamic theory of plates determines the propagation of waves in the plates, and the study of standing waves and vibration modes. The equations that govern the dynamic bending of Kirchhoff plates are

$$M_{\alpha\beta,\alpha\beta} - q(x,t) = J_1 \; \ddot{w}^0 - J_3 \; \ddot{w}^0_{,\alpha\alpha}$$

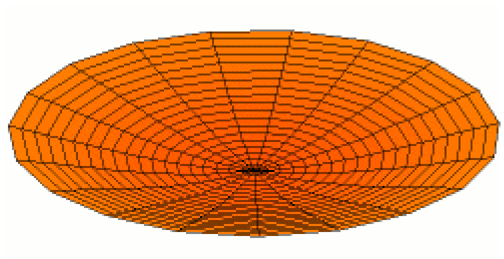
where, for a plate with density ρ = ρ(*x*),

$$J_1 := \int_{-h}^{h} \rho \; dx_3 \; ; \quad J_3 := \int_{-h}^{h} x_3^2 \; \rho \; dx_3$$

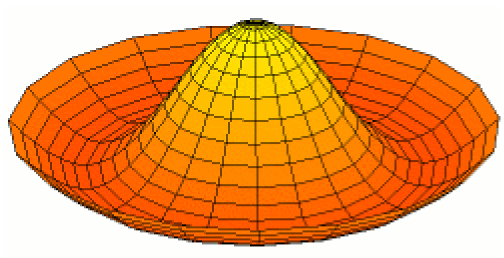
and

$$\ddot{w}^0 = \frac{\partial^2 w^0}{\partial t^2} \;\; ; \;\; \ddot{w}^0_{,\alpha\beta} = \frac{\partial^2 \ddot{w}^0}{\partial x_\alpha \partial x_\beta}$$

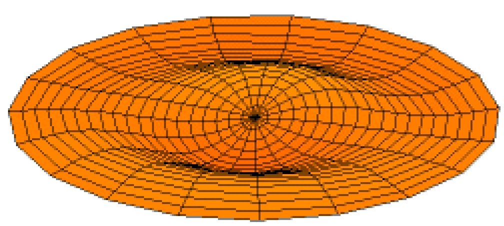
The figures below show some vibrational modes of a circular plate.



mode $k = 0$, $p = 1$



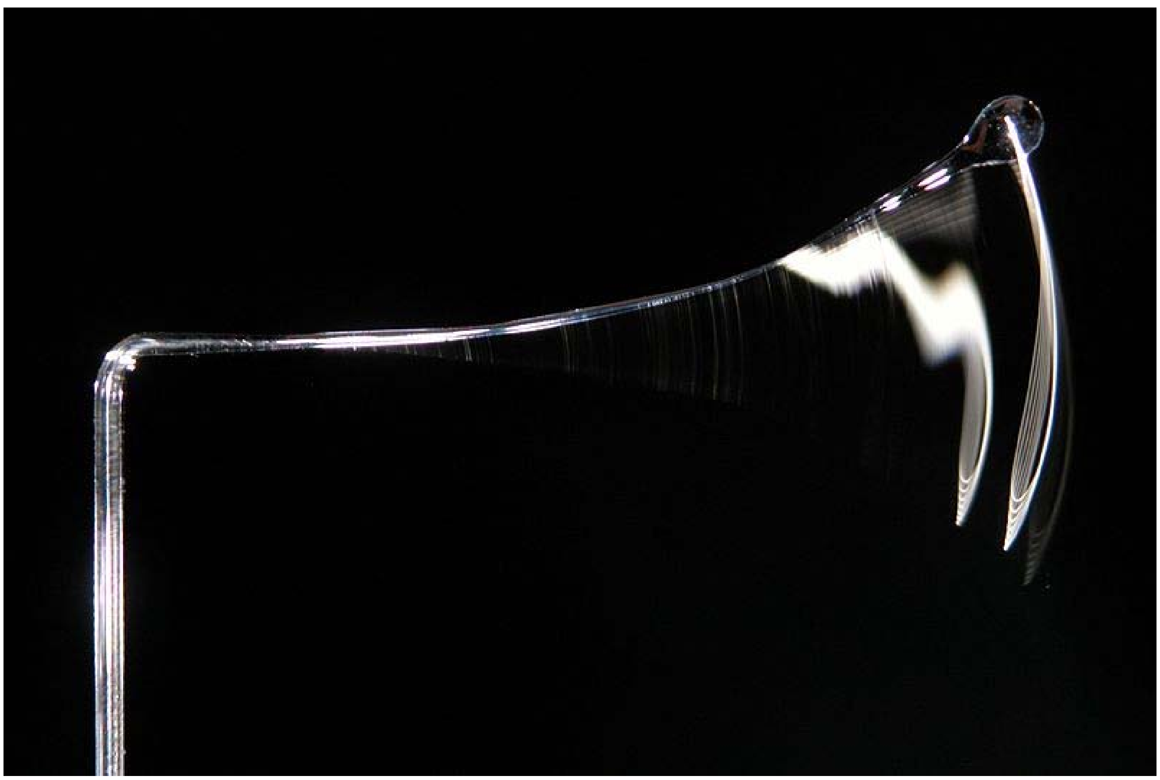
mode $k = 0$, $p = 2$



mode $k = 1$, $p = 2$

## Chapter 2

# Euler–Bernoulli Beam Equation

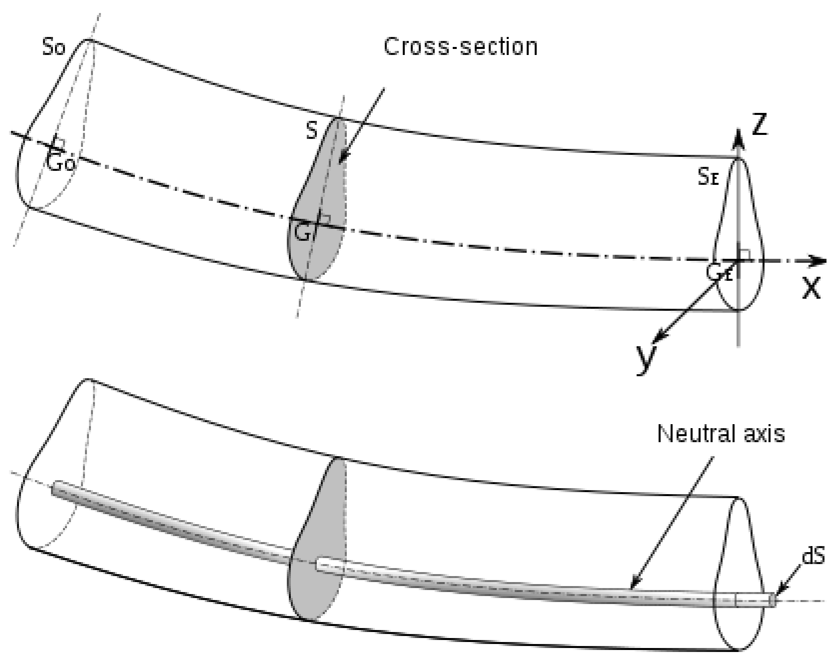


This vibrating glass beam may be modeled as a cantilever beam with acceleration, variable linear density, variable section modulus, some kind of dissipation, springy end loading, and possibly a point mass at the free end.

**Euler–Bernoulli beam theory** (also known as **engineer's beam theory**, **classical beam theory** or just **beam theory**) is a simplification of the linear theory of elasticity which provides a means of calculating the load-carrying and deflection characteristics of beams. It covers the case for small deflections of a beam which is subjected to lateral loads only. It is thus a special case of Timoshenko beam theory which accounts for shear deformation and is applicable for thick beams. It was first enunciated circa 1750 , but was not applied on a large scale until the development of the Eiffel Tower and the Ferris wheel in the late 19th century. Following these successful demonstrations, it quickly became a cornerstone of engineering and an enabler of the Second Industrial Revolution.

Additional analysis tools have been developed such as plate theory and finite element analysis, but the simplicity of beam theory makes it an important tool in the sciences, especially structural and mechanical engineering.

# History

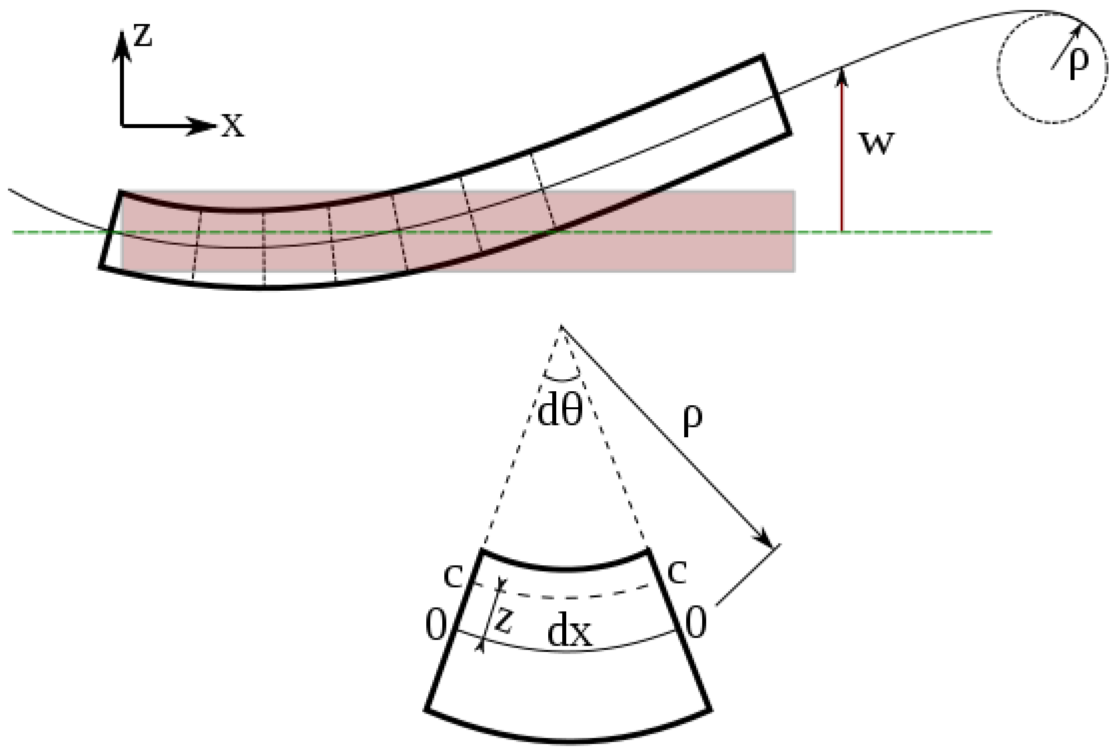



Schematic of cross-section of a bent beam showing the neutral axis.

Prevailing consensus is that Galileo Galilei made the first attempts at developing a theory of beams, but recent studies argue that Leonardo da Vinci was the first to make the crucial observations. Da Vinci lacked Hooke's law and calculus to complete the theory, whereas Galileo was held back by an incorrect assumption he made.

The Bernoulli beam is named after Jacob Bernoulli, who made the significant discoveries. Leonhard Euler and Daniel Bernoulli were the first to put together a useful theory circa 1750. At the time, science and engineering were generally seen as very distinct fields, and there was considerable doubt that a mathematical product of academia could be trusted for practical safety applications. Bridges and buildings continued to be designed by precedent until the late 19th century, when the Eiffel Tower and Ferris wheel demonstrated the validity of the theory on large scales.

## Static beam equation




Bending of an Euler-Bernoulli beam. Each cross-section of the beam is at 90 degrees to the neutral axis.

The Euler-Bernoulli equation describes the relationship between the beam's deflection and the applied load:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) = q.$$

The curve $w(x)$ describes the deflection $w$ of the beam at some position $x$ (recall that the beam is modeled as a one-dimensional object). $q$ is a distributed load, in other words a force per unit length (analogous to pressure being a force per area); it may be a function of $x$, $w$, or other variables.

Note that $E$ is the elastic modulus and that $I$ is the second moment of area. $I$ must be calculated with respect to the centroidal axis perpendicular to the applied loading. For an Euler-Bernoulli beam not under any axial loading this axis is called the neutral axis.

Often, $w = w(x)$, $q = q(x)$, and EI is a constant, so that:

$$EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x).$$

This equation, describing the deflection of a uniform, static beam, is used widely in engineering practice. Tabulated expressions for the deflection *w* for common beam configurations can be found in engineering handbooks. For more complicated situations the deflection can be determined by solving the Euler-Bernoulli equation using techniques such as the "slope deflection method", "moment distribution method", "moment area method, "conjugate beam method", "the principle of virtual work", "direct integration", "Castigliano's method", "Macaulay's method" or the "direct stiffness method".

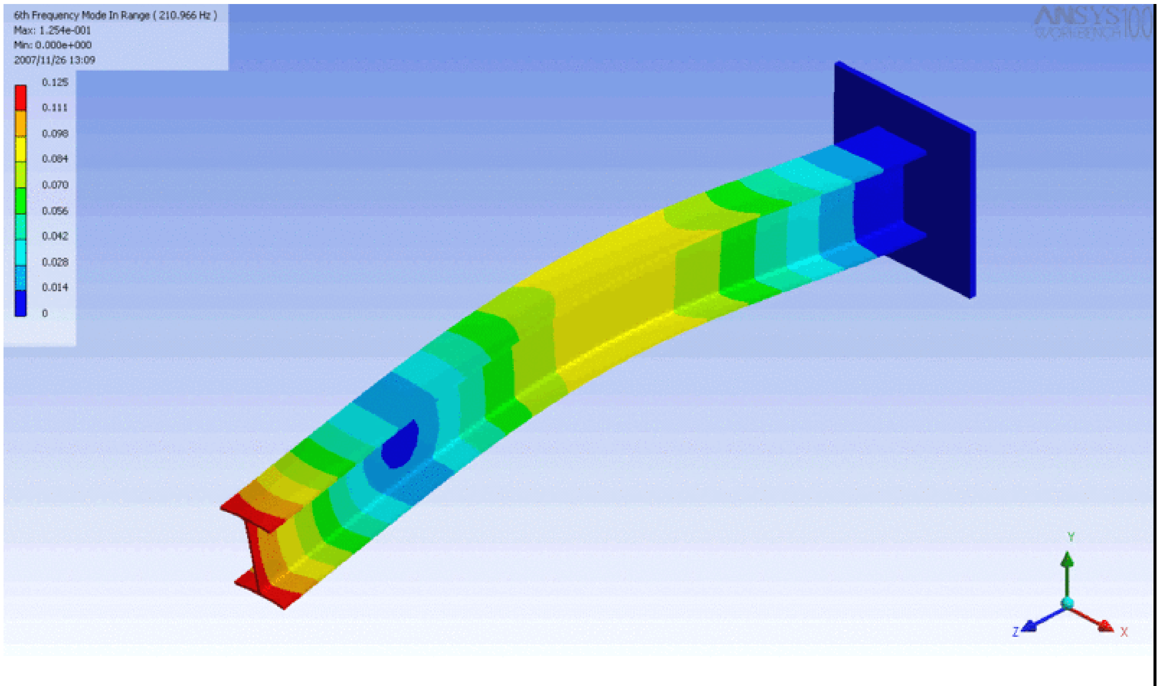
Successive derivatives of *w* have important meanings:

- $w$is the deflection.
- $\frac{\mathrm{d}w}{\mathrm{d}x} = \varphi$is the slope of the beam.
- $-EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2} = M$is the bending moment in the beam.
- $-\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) = Q$is the shear force in the beam.

The stresses in a beam can be calculated from the above expressions after the deflection due to a given load has been determined.

A number of different sign conventions can be found in the literature on the bending of beams and care should be take to maintain consistency. In this chapter, the sign convention has been chosen so the coordinate system is right handed. Forces acting in the positive *x* and *z* directions are assumed positive. The sign of the bending moment is chosen so that a positive value leads to a tensile stress at the bottom cords. The sign of the shear force has been chosen such that it matches the sign of the bending moment.

## Dynamic beam equation




Vibration of a wide-flange beam (I-beam).

The dynamic beam equation is the Euler-Lagrange equation for the following action

$$S = \int_0^L \left[ \frac{1}{2}\mu \left( \frac{\partial w}{\partial t} \right)^2 - \frac{1}{2} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + q(x) w(x,t) \right] dx$$

The first term represents the kinetic energy where μ is the mass per unit length; the second one represents the potential energy due to internal forces (when considered with a negative sign) and the third term represents the potential energy due to the external load $q(x)$. The Euler-Lagrange equation is used to determine the function that minimizes the functional $S$. For a dynamic Euler-Bernoulli beam, the Euler-Lagrange equation is

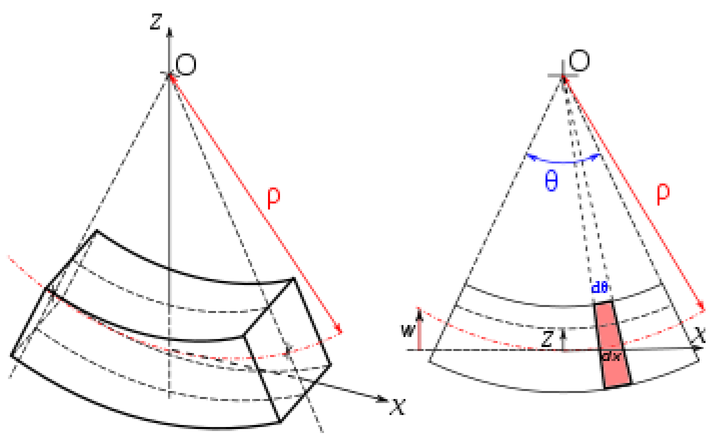
$$\frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 w}{\partial x^2} \right) = -\mu \frac{\partial^2 w}{\partial t^2} + q(x)$$

## Stress

Besides deflection, the beam equation describes forces and moments and can thus be used to describe stresses. For this reason, the Euler–Bernoulli beam equation is widely used in engineering, especially civil and mechanical, to determine the strength (as well as deflection) of beams under bending.

Both the bending moment and the shear force cause stresses in the beam. The stress due to shear force is maximum along the neutral axis of the beam (when the width of the beam, t, is constant along the cross section of the beam; otherwise an integral involving the first moment and the beam's width needs to be evaluated for the particular cross section), and the maximum tensile stress is at either the top or bottom surfaces. Thus the maximum principal stress in the beam may be neither at the surface nor at the center but in some general area. However, shear force stresses are negligible in comparison to bending moment stresses in all but the stockiest of beams as well as the fact that stress concentrations commonly occur at surfaces, meaning that the maximum stress in a beam is likely to be at the surface.

## Simple or symmetrical bending



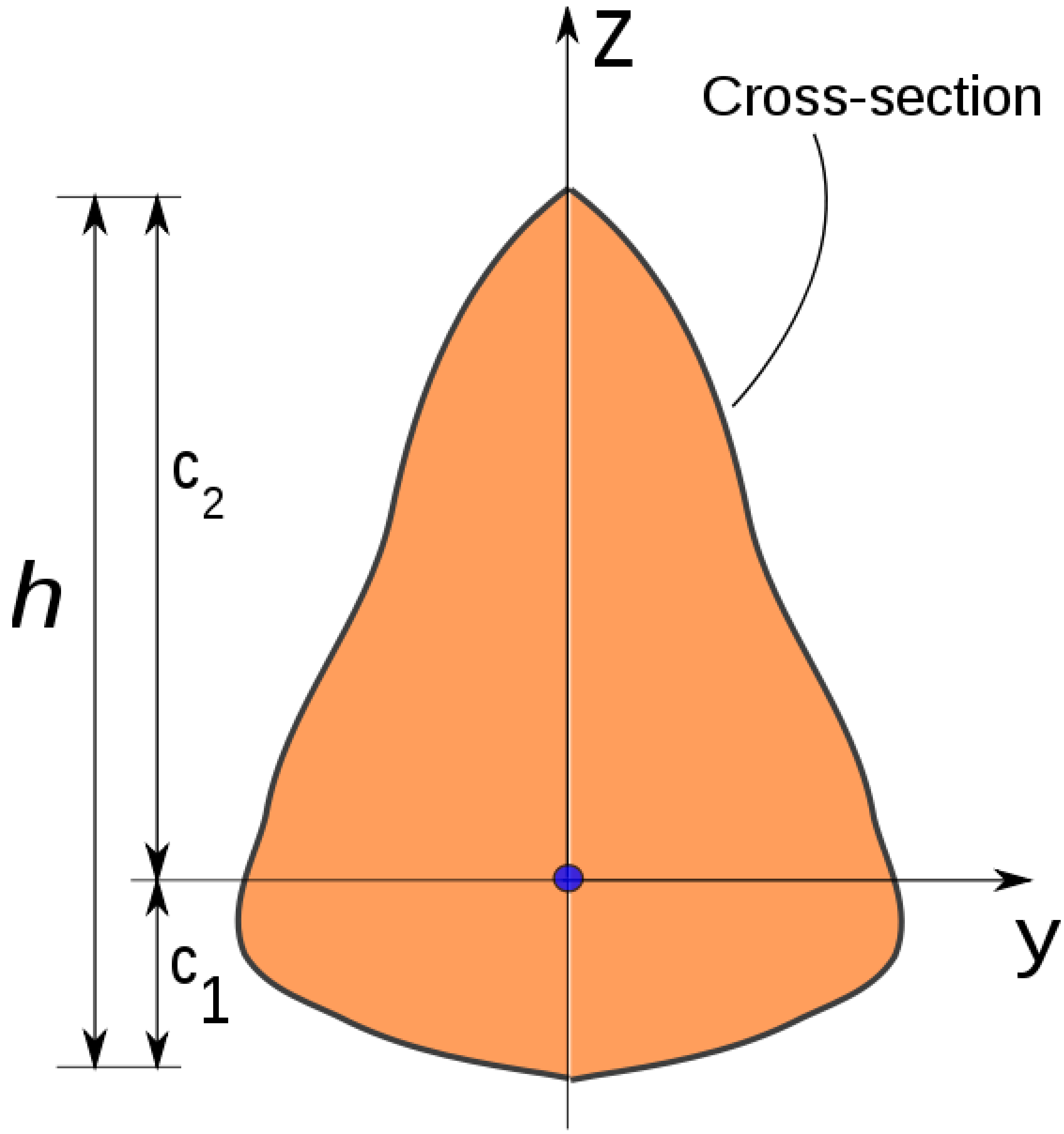

Element of a bent beam: the fibers form concentric arcs, the top fibers are compressed and bottom fibers stretched.

For beam cross-sections that are symmetrical about a plane perpendicular to the neutral plane, it can be shown that the tensile stress experienced by the beam may be expressed as:

$$\sigma = \frac{Mz}{I} = -zE\ \frac{\mathrm{d}^2 w}{\mathrm{d}x^2}.$$

Here, $z$ is the distance from the neutral axis to a point of interest; and $M$ is the bending moment. Note that this equation implies that pure bending (of positive sign) will cause zero stress at the neutral axis, positive (tensile) stress at the "top" of the beam, and negative (compressive) stress at the bottom of the beam; and also implies that the maximum stress will be at the top surface and the minimum at the bottom. This bending stress may be superimposed with axially applied stresses, which will cause a shift in the neutral (zero stress) axis.

**Maximum stresses at a cross-section**




Quantities used in the definition of the section modulus of a beam.

The maximum tensile stress at a cross-section is at the location $z = c_1$ and the maximum compressive stress is at the location $z = -c_2$ where the height of the cross-section is $h = c_1 + c_2$. These stresses are

$$\sigma_1 = \frac{Mc_1}{I} = \frac{M}{S_1} ; \quad \sigma_2 = -\frac{Mc_2}{I} = -\frac{M}{S_2}$$

The quantities $S_1, S_2$ are the section moduli and are defined as

$$S_1 = \frac{I}{c_1} \; ; \quad S_2 = \frac{I}{c_2}$$

The section modulus combines all the important geometric information about a beam's section into one quantity. For the case where a beam is doubly-symmetric, $c_1 = c_2$ and we have one section modulus $S = I / c$.

## Strain in an Euler–Bernoulli beam

We need an expression for the strain in terms of the deflection of the neutral surface to relate the stresses in an Euler-Bernoulli beam to the deflection. To obtain that expression we use the assumption that normals to the neutral surface remain normal during the deformation and that deflections are small. These assumptions imply that the beam bends into an arc of a circle of radius ρ and that the neutral surface does not change in length during the deformation.

Let d$x$ be the length of an element of the neutral surface in the undeformed state. For small deflections, the element does not change its length after bending but deforms into an arc of a circle of radius ρ. If dθ is the angle subtended by this arc, then $\mathrm{d}x = \rho\ \mathrm{d}\theta$.

Let us now consider another segment of the element at a distance $z$ above the neutral surface. The initial length of this element is d$x$. However, after bending, the length of the element becomes $\mathrm{d}x' = (\rho - z)\ \mathrm{d}\theta = \mathrm{d}x - z\ \mathrm{d}\theta$. The strain in that segment of the beam is given by

$$\varepsilon_x = \frac{\mathrm{d}x' - \mathrm{d}x}{\mathrm{d}x} = -\frac{z}{\rho} = -\kappa\ z$$

where κ is the curvature of the beam. This gives us the axial strain in the beam as a function of distance from the neutral surface. However, we still need to find a relation between the radius of curvature and the beam deflection $w$.

## Relation between curvature and beam deflection

Let P be a point on the neutral surface of the beam at a distance $x$ from the origin of the ($x$,$z$) coordinate system. The slope of the beam, i.e., the angle made by the neutral surface with the $x$-axis, at this point is

$$\theta(x) = \frac{\mathrm{d}w}{\mathrm{d}x}$$

Therefore, for an infinitesimal element d$x$, the relation $\mathrm{d}x = \rho\ \mathrm{d}\theta$ can be written as

$$\frac{1}{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \kappa$$

Hence the strain in the beam may be expressed as

$$\varepsilon_x = -z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

### Stress-strain relations

For a one-dimensional linear elastic material, the stress is related to be strain by $\sigma = E\varepsilon$ where *E* is the Young's modulus. Hence the stress in an Euler-Bernoulli beam is given by

$$\sigma_x = -zE\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Note that the above relation, when compared with the relation between the axial stress and the bending moment, leads to

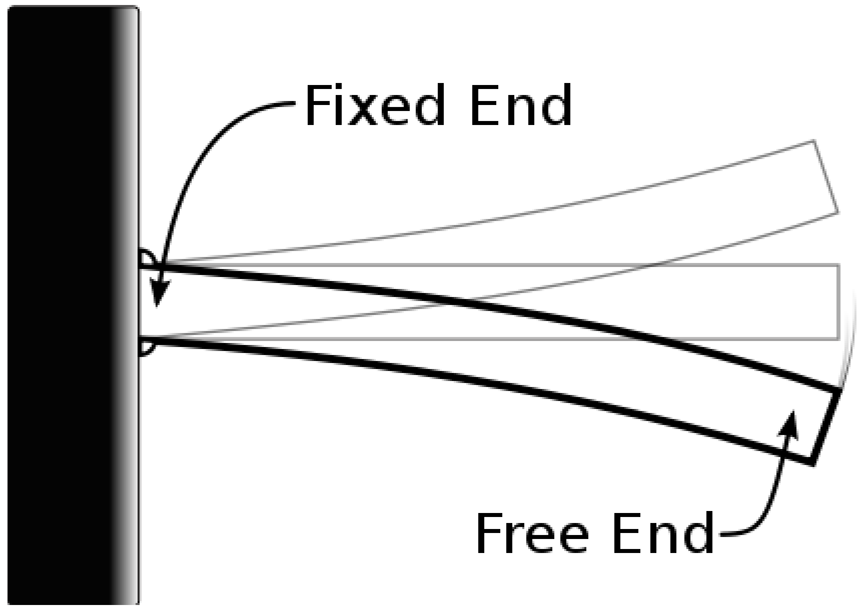
$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$$

Since the shear force is given by $Q = \mathrm{d}M / \mathrm{d}x$, we also have

$$Q = -EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}$$

## Boundary considerations

The beam equation contains a fourth-order derivative in *x*. To find a unique solution $w(x,t)$ we need four boundary conditions. The boundary conditions usually model *supports*, but they can also model point loads, distributed loads and moments. The *support* or displacement boundary conditions are used to fix values of displacement (*w*) and rotations ($\mathrm{d}w / \mathrm{d}x$) on the boundary. Such boundary conditions are also called Dirichlet boundary conditions. Load and moment boundary conditions involve higher derivatives of *w* and represent momentum flux. Flux boundary conditions are also called Neumann boundary conditions.




A cantilever beam.

As an example consider a cantilever beam that is built-in at one end and free at the other as shown in the adjacent figure. At the built-in end of the beam there cannot be any displacement or rotation of the beam. This means that at the left end both deflection and slope are zero. Since no external bending moment is applied at the free end of the beam, the bending moment at that location is zero. In addition, if there is no external force applied to the beam, the shear force at the free end is also zero.

Taking the *x* coordinate of the left end as 0 and the right end as *L* (the length of the beam), these statements translate to the following set of boundary conditions (assume *EI* is a constant):

$$w|_{x=0} = 0 \quad ; \quad \left.\frac{\partial w}{\partial x}\right|_{x=0} = 0 \qquad \text{(fixed end)}$$
$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=L} = 0 \quad ; \quad \left.\frac{\partial^3 w}{\partial x^3}\right|_{x=L} = 0 \qquad \text{(free end)}$$

A simple support (pin or roller) is equivalent to a point force on the beam which is adjusted in such a way as to fix the position of the beam at that point. A fixed support or clamp, is equivalent to the combination of a point force and a point torque which is adjusted in such a way as to fix both the position and slope of the beam at that point. Point forces and torques, whether from supports or directly applied, will divide a beam into a set of segments, between which the beam equation will yield a continuous solution, given four boundary conditions, two at each end of the segment. Assuming that the product *EI* is a constant, and defining $\lambda = F / EI$ where *F* is the magnitude of a point force, and $\tau = M / EI$ where *M* is the magnitude of a point torque, the boundary conditions appropriate for some common cases is given in the table below. The change in a particular derivative of $w$ across the boundary as $x$ increases is denoted by $\Delta$ followed by that derivative. For example, $\Delta w'' = w''(x+) - w''(x-)$ where $w''(x+)$ is the value of $w''$ at the lower boundary of the upper segment, while $w''(x-)$ is the value of $w''$ at the upper boundary of the lower segment. When the values of the particular derivative are not only continuous across the boundary, but fixed as well, the boundary condition is written e.g. $\Delta w'' = 0^*$ which actually constitutes two separate equations (e.g. $w''(x-) = w''(x+) =$ fixed).

| **Boundary** | $w'''$ | $w''$ | $w'$ | $w$ |
|---|---|---|---|---|
| Clamp | | | $\Delta w' = 0^*$ | $\Delta w = 0^*$ |
| Simple support | | $\Delta w'' = 0$ | $\Delta w' = 0$ | $\Delta w = 0^*$ |
| Point force | $\Delta w''' = \lambda$ | $\Delta w'' = 0$ | $\Delta w' = 0$ | $\Delta w = 0$ |
| Point torque | $\Delta w''' = 0$ | $\Delta w'' = \tau$ | $\Delta w' = 0$ | $\Delta w = 0$ |
| Free end | $w''' = 0$ | $w'' = 0$ | | |
| Clamp at end | | | $w'$ fixed | $w$ fixed |
| Simply supported end | | $w'' = 0$ | | $w$ fixed |
| Point force at end | $w''' = \lambda$ | $w'' = 0$ | | |
| Point torque at end | $w''' = 0$ | $w'' = \tau$ | | |

Note that in the first cases, in which the point forces and torques are located between two segments, there are four boundary conditions, two for the lower segment, and two for the upper. When forces and torques are applied to an end of the beam, there are two boundary conditions given which apply at that end.

# Loading considerations

Applied loads may be represented either through boundary conditions or through the function $q(x,t)$ which represents an external distributed load. Using distributed loading is often favorable for simplicity. Boundary conditions are, however, often used to model loads depending on context; this practice being especially common in vibration analysis.

By nature, the distributed load is very often represented in a piecewise manner, since in practice a load isn't typically a continuous function. Point loads can be modeled with help of the Dirac delta function. For example, consider a static uniform cantilever beam of length $L$ with an upward point load $F$ applied at the free end. Using boundary conditions, this may be modeled in two ways. In the first approach, the applied point load is approximated by a shear force applied at the free end. In that case the governing equation and boundary conditions are:

$$EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} = 0$$

$$w|_{x=0} = 0 \quad ; \quad \frac{\mathrm{d}w}{\mathrm{d}x}\bigg|_{x=0} = 0 \quad ; \quad \frac{\mathrm{d}^2w}{\mathrm{d}x^2}\bigg|_{x=L} = 0 \quad ; \quad -EI\frac{\mathrm{d}^3w}{\mathrm{d}x^3}\bigg|_{x=L} = F$$

Alternatively we can represent the point load as a distribution using the Dirac function. In that case the equation and boundary conditions are

$$EI\frac{\mathrm{d}^4w}{\mathrm{d}x^4} = F\delta(x-L)$$

$$w|_{x=0} = 0 \quad ; \quad \frac{\mathrm{d}w}{\mathrm{d}x}\bigg|_{x=0} = 0 \quad ; \quad \frac{\mathrm{d}^2w}{\mathrm{d}x^2}\bigg|_{x=L} = 0$$

Note that shear force boundary condition (third derivative) is removed, otherwise there would be a contradiction. These are equivalent boundary value problems, and both yield the solution

$$w = \frac{F}{6EI}(3Lx^2 - x^3) \ .$$

The application of several point loads at different locations will lead to *w*(*x*) being a piecewise function. Use of the Dirac function greatly simplifies such situations; otherwise the beam would have to be divided into sections, each with four boundary conditions solved separately. A well organized family of functions called Singularity functions are often used as a shorthand for the Dirac function, its derivative, and its antiderivatives.

Dynamic phenomena can also be modeled using the static beam equation by choosing appropriate forms of the load distribution. As an example, the free vibration of a beam can be accounted for by using the load function:

$$q(x,t) = \mu\frac{\partial^2w}{\partial t^2}$$

where $\mu$ is the linear mass density of the beam, not necessarily a constant. With this time-dependent loading, the beam equation will be a partial differential equation:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2w}{\partial x^2}\right) = \mu\frac{\partial^2w}{\partial t^2}.$$

Another interesting example describes the deflection of a beam rotating with a constant angular frequency of $\omega$:

$$q(x) = \mu\omega^2 w(x)$$

This is a centripetal force distribution. Note that in this case, $q$ is a function of the displacement (the dependent variable), and the beam equation will be an autonomous ordinary differential equation.

# Examples

## Three-point bending

The three point bending test is a classical experiment in mechanics. It represents the case of a beam resting on two roller supports and subjected to a concentrated load applied in the middle of the beam. The shear is constant in absolute value: it is half the central load, P / 2. It changes sign in the middle of the beam. The bending moment varies linearly from one end, where it is 0, and the center where its absolute value is PL / 4, is where the risk of rupture is the most important. The deformation of the beam is described by a polynomial of third degree over a half beam (the other half being symmetrical). The bending moments (*M*) , shear forces (*Q*), and deflections (*w*) for a beam subjected to a central point load and an asymmetric point load are given in the table below.

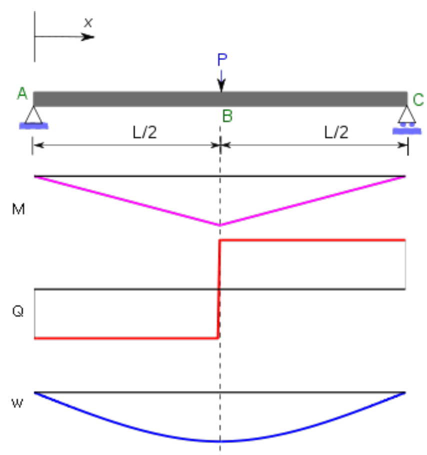
**Distribution** **Max. value**

**Simply supported beam with central load**

$$M(x) = \begin{cases} \frac{Px}{2}, & \text{for } 0 \le x \le \frac{L}{2} \\ \frac{P(L-x)}{2}, & \text{for } \frac{L}{2} < x \le L \end{cases} \qquad M_{L/2} = \frac{PL}{4}$$

$$Q(x) = \begin{cases} \frac{P}{2}, & \text{for } 0 \le x \le \frac{L}{2} \\ \frac{P}{2}, & \text{for } \frac{L}{2} < x \le L \end{cases} \qquad |Q_0| = |Q_L| = \frac{P}{2}$$

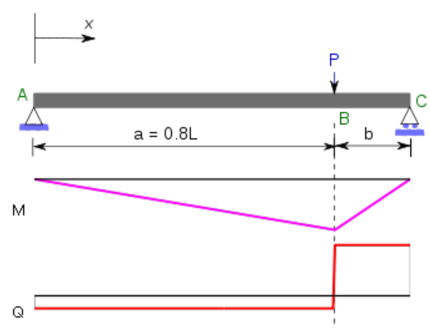
$$w(x) = \begin{cases} \frac{Px(4x^2-3L^2)}{48EI}, & \text{for } 0 \le x \le \frac{L}{2} \\ \frac{P(x-L)(L^2-8Lx+4x^2)}{48EI}, & \text{for } \frac{L}{2} < x \le L \end{cases} \qquad w_{L/2} = \frac{PL^3}{48EI}$$




**Simply supported beam with asymmetric load**

$$M(x) = \begin{cases} \frac{Pbx}{L}, & \text{for } 0 \le x \le a \\ \frac{Pbx}{L} - P(x-a), & \text{for } a < x \le L \end{cases} \qquad M_B = \frac{Pab}{L}$$

$$Q(x) = \begin{cases} \frac{Pb}{L}, & \text{for } 0 \le x \le a \\ -\frac{Pb}{L} - P, & \text{for } a < x \le L \end{cases} \qquad Q_A = \frac{Pb}{L} \qquad Q_C = \frac{P(b-L)}{L}$$



$$w(x) = \begin{cases} \frac{Pbx(L^2-b^2-x^2)}{6LEI}, & 0 \le x \le a \\ \frac{Pbx(L^2-b^2-x^2)}{6LEI} + \frac{P(x-a)^3}{6EI}, & a < x \le L \end{cases} \quad w_{\max} = \frac{\sqrt{3}Pb(L^2-b^2)^{\frac{3}{2}}}{27LEI}$$

at $x = \sqrt{\frac{L^2-b^2}{3}}$

## Cantilever beams

Another important class of problems involves cantilever beams. The bending moments ($M$) , shear forces ($Q$), and deflections ($w$) for a cantilever beam subjected to a point load at the free end and a uniformly distributed load are given in the table below.

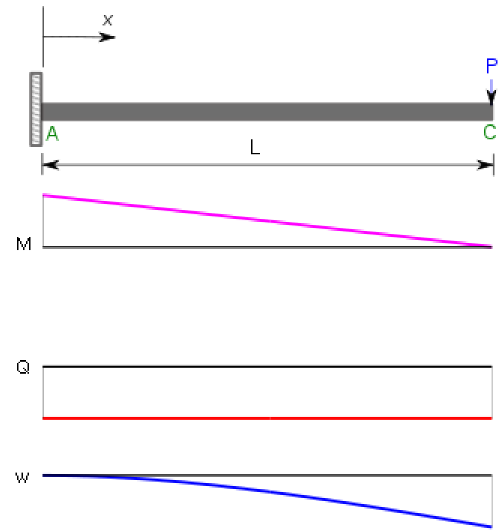
**Distribution** **Max. value**

**Cantilever beam with end load**

$M(x) = P(x - L)$ $M_A = PL$

$Q(x) = P$ $Q_{\max} = P$

$$w(x) = \frac{Px^2(3L-x)}{6EI} \qquad w_C = \frac{PL^3}{3EI}$$

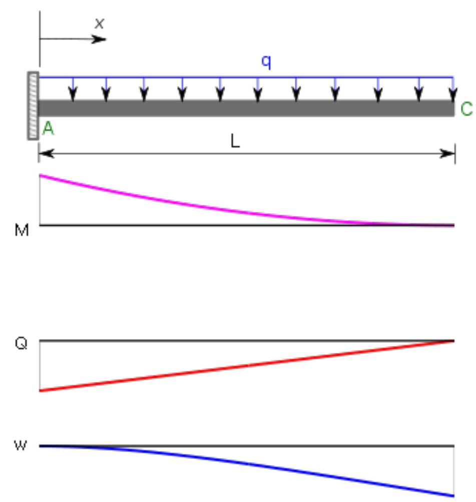



**Cantilever beam with uniformly distributed load**

$$M(x) = -\frac{q(L^2-2Lx+x^2)}{2} \qquad M_A = \frac{qL^2}{2}$$

$Q(x) = q(L - x)$, $Q_A = qL$

$$w(x) = \frac{qx^2(6L^2-4Lx+x^2)}{24EI} \qquad w_C = \frac{qL^4}{8EI}$$



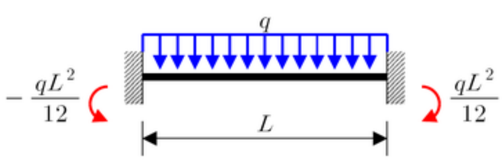

Solutions for several other commonly encountered configurations are readily available in textbooks on mechanics of materials and engineering handbooks.

## Statically indeterminate beams

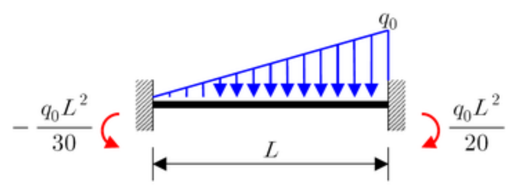
The bending moments and shear forces in Euler-Bernoulli beams can often be determined directly using static balance of forces and moments. However, for certain boundary conditions, the number of reactions can exceed the number of independent equilibrium equations. Such beams are called **statically indeterminate**.

The built-in beams shown in the figure below are statically indeterminate. To determine the stresses and deflections of such beams, the most direct method is to solve the Euler-Bernoulli beam equation with appropriate boundary conditions. But direct analytical solutions of the beam equation are possible only for the simplest cases. Therefore, additional techniques such as linear superposition are often used to solve statically indeterminate beam problems.

The superposition method involves adding the solutions of a number of statically determinate problems which are chosen such that the boundary conditions for the sum of the individual problems add up to those of the original problem.

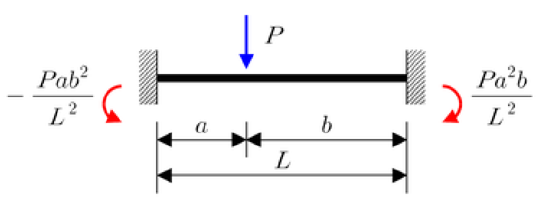



(a) Uniformly distributed load $q$.

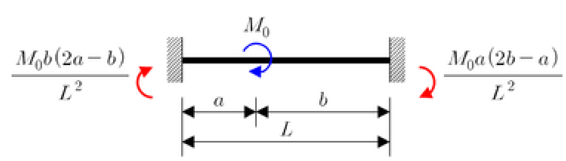



(b) Linearly distributed load with maximum $q_0$

$$M_{\max} = \frac{qL^2}{12}\,; \quad w_{\max} = \frac{qL^4}{384EI}$$



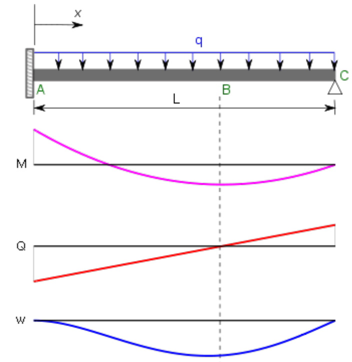

(c) Concentrated load P




(d) Moment $M_0$

Another commonly encountered statically indeterminate beam problem is the cantilevered beam with the free end supported on a roller. The bending moments, shear forces, and deflections of such a beam are listed below.

| Distribution | Max. value |
|---|---|
| $M(x) = -\frac{q}{8}(L^2 - 5Lx + 4x^2)$ | $M_B = -\frac{9qL^2}{128}$ at $x = \frac{5L}{8}$ <br> $M_A = \frac{qL^2}{8}$ |
| $Q(x) = -\frac{q}{8}(8x - 5L)$ | $Q_A = -\frac{5qL}{8}$ |
| $w(x) = \frac{qx^2}{48EI}(3L^2 - 5Lx + 2x^2)$ | $w_{\max} = \frac{qL^4}{185EI}$ at $x = 0.5785L$ |

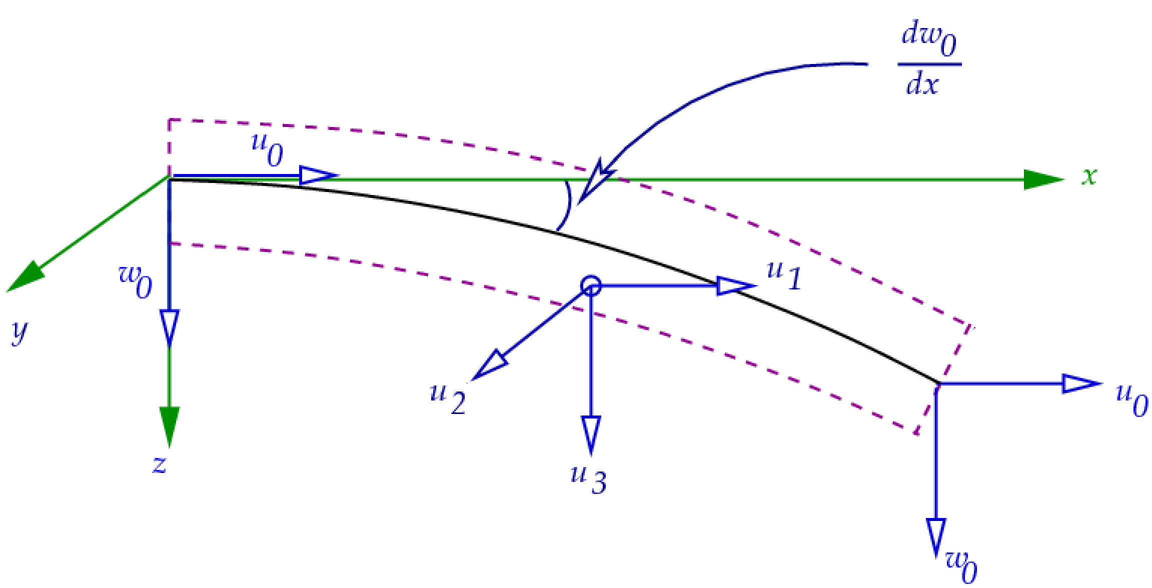



# Extensions

The kinematic assumptions upon which the Euler-Bernoulli beam theory is founded allow it to be extended to more advanced analysis. Simple superposition allows for three-dimensional transverse loading. Using alternative constitutive equations can allow for viscoelastic or plastic beam deformation. Euler-Bernoulli beam theory can also be extended to the analysis of curved beams, beam buckling, composite beams, and geometrically nonlinear beam deflection.

Euler-Bernoulli beam theory does not account for the effects of transverse shear strain. As a result it underpredicts deflections and overpredicts natural frequencies. For thin beams (beam length to thickness ratios of the order 20 or more) these effects are of minor importance. For thick beams, however, these effects can be significant. More advanced beam theories such as the Timoshenko beam theory (developed by the Russian-born scientist Stephen Timoshenko) have been developed to account for these effects.

## Large deflections




Euler-Bernoulli beam

The original Euler-Bernoulli theory is valid only for infinitesimal strains and small rotations. The theory can be extended in a straightforward manner to problems involving moderately large rotations provided that the strain remains small by using the von Karman strains.

The Euler-Bernoulli hypotheses that plane sections remain plane and normal to the axis of the beam lead to displacements of the form

$$u_1 = u_0(x) - z\frac{dw_0}{dx} \; ; \quad u_2 = 0 \; ; \quad u_3 = w_0(x)$$

Using the definition of the Lagrangian Green strain from finite strain theory, we can find the **von Karman strains** for the beam that are valid for large rotations but small strains. These strains have the form

$$\begin{aligned}
\varepsilon_{11} &= \frac{du_0}{dx_1} - x_3\frac{d^2w_0}{dx_1^2} + \frac{1}{2}\left[\left(\frac{du_0}{dx_1} - x_3\frac{d^2w_0}{dx_1^2}\right)^2 + \left(\frac{dw_0}{dx_1}\right)^2\right] \\
\varepsilon_{22} &= 0 \\
\varepsilon_{33} &= \frac{1}{2}\left(\frac{dw_0}{dx_1}\right)^2 \\
\varepsilon_{23} &= 0 \\
\varepsilon_{31} &= \frac{1}{2}\left(\frac{dw_0}{dx_1} - \frac{dw_0}{dx_1}\right) - \frac{1}{2}\left[\left(\frac{du_0}{dx_1} - x_3\frac{d^2w_0}{dx_1^2}\right)\left(\frac{dw_0}{dx_1}\right)\right] \\
\varepsilon_{12} &= 0
\end{aligned}$$

From the principle of virtual work, the balance of forces and moments in the beams gives us the equilibrium equations

$$\begin{aligned}
\frac{dN_{xx}}{dx} + f(x) &= 0 \\
\frac{d^2M_{xx}}{dx^2} + q(x) + \frac{d}{dx}\left(N_{xx}\frac{dw_0}{dx}\right) &= 0
\end{aligned}$$

where $f(x)$ is the axial load, $q(x)$ is the transverse load, and

$$N_{xx} = \int_A \sigma_{xx}\, dA \; ; \quad M_{xx} = \int_A z\sigma_{xx}\, dA$$

To close the system of equations we need the constitutive equations that relate stresses to strains (and hence stresses to displacements). For large rotations and small strains these relations are

$$N_{xx} = A_{xx}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right] - B_{xx}\frac{d^2w_0}{dx^2}$$
$$M_{xx} = B_{xx}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw_0}{dx}\right)^2\right] - D_{xx}\frac{d^2w_0}{dx^2}$$

where

$$A_{xx} = \int_A E\ dA\ ;\quad B_{xx} = \int_A zE\ dA\ ;\quad D_{xx} = \int_A z^2E\ dA\ .$$

The quantity $A_{xx}$ is the **extensional stiffness**, $B_{xx}$ is the coupled **extensional-bending stiffness**, and $D_{xx}$ is the **bending stiffness**.

For the situation where the beam has a uniform cross-section and no axial load, the governing equation for a large-rotation Euler-Bernoulli beam is

$$EI\ \frac{\mathrm{d}^4w}{\mathrm{d}x^4} - \frac{3}{2}\ EA\ \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2}\right) = q(x)$$

Chapter 3

# Stress (Mechanics)

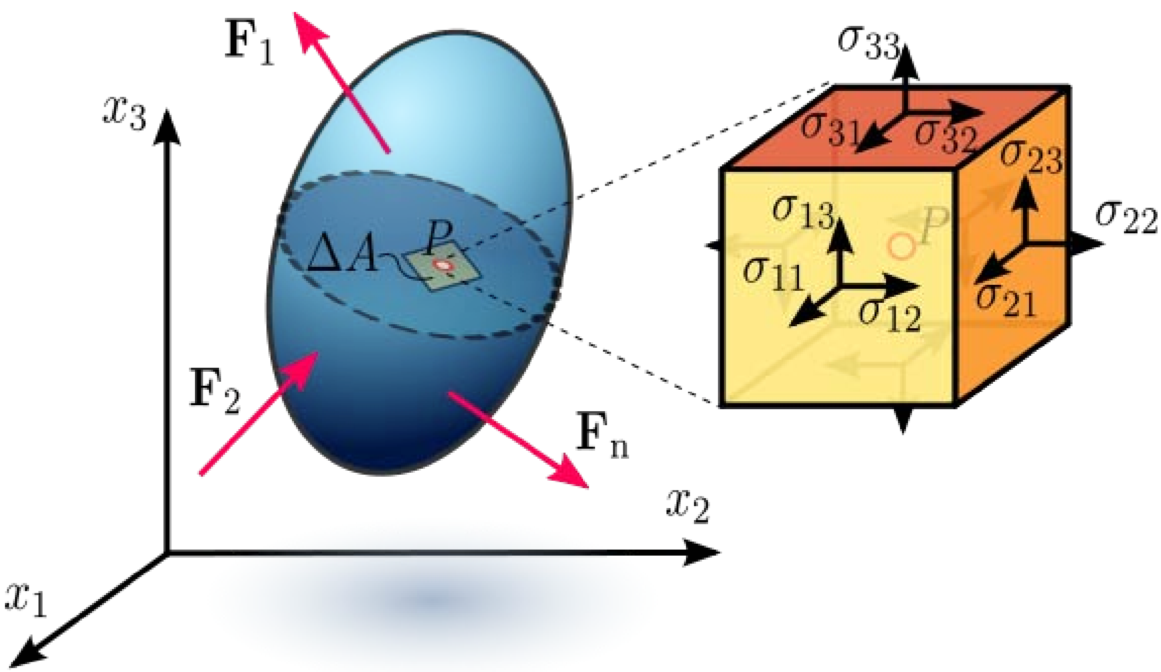



Figure 1.1 Stress in a loaded deformable material body assumed as a continuum.

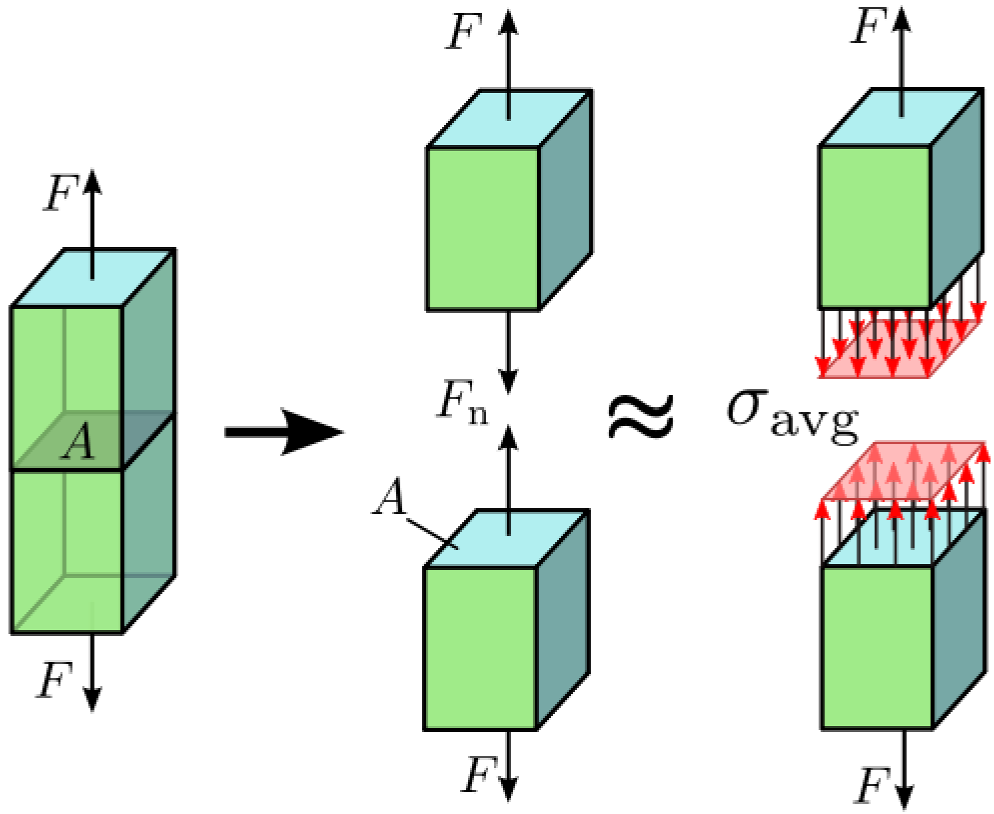



Figure 1.2 Axial stress in a prismatic bar axially loaded

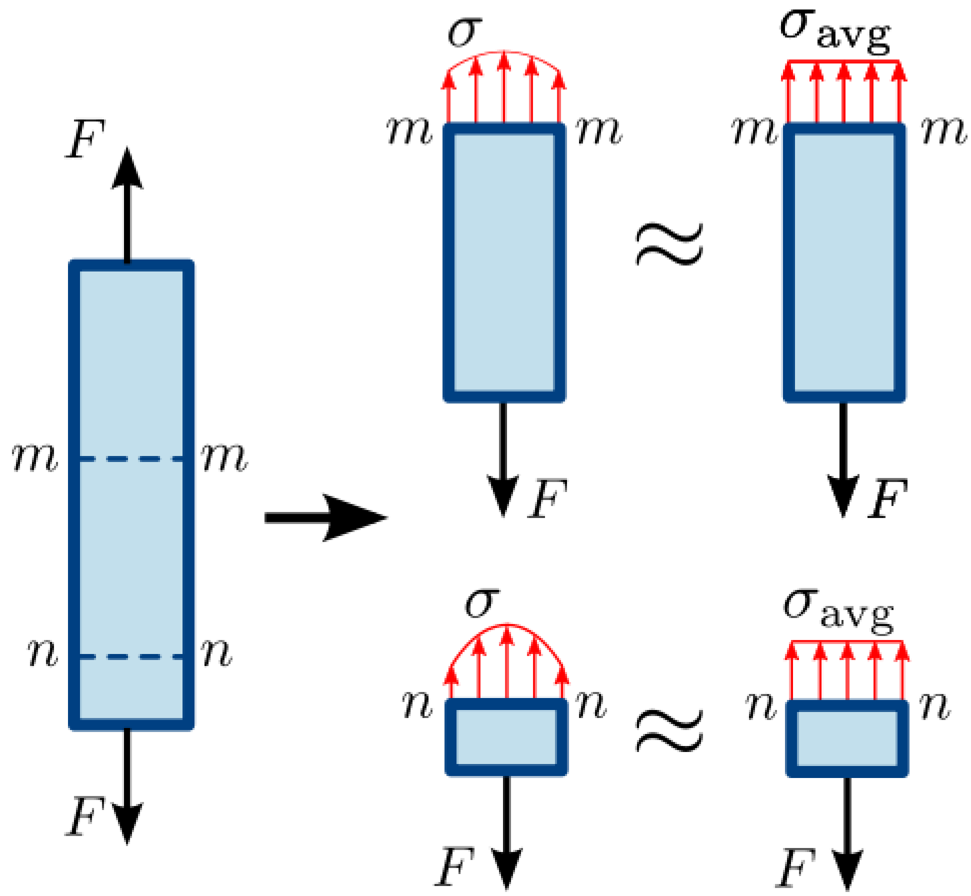



Figure 1.3 Normal stress in a prismatic (straight member of uniform cross-sectional area) bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. However, an average normal stress $\sigma_{\text{avg}}$can be used

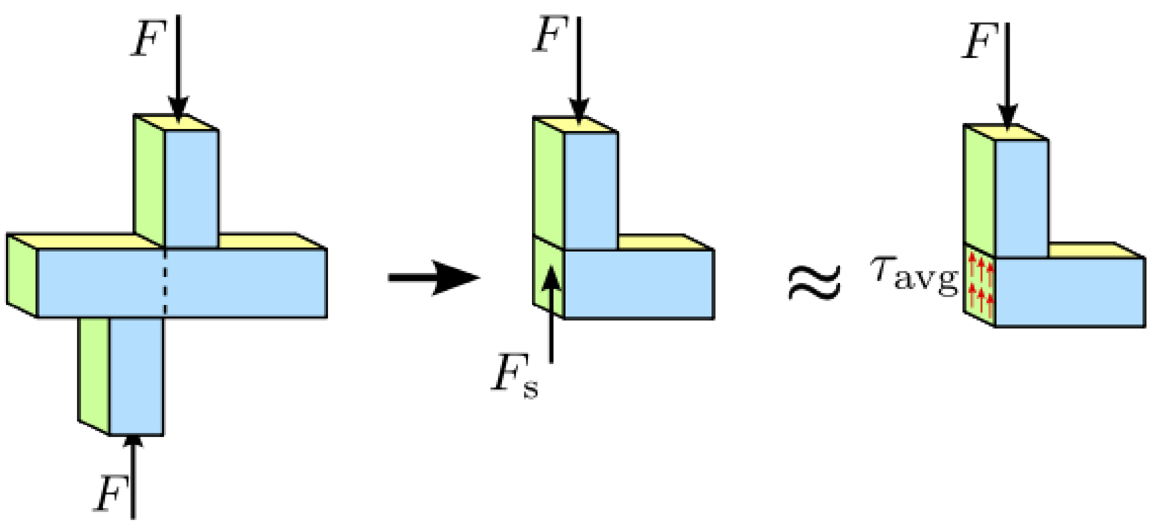



Figure 1.4 Shear stress in a prismatic bar. The stress or force distribution in the cross section of the bar is not necessarily uniform. Nevertheless, an average shear stress $\tau_{\text{avg}}$is a reasonable approximation.

In continuum mechanics, **stress** is a measure of the internal forces acting within a deformable body. Quantitatively, it is a measure of the average force per unit area of a surface within the body on which internal forces act. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, and result in deformation of the body's shape. Beyond certain limits of material strength, this can lead to a permanent change of shape or physical failure.

However, treating physical force as a "one dimensional entity", as it is often done in mechanics, creates a few problems. Any model of continuum mechanics which explicitly expresses force as a variable generally fails to merge and describe deformation of matter and solid bodies, because the attributes of matter and solids are three dimensional. Classical models of continuum mechanics assume an average force and fail to properly incorporate "geometrical factors", which are important to describe stress distribution and accumulation of energy during the continuum.

The dimension of stress is that of pressure, and therefore the SI unit for stress is the pascal (symbol Pa), which is equivalent to one newton (force) per square meter (unit area), that is $N/m^2$. In Imperial units, stress is measured in pound-force per square inch, which is abbreviated as psi.

# Introduction

Stress is a measure of the average force per unit area of a surface within a deformable body on which internal forces act. It is a measure of the intensity of the internal forces acting between particles of a deformable body across imaginary internal surfaces. These internal forces are produced between the particles in the body as a reaction to external forces applied on the body. External forces are either surface forces or body forces. Because the loaded deformable body is assumed to behave as a continuum, these internal forces are distributed continuously within the volume of the material body, *i.e.* the stress distribution in the body is expressed as a piecewise continuous function of space coordinates and time.

### Normal , shear stresses and virial stresses

For the simple case of a body axially loaded, e.g., a prismatic bar subjected to tension or compression by a force passing through its centroid (Figures 1.2 and 1.3) the stress $\sigma$, or intensity of internal forces, can be obtained by dividing the total *normal force* $F_{\mathrm{n}}$, determined from the equilibrium of forces, by the cross-sectional area $A$ of the prism it is acting upon. The normal force can be a *tensile force* if acting outward from the plane, or *compressive force* if acting inward to the plane. In the case of a prismatic bar axially loaded, the stress $\sigma$ is represented by a scalar called *engineering stress* or *nominal stress* that represents an average stress ($\sigma_{\mathrm{avg}}$) over the area, meaning that the stress in the cross section is uniformly distributed. Thus, we have

$$\sigma_{\mathrm{avg}} = \frac{F_{\mathrm{n}}}{A} \approx \sigma$$

A different type of stress is obtained when transverse forces $F$ are applied to the prismatic bar as shown in Figure 1.4. Considering the same cross-section as before, from static equilibrium the internal force has a magnitude equal to $F_{\mathrm{s}}$ and in opposite direction parallel to the cross-section. $F_{\mathrm{s}}$ is called the *shear force*. Dividing the shear force $F_{\mathrm{s}}$ by the area $A$ of the cross section we obtain the *shear stress*. In this case the shear stress $\tau$ is a scalar quantity representing an average shear stress ($\tau_{\mathrm{avg}}$) in the section, *i.e.* the stress in the cross-section is uniformly distributed. In materials science and in engineering aspects the average of the ""scalar"" shear force ($\tau_{\mathrm{avg}}$) are true for crystallized materials during brittle fracture and operates through the fractured cross-section or stress plane.

$$\tau_{\mathrm{avg}} = \frac{F_{\mathrm{s}}}{A} \approx \tau$$

In Figure 1.3, the normal stress is observed in two planes $m - m$ and $n - n$ of the axially loaded prismatic bar. The stress on plane $n - n$, which is closer to the point of application of the load $F$, varies more across the cross-section than that of plane $m - m$. However, if the cross-sectional area of the bar is very small, *i.e.* the bar is slender, the variation of stress across the area is small and the normal stress can be approximated by $\sigma_{\mathrm{avg}}$. On the other hand, the variation of shear stress across the section of a prismatic bar cannot be assumed to be uniform.

Virial stress is a measure of stress on an atomic scale. It is given by

$$\tau_{ij} = \frac{1}{\Omega}\sum_{k\in\Omega}\left(-m^{(k)}(u_i^{(k)} - \bar{u}_i)(u_j^{(k)} - \bar{u}_j) + \frac{1}{2}\sum_{\ell\in\Omega}(x_i^{(\ell)} - x_i^{(k)})f_j^{(k\ell)}\right)$$

where

- *k* and $\ell$ are atoms in the domain,
- Ω is the volume of the domain,
- $m^{(k)}$ is the mass of atom *k*,
- $u_i^{(k)}$ is the $i^{\text{th}}$ component of the velocity of atom *k*,
- $\bar{u}_j$ is the $j^{\text{th}}$ component of the average velocity of atoms in the volume,
- $x_i^{(k)}$ is the $i^{\text{th}}$ component of the position of atom *k*, and
- $f_i^{(k\ell)}$ is the $i^{\text{th}}$ component of the force between atom *k* and $\ell$.

At zero kelvin, all velocities are zero so we have

$$\tau_{ij} = \frac{1}{2\Omega} \sum_{k,\ell \in \Omega} (x_i^{(\ell)} - x_i^{(k)}) f_j^{(k\ell)}$$

This can be thought of as follows. The $\tau_{11}$ component of stress is the force in the 1 direction divided by the area of a plane perpendicular to that direction. Consider two adjacent volumes separated by such a plane. The 11-component of stress on that interface is the sum of all pairwise forces between atoms on the two sides....

## Stress modeling (Cauchy)

In general, stress is not uniformly distributed over the cross-section of a material body, and consequently the stress at a point in a given region is different from the average stress over the entire area. Therefore, it is necessary to define the stress not over a given area but at a specific point in the body (Figure 1.1). According to Cauchy, the *stress at any point* in an object, assumed to behave as a continuum, is completely defined by the nine components $\sigma_{ij}$ of a second-order tensor of type (0,2) known as the Cauchy stress tensor, $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{bmatrix}$$

The Cauchy stress tensor obeys the tensor transformation law under a change in the system of coordinates. A graphical representation of this transformation law is the Mohr's circle of stress distribution.

The Cauchy stress tensor is used for stress analysis of material bodies experiencing small deformations where the differences in stress distribution in most cases can be neglected. For large deformations, also called finite deformations, other measures of stress, such as the first and second Piola-Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor, are required.

According to the principle of conservation of linear momentum, if a continuous body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations (Cauchy's equations of motion for zero acceleration). At the same time, according to the principle of conservation of angular momentum, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, thus having only six independent stress components instead of the original nine.

There are certain invariants associated with the stress tensor, whose values do not depend upon the coordinate system chosen or the area element upon which the stress tensor operates. These are the three eigenvalues of the stress tensor, which are called the principal stresses. Solids, liquids, and gases have stress fields. Static fluids support normal stress but will flow under shear stress. Moving viscous fluids can support shear stress (dynamic pressure). Solids can support both shear and normal stress, with ductile materials failing under shear and brittle materials failing under normal stress. All materials have temperature dependent variations in stress-related properties, and non-Newtonian materials have rate-dependent variations.

### Stress analysis

Stress analysis means the determination of the internal distribution of stresses in a structure. It is needed in engineering for the study and design of structures such as tunnels, dams, mechanical parts, and structural frames, under prescribed or expected loads. To determine the distribution of stress in a structure, the engineer needs to solve a boundary-value problem by specifying the boundary conditions. These are displacements and forces on the boundary of the structure.

Constitutive equations, such as Hooke's Law for linear elastic materials, describe the stress-strain relationship in these calculations.

When a structure is expected to deform elastically (and resume its original shape), a boundary-value problem based on the theory of elasticity is applied, with infinitesimal strains, under design loads.

When the applied loads permanently deform the structure, the theory of plasticity is used.

The stress analysis can be simplified when the physical dimensions and the distribution of loads allow the structure to be treated as one-dimensional or two-dimensional. For a two-dimensional analysis a plane stress or a plane strain condition can be assumed. Alternatively, experimental determination of stresses can be carried out.

Approximate computer-based solutions for boundary-value problems can be obtained through numerical methods such as the Finite Element Method, the Finite Difference Method, and the Boundary Element Method. Analytical or closed-form solutions can be obtained for simple geometries, constitutive relations, and boundary conditions.

## Theoretical background

Continuum mechanics deals with deformable bodies, as opposed to rigid bodies. The stresses considered in continuum mechanics are only those produced by deformation of the body, *sc.* only relative changes in stress are considered, not the absolute values. A body is considered stress-free if the only forces present are those inter-atomic forces (ionic, metallic, and van der Waals forces) required to hold the body together and to keep its shape in the absence of all external influences, including gravitational attraction.

Stresses generated during manufacture of the body to a specific configuration are also excluded.

Following the classical dynamics of Newton and Euler, the motion of a material body is produced by the action of externally applied forces which are assumed to be of two kinds: surface forces and body forces.

Surface forces, or contact forces, can act either on the bounding surface of the body, as a result of mechanical contact with other bodies, or on imaginary internal surfaces that bound portions of the body, as a result of the mechanical interaction between the parts of the body to either side of the surface (Euler-Cauchy's stress principle). When a body is acted upon by external contact forces, internal contact forces are then transmitted from point to point inside the body to balance their action, according to Newton's second law of motion of conservation of linear momentum and angular momentum (for continuous bodies these laws are called the Euler's equations of motion). The internal contact forces are related to the body's deformation through constitutive equations. This is concerned with the manner in which internal contact forces are mathematically described and how they relate to the motion of the body, independent of the body's material makeup.

The concept of stress can then be thought as a measure of the intensity of the internal contact forces acting between particles of the body across imaginary internal surfaces. In other words, stress is a measure of the average quantity of force exerted per unit area of the surface on which these internal forces act. The intensity of contact forces is related, specifically in an inverse proportion, to the area of contact. For example, if a force applied to a small area is compared to a distributed load of the same resultant magnitude applied to a larger area, one finds that the effects or intensities of these two forces are locally different because the stresses are not the same.

Body forces are forces originating from sources outside of the body that act on the volume (or mass) of the body. Saying that body forces are due to outside sources implies that the *internal forces* are manifested through the contact forces alone. These forces arise from the presence of the body in force fields, (*e.g.*, a gravitational field). As the mass of a continuous body is assumed to be continuously distributed, any force originating from the mass is also continuously distributed. Thus, body forces are assumed to be continuous over the entire volume of the body.

The density of internal forces at every point in a deformable body are not necessarily equal, *i.e.* there is a distribution of stresses throughout the body. This variation of internal forces throughout the body is governed by Newton's second law of motion of conservation of linear momentum and angular momentum, which normally are applied to a mass particle but are extended in continuum mechanics to a body of continuously distributed mass. For continuous bodies these laws are called Euler’s equations of motion. If a body is represented as an assemblage of discrete particles, each governed by Newton’s laws of motion, then Euler’s equations can be derived from Newton’s laws. Euler’s equations can, however, be taken as axioms describing the laws of motion for extended bodies, independently of any particle structure.

## Euler–Cauchy stress principle

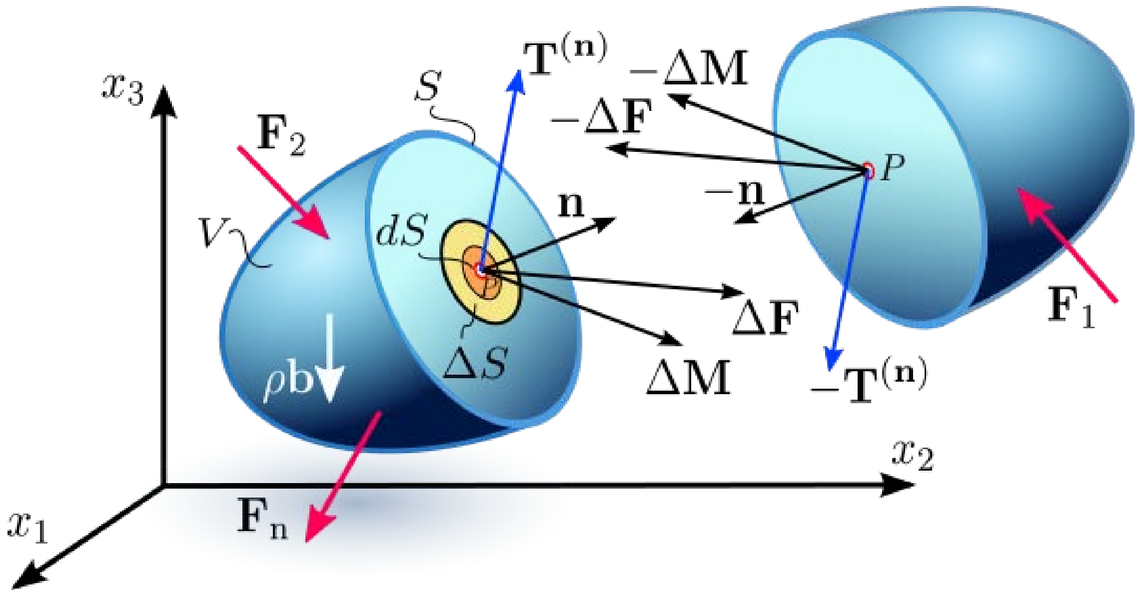



Figure 2.1a Internal distribution of contact forces and couple stresses on a differential $dS$ of the internal surface $S$ in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

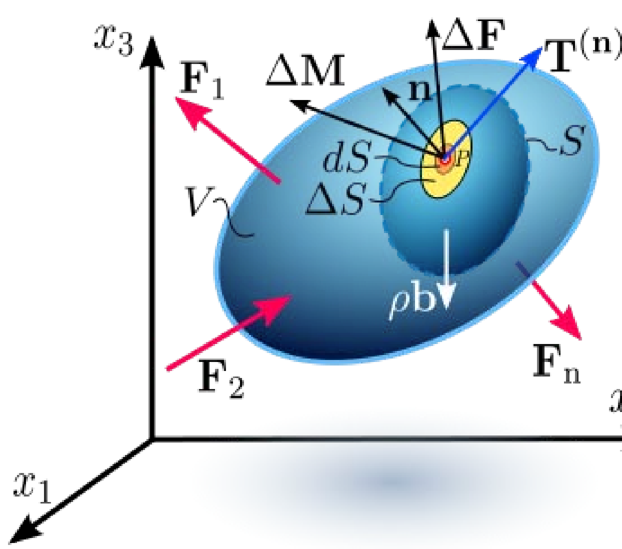



Figure 2.1b Internal distribution of contact forces and couple stresses on a differential $dS$ of the internal surface $S$ in a continuum, as a result of the interaction between the two portions of the continuum separated by the surface

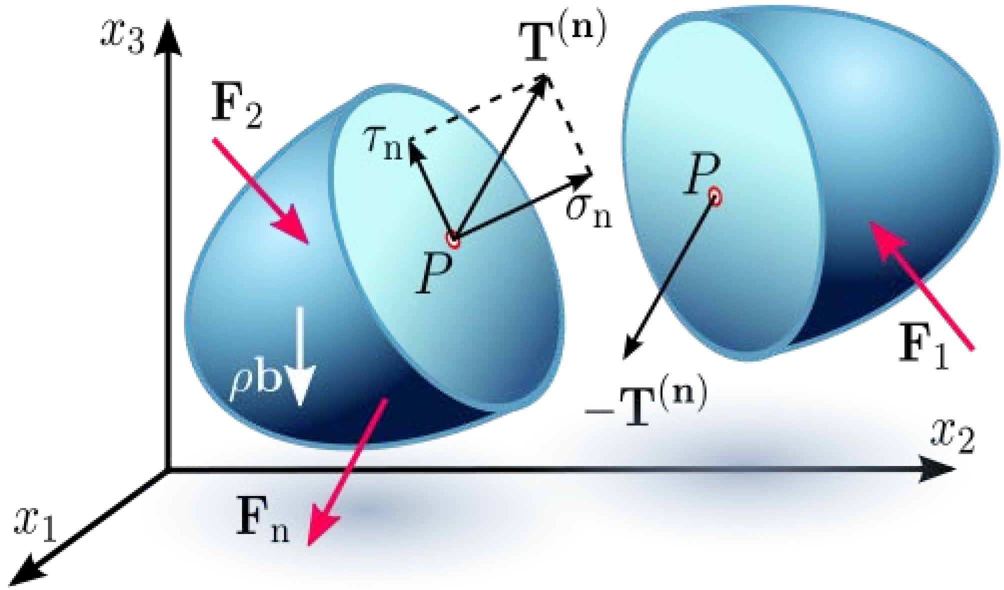



Figure 2.1c Stress vector on an internal surface S with normal vector n. Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to $\mathbf{n}$, and can be resolved into two components: one component normal to the plane, called *normal stress* $\sigma_\mathbf{n}$, and another component parallel to this plane, called the *shearing stress* $\tau$.

The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equivalent (equipollent) to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field $\mathbf{T}^{(\mathbf{n})}$, called the stress vector, defined on the surface $S$ and assumed to depend continuously on the surface's unit vector $\mathbf{n}$.

To explain this principle, we consider an imaginary surface $S$ passing through an internal material point $P$ dividing the continuous body into two segments, as seen in Figure 2.1a or 2.1b (some authors use the cutting plane diagram and others use the diagram with the arbitrary volume inside the continuum enclosed by the surface $S$). The body is subjected to external surface forces $\mathbf{F}$ and body forces $\mathbf{b}$. The internal contact forces being transmitted from one segment to the other through the dividing plane, due to the action of one portion of the continuum onto the other, generate a force distribution on a small area $\Delta S$, with a normal unit vector $\mathbf{n}$, on the dividing plane $S$. The force distribution is equipollent to a contact force $\Delta\mathbf{F}$ and a couple stress $\Delta\mathbf{M}$, as shown in Figure 2.1a and 2.1b. Cauchy's stress principle asserts that as $\Delta S$ becomes very small and tends to zero the ratio $\Delta\mathbf{F}/\Delta S$ becomes $d\mathbf{F}/dS$ and the couple stress vector $\Delta\mathbf{M}$ vanishes. In specific fields of continuum mechanics the couple stress is assumed not to vanish; however, as stated previously, in classical branches of continuum mechanics we deal with non-polar materials which do not consider couple stresses and body moments. The resultant vector $d\mathbf{F}/dS$ is defined as the *stress vector* or *traction vector* given by $\mathbf{T}^{(\mathbf{n})} = T_i^{(\mathbf{n})}\,\mathbf{e}_i$ at the point $P$ associated with a plane with a normal vector $\mathbf{n}$:

$$T_i^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta F_i}{\Delta S} = \frac{dF_i}{dS}.$$

This equation means that the stress vector depends on its location in the body and the orientation of the plane on which it is acting.

Depending on the orientation of the plane under consideration, the stress vector may not necessarily be perpendicular to that plane, *i.e.* parallel to **n**, and can be resolved into two components:

- one normal to the plane, called *normal stress*

  $$\sigma_{\mathbf{n}} = \lim_{\Delta S \to 0} \frac{\Delta F_{\mathbf{n}}}{\Delta S} = \frac{dF_{\mathbf{n}}}{dS},$$

  where $dF_n$ is the normal component of the force d**F** to the differential area d$S$

- and the other parallel to this plane, called the *shear stress*

  $$\tau = \lim_{\Delta S \to 0} \frac{\Delta F_{\mathbf{s}}}{\Delta S} = \frac{dF_{\mathbf{s}}}{dS},$$

  where $dF_s$ is the tangential component of the force d**F** to the differential surface area d$S$. The shear stress can be further decomposed into two mutually perpendicular vectors.

## Cauchy's postulate

According to the *Cauchy Postulate*, the stress vector $\mathbf{T}^{(\mathbf{n})}$ remains unchanged for all surfaces passing through the point $P$ and having the same normal vector **n** at $P$, *i.e.* having a common tangent at $P$. This means that the stress vector is a function of the normal vector **n** only, and it is not influenced by the curvature of the internal surfaces.

## Cauchy's fundamental lemma

A consequence of Cauchy's postulate is *Cauchy's Fundamental Lemma*, also called the *Cauchy reciprocal theorem*, which states that the stress vectors acting on opposite sides of the same surface are equal in magnitude and opposite in direction. Cauchy's fundamental lemma is equivalent to Newton's third law of motion of action and reaction, and it is expressed as

$$-\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(-\mathbf{n})}.$$

## Cauchy's stress theorem – stress tensor

*The state of stress at a point* in the body is then defined by all the stress vectors $\mathbf{T}^{(\mathbf{n})}$ associated with all planes (infinite in number) that pass through that point. However, according to *Cauchy's fundamental theorem*, also called *Cauchy's stress theorem*, merely by knowing the stress vectors on three mutually perpendicular planes, the stress vector on any other plane passing through that point can be found through coordinate transformation equations.

Cauchy's stress theorem states that there exists a second-order tensor field **σ**(**x**, t), called the *Cauchy stress tensor*, independent of **n**, such that **T** is a linear function of **n**:

$$\mathbf{T}^{(\mathbf{n})} = \boldsymbol{\sigma} \cdot \mathbf{n} \quad \text{or} \quad T_j^{(n)} = \sigma_{ij} n_i.$$

This equation implies that the stress vector $\mathbf{T}^{(\mathbf{n})}$ at any point *P* in a continuum associated with a plane with normal vector **n** can be expressed as a function of the stress vectors on the planes perpendicular to the coordinate axes, *i.e.* in terms of the components $\sigma_{ij}$ of the stress tensor **σ**.

To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal area d*A* oriented in an arbitrary direction specified by a normal vector **n** (Figure 2.2). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane **n**. The stress vector on this plane is denoted by $\mathbf{T}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$, and are by definition the components $\sigma_{ij}$ of the stress tensor **σ**. This tetrahedron is sometimes called the *Cauchy tetrahedron*. From equilibrium of forces, *i.e.* Euler's first law of motion (Newton's second law of motion), we have

$$\mathbf{T}^{(\mathbf{n})}\,dA - \mathbf{T}^{(\mathbf{e}_1)}\,dA_1 - \mathbf{T}^{(\mathbf{e}_2)}\,dA_2 - \mathbf{T}^{(\mathbf{e}_3)}\,dA_3 = \rho\left(\frac{h}{3}dA\right)\mathbf{a},$$

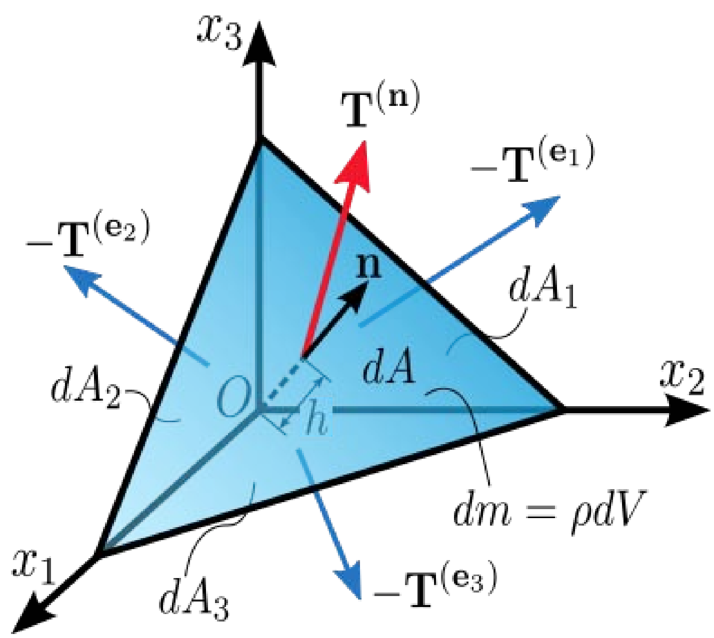


Figure 2.2. Stress vector acting on a plane with normal vector **n**.

**A note on the sign convention:** The tetrahedron is formed by slicing a parallelepiped along an arbitrary plane **n**. So, the force acting on the plane **n** is the reaction exerted by the other half of the parallelepiped and has an opposite sign.

where the right-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: $\rho$ is the density, **a** is the acceleration, and $h$ is the height of the tetrahedron, considering the plane **n** as the base. The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting d$A$ into each face (using the dot product):

$$\begin{aligned} dA_1 &= (\mathbf{n}\cdot\mathbf{e}_1)\,dA = n_1\,dA,\\ dA_2 &= (\mathbf{n}\cdot\mathbf{e}_2)\,dA = n_2\,dA,\\ dA_3 &= (\mathbf{n}\cdot\mathbf{e}_3)\,dA = n_3\,dA, \end{aligned}$$

and then substituting into the equation to cancel out d$A$:

$$\mathbf{T}^{(\mathbf{n})} - \mathbf{T}^{(\mathbf{e}_1)}n_1 - \mathbf{T}^{(\mathbf{e}_2)}n_2 - \mathbf{T}^{(\mathbf{e}_3)}n_3 = \rho\left(\frac{h}{3}\right)\mathbf{a}.$$

To consider the limiting case as the tetrahedron shrinks to a point, $h$ must go to 0 (intuitively, the plane **n** is translated along **n** toward $O$). As a result, the right-hand-side of the equation approaches 0, so

$$\mathbf{T}^{(\mathbf{n})} = \mathbf{T}^{(\mathbf{e}_1)}n_1 + \mathbf{T}^{(\mathbf{e}_2)}n_2 + \mathbf{T}^{(\mathbf{e}_3)}n_3.$$

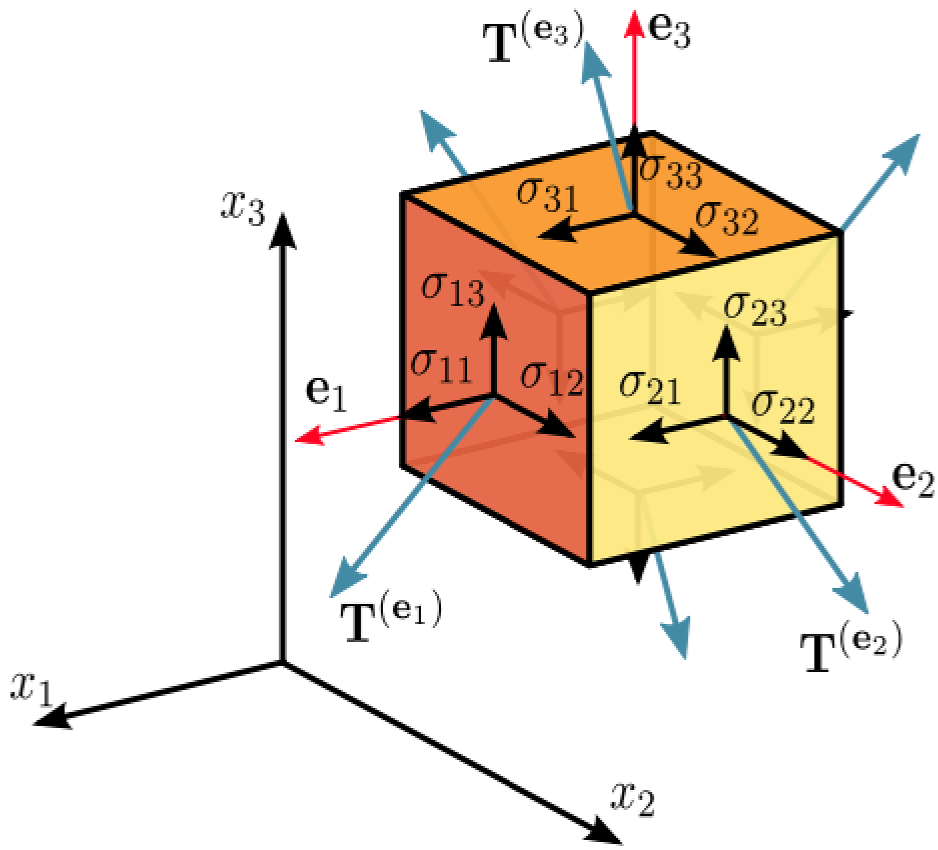



Figure 2.3 Components of stress in three dimensions

Assuming a material element (Figure 2.3) with planes perpendicular to the coordinate axes of a Cartesian coordinate system, the stress vectors associated with each of the element planes, *i.e.* $\mathbf{T}^{(\mathbf{e}_1)}$, $\mathbf{T}^{(\mathbf{e}_2)}$, and $\mathbf{T}^{(\mathbf{e}_3)}$ can be decomposed into a normal component and two shear components, *i.e.* components in the direction of the three coordinate axes. For the particular case of a surface with normal unit vector oriented in the direction of the $x_1$-axis, the normal stress is denoted by $\sigma_{11}$, and the two shear stresses are denoted as $\sigma_{12}$ and $\sigma_{13}$:

$$\mathbf{T}^{(\mathbf{e}_1)} = T_1^{(\mathbf{e}_1)}\mathbf{e}_1 + T_2^{(\mathbf{e}_1)}\mathbf{e}_2 + T_3^{(\mathbf{e}_1)}\mathbf{e}_3 = \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3,$$
$$\mathbf{T}^{(\mathbf{e}_2)} = T_1^{(\mathbf{e}_2)}\mathbf{e}_1 + T_2^{(\mathbf{e}_2)}\mathbf{e}_2 + T_3^{(\mathbf{e}_2)}\mathbf{e}_3 = \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3,$$
$$\mathbf{T}^{(\mathbf{e}_3)} = T_1^{(\mathbf{e}_3)}\mathbf{e}_1 + T_2^{(\mathbf{e}_3)}\mathbf{e}_2 + T_3^{(\mathbf{e}_3)}\mathbf{e}_3 = \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3,$$

In index notation this is

$$\mathbf{T}^{(\mathbf{e}_i)} = T_j^{(\mathbf{e}_i)}\mathbf{e}_j = \sigma_{ij}\mathbf{e}_j.$$

The nine components $\sigma_{ij}$ of the stress vectors are the components of a second-order Cartesian tensor called the *Cauchy stress tensor*, which completely defines the state of stress at a point and is given by

$$\boldsymbol{\sigma} = \sigma_{ij} = \begin{bmatrix} \mathbf{T}^{(\mathbf{e}_1)} \\ \mathbf{T}^{(\mathbf{e}_2)} \\ \mathbf{T}^{(\mathbf{e}_3)} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \equiv \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix},$$

where $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{33}$ are normal stresses, and $\sigma_{12}$, $\sigma_{13}$, $\sigma_{21}$, $\sigma_{23}$, $\sigma_{31}$, and $\sigma_{32}$ are shear stresses. The first index $i$ indicates that the stress acts on a plane normal to the $x_i$-axis, and the second index $j$ denotes the direction in which the stress acts. A stress component is positive if it acts in the positive direction of the coordinate axes, and if the plane where it acts has an outward normal vector pointing in the positive coordinate direction.

Thus, using the components of the stress tensor

$$\begin{aligned} \mathbf{T}^{(\mathbf{n})} &= \mathbf{T}^{(\mathbf{e}_1)} n_1 + \mathbf{T}^{(\mathbf{e}_2)} n_2 + \mathbf{T}^{(\mathbf{e}_3)} n_3 \\ &= \sum_{i=1}^{3} \mathbf{T}^{(\mathbf{e}_i)} n_i \\ &= \left(\sigma_{ij} \mathbf{e}_j\right) n_i \\ &= \sigma_{ij} n_i \mathbf{e}_j \end{aligned}$$

or, equivalently,

$$T_j^{(\mathbf{n})} = \sigma_{ij} n_i.$$

Alternatively, in matrix form we have

$$\begin{bmatrix} T_1^{(\mathbf{n})} & T_2^{(\mathbf{n})} & T_3^{(\mathbf{n})} \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

The Voigt notation representation of the Cauchy stress tensor takes advantage of the symmetry of the stress tensor to express the stress as a six-dimensional vector of the form:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & \sigma_5 & \sigma_6 \end{bmatrix}^T \equiv \begin{bmatrix} \sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{31} & \sigma_{12} \end{bmatrix}^T.$$

The Voigt notation is used extensively in representing stress-strain relations in solid mechanics and for computational efficiency in numerical structural mechanics software.

## Transformation rule of the stress tensor

It can be shown that the stress tensor is a contravariant second order tensor, which is a statement of how it transforms under a change of the coordinate system. From an $x_i$-system to an $x_i'$-system, the components $\sigma_{ij}$ in the initial system are transformed into the components $\sigma_{ij}'$ in the new system according to the tensor transformation rule (Figure 2.4):

$$\sigma_{ij}' = a_{im}a_{jn}\sigma_{mn} \quad \text{or} \quad \boldsymbol{\sigma}' = \mathbf{A}\boldsymbol{\sigma}\mathbf{A}^T,$$

where **A** is a rotation matrix with components $a_{ij}$. In matrix form this is

$$\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

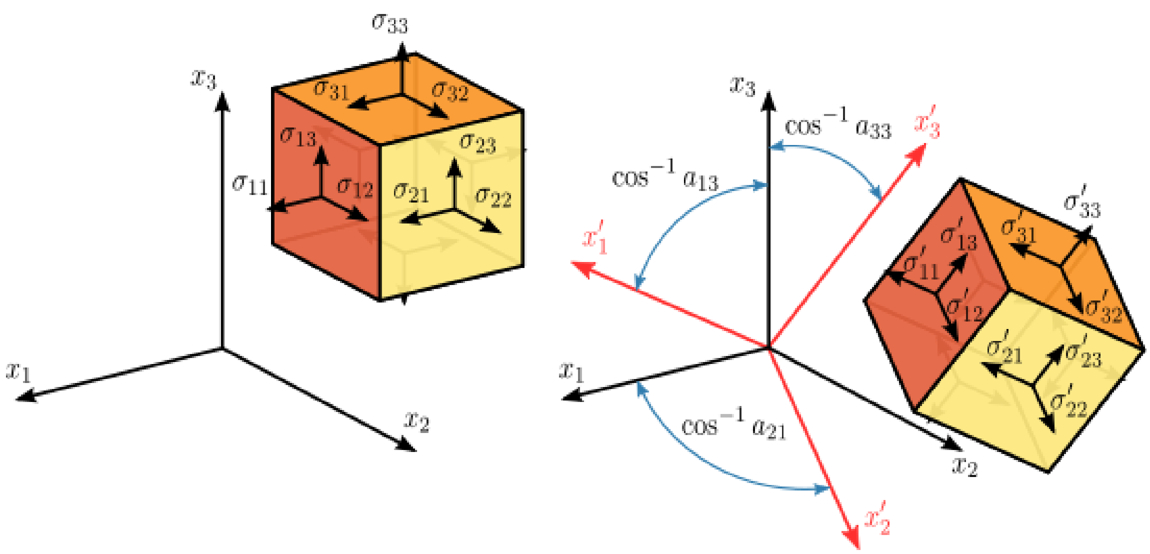



Figure 2.4 Transformation of the stress tensor

Expanding the matrix operation, and simplifying some terms by taking advantage of the symmetry of the stress tensor, gives

$$\begin{aligned}
\sigma_{11}' &= a_{11}^2\sigma_{11} + a_{12}^2\sigma_{22} + a_{13}^2\sigma_{33} + 2a_{11}a_{12}\sigma_{12} + 2a_{11}a_{13}\sigma_{13} + 2a_{12}a_{13}\sigma_{23}, \\
\sigma_{22}' &= a_{21}^2\sigma_{11} + a_{22}^2\sigma_{22} + a_{23}^2\sigma_{33} + 2a_{21}a_{22}\sigma_{12} + 2a_{21}a_{23}\sigma_{13} + 2a_{22}a_{23}\sigma_{23}, \\
\sigma_{33}' &= a_{31}^2\sigma_{11} + a_{32}^2\sigma_{22} + a_{33}^2\sigma_{33} + 2a_{31}a_{32}\sigma_{12} + 2a_{31}a_{33}\sigma_{13} + 2a_{32}a_{33}\sigma_{23}, \\
\sigma_{12}' &= a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + a_{13}a_{23}\sigma_{33} \\
&\quad + (a_{11}a_{22} + a_{12}a_{21})\sigma_{12} + (a_{12}a_{23} + a_{13}a_{22})\sigma_{23} + (a_{11}a_{23} + a_{13}a_{21})\sigma_{13},
\end{aligned}$$

$$\begin{aligned}\sigma'_{23} =& a_{21}a_{31}\sigma_{11} + a_{22}a_{32}\sigma_{22} + a_{23}a_{33}\sigma_{33}\\ &+ (a_{21}a_{32} + a_{22}a_{31})\sigma_{12} + (a_{22}a_{33} + a_{23}a_{32})\sigma_{23} + (a_{21}a_{33} + a_{23}a_{31})\sigma_{13},\\ \sigma'_{13} =& a_{11}a_{31}\sigma_{11} + a_{12}a_{32}\sigma_{22} + a_{13}a_{33}\sigma_{33}\\ &+ (a_{11}a_{32} + a_{12}a_{31})\sigma_{12} + (a_{12}a_{33} + a_{13}a_{32})\sigma_{23} + (a_{11}a_{33} + a_{13}a_{31})\sigma_{13}.\end{aligned}$$

The Mohr circle for stress is a graphical representation of this transformation of stresses.

## Normal and shear stresses

The magnitude of the normal stress component $\sigma_\mathrm{n}$ of any stress vector $\mathbf{T}^{(\mathbf{n})}$ acting on an arbitrary plane with normal vector **n** at a given point, in terms of the components $\sigma_{ij}$ of the stress tensor $\boldsymbol{\sigma}$, is the dot product of the stress vector and the normal vector:

$$\begin{aligned}\sigma_\mathrm{n} &= \mathbf{T}^{(\mathbf{n})} \cdot \mathbf{n}\\ &= T_i^{(\mathbf{n})} n_i\\ &= \sigma_{ij} n_i n_j.\end{aligned}$$

The magnitude of the shear stress component $\tau_\mathrm{n}$, acting in the plane spanned by the two vectors $\mathbf{T}^{(\mathbf{n})}$ and **n**, can then be found using the Pythagorean theorem:

$$\begin{aligned}\tau_\mathrm{n} &= \sqrt{\left(T^{(\mathbf{n})}\right)^2 - \sigma_\mathrm{n}^2}\\ &= \sqrt{T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} - \sigma_\mathrm{n}^2},\end{aligned}$$

where

$$\left(T^{(\mathbf{n})}\right)^2 = T_i^{(\mathbf{n})} T_i^{(\mathbf{n})} = \left(\sigma_{ij} n_j\right)\left(\sigma_{ik} n_k\right) = \sigma_{ij}\sigma_{ik} n_j n_k.$$

## Equilibrium equations and symmetry of the stress tensor

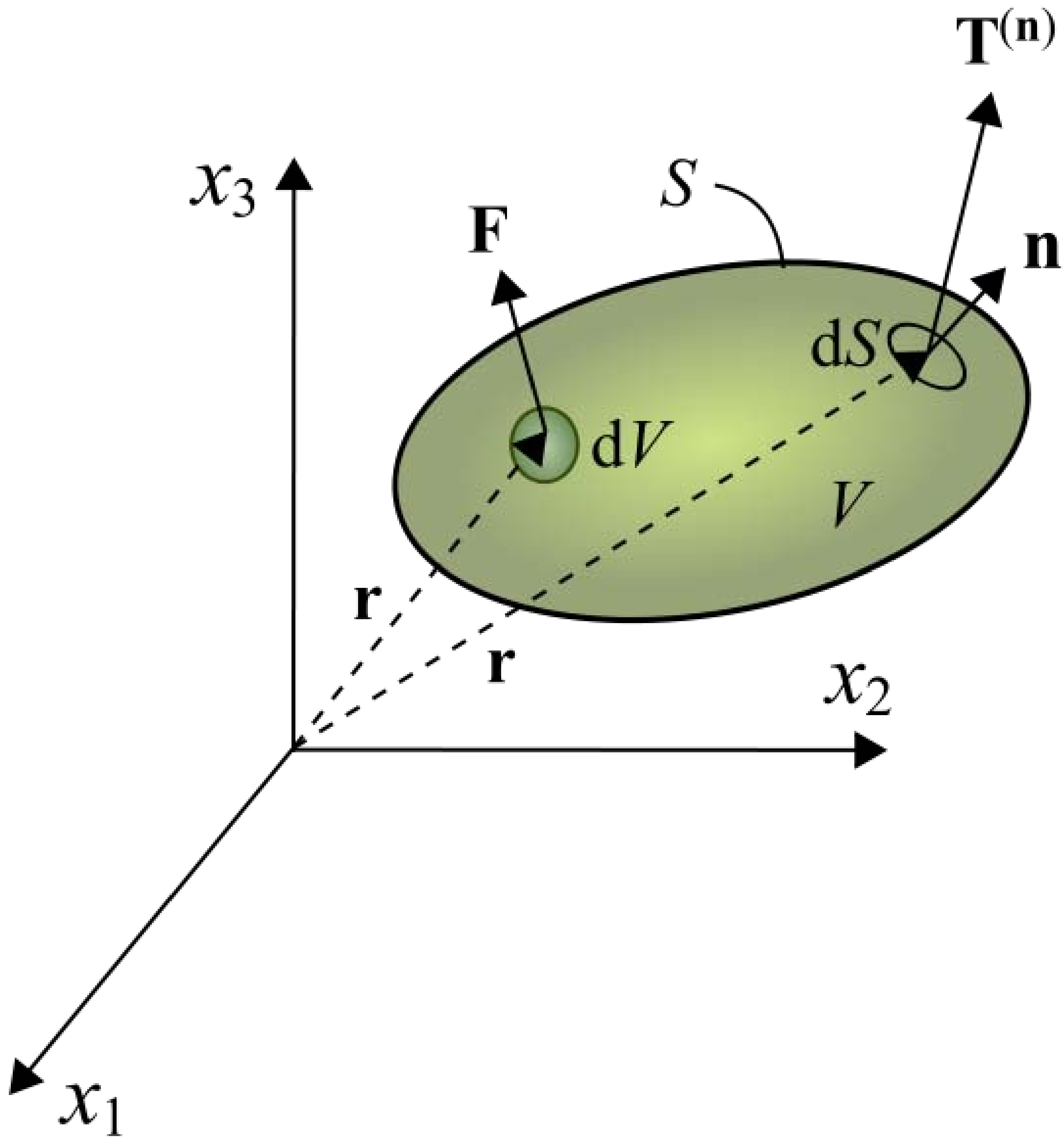



Figure 4. Continuum body in equilibrium

When a body is in equilibrium the components of the stress tensor in every point of the body satisfy the equilibrium equations,

$$\sigma_{ji,j} + F_i = 0$$

For example, for a hydrostatic fluid in equilibrium conditions, the stress tensor takes on the form:

$$\sigma_{ij} = -p\delta_{ij},$$

where $p$ is the hydrostatic pressure, and $\delta_{ij}$ is the kronecker delta.

At the same time, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}$$

However, in the presence of couple-stresses, i.e. moments per unit volume, the stress tensor is non-symmetric. This also is the case when the Knudsen number is close to one, $K_n \rightarrow 1$, or the continuum is a non-Newtonian fluid, which can lead to rotationally non-invariant fluids, such as polymers.

## Principal stresses and stress invariants

At every point in a stressed body there are at least three planes, called *principal planes*, with normal vectors $\mathbf{n}$, called *principal directions*, where the corresponding stress vector is perpendicular to the plane, i.e., parallel or in the same direction as the normal vector $\mathbf{n}$, and where there are no normal shear stresses $\tau_{\mathbf{n}}$. The three stresses normal to these principal planes are called *principal stresses*.

The components $\sigma_{ij}$ of the stress tensor depend on the orientation of the coordinate system at the point under consideration. However, the stress tensor itself is a physical quantity and as such, it is independent of the coordinate system chosen to represent it. There are certain invariants associated with every tensor which are also independent of the coordinate system. For example, a vector is a simple tensor of rank one. In three dimensions, it has three components. The value of these components will depend on the coordinate system chosen to represent the vector, but the length of the vector is a physical quantity (a scalar) and is independent of the coordinate system chosen to represent the vector. Similarly, every second rank tensor (such as the stress and the strain tensors) has three independent invariant quantities associated with it. One set of such invariants are the principal stresses of the stress tensor, which are just the eigenvalues of the stress tensor. Their direction vectors are the principal directions or eigenvectors.

A stress vector parallel to the normal vector $\mathbf{n}$ is given by:

$$\mathbf{T}^{(\mathbf{n})} = \lambda\mathbf{n} = \sigma_{\mathbf{n}}\mathbf{n}$$

where $\lambda$ is a constant of proportionality, and in this particular case corresponds to the magnitudes $\sigma_{\mathbf{n}}$ of the normal stress vectors or principal stresses.

Knowing that $T_i^{(n)} = \sigma_{ij}n_j$ and $n_i = \delta_{ij}n_j$, we have

$$
\begin{aligned}
T_i^{(n)} &= \lambda n_i \\
\sigma_{ij} n_j &= \lambda n_i \\
\sigma_{ij} n_j - \lambda n_i &= 0 \\
(\sigma_{ij} - \lambda \delta_{ij})\, n_j &= 0
\end{aligned}
$$

This is a homogeneous system, i.e. equal to zero, of three linear equations where $n_j$ are the unknowns. To obtain a nontrivial (non-zero) solution for $n_j$, the determinant matrix of the coefficients must be equal to zero, i.e. the system is singular. Thus,

$$
|\sigma_{ij} - \lambda \delta_{ij}| = \begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \lambda \end{vmatrix} = 0
$$

Expanding the determinant leads to the *characteristic equation*

$$
|\sigma_{ij} - \lambda \delta_{ij}| = -\lambda^3 + I_1 \lambda^2 - I_2 \lambda + I_3 = 0
$$

where

$$
\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
&= \sigma_{kk} \\
I_2 &= \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} \\
&= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) \\
I_3 &= \det(\sigma_{ij}) \\
&= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{13}^2\sigma_{22}
\end{aligned}
$$

The characteristic equation has three real roots $\lambda$, i.e. not imaginary due to the symmetry of the stress tensor. The three roots $\lambda_1 = \sigma_1$, $\lambda_2 = \sigma_2$, and $\lambda_3 = \sigma_3$ are the eigenvalues or principal stresses, and they are the roots of the Cayley–Hamilton theorem. The principal stresses are unique for a given stress tensor. Therefore, from the characteristic equation it is seen that the coefficients $I_1$, $I_2$ and $I_3$, called the first, second, and third *stress invariants*, respectively, have always the same value regardless of the orientation of the coordinate system chosen.

For each eigenvalue, there is a non-trivial solution for $n_j$ in the equation $(\sigma_{ij} - \lambda\delta_{ij})\, n_j = 0$. These solutions are the principal directions or eigenvectors defining the plane where the principal stresses act. The principal stresses and principal directions characterize the stress at a point and are independent of the orientation of the coordinate system.

If we choose a coordinate system with axes oriented to the principal directions, then the normal stresses will be the principal stresses and the stress tensor is represented by a diagonal matrix:

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

The principal stresses may be combined to form the stress invariants, $I_1$, $I_2$, and $I_3$.The first and third invariant are the trace and determinant respectively, of the stress tensor. Thus,

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$
$$I_3 = \sigma_1\sigma_2\sigma_3$$

Because of its simplicity, working and thinking in the principal coordinate system is often very useful when considering the state of the elastic medium at a particular point.

Principal stresses are often expressed in the following equation for evaluating stresses in the x and y directions or axial and bending stresses on a part. The principal normal stresses can then be used to calculate the Von Mises stress and ultimately the safety factor and margin of safety.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Using just the part of the equation under the square root is equal to the maximum and minimum shear stress for plus and minus. This is shown as:

$$\tau_{max}, \tau_{min} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

## Maximum and minimum shear stresses

The maximum shear stress or maximum principal shear stress is equal to one-half the difference between the largest and smallest principal stresses, and acts on the plane that bisects the angle between the directions of the largest and smallest principal stresses, i.e. the plane of the maximum shear stress is oriented $45^\circ$ from the principal stress planes. The maximum shear stress is expressed as

$$\tau_{\max} = \frac{1}{2}\left|\sigma_{\max} - \sigma_{\min}\right|$$

Assuming $\sigma_1 \ge \sigma_2 \ge \sigma_3$ then

$$\tau_{\max} = \frac{1}{2}\left|\sigma_1 - \sigma_3\right|$$

The normal stress component acting on the plane for the maximum shear stress is non-zero and it is equal to

$$\sigma_{\mathrm{n}} = \frac{1}{2}\left(\sigma_1 + \sigma_3\right)$$

## Stress deviator tensor

The stress tensor $\sigma_{ij}$ can be expressed as the sum of two other stress tensors:

1. a *mean hydrostatic stress tensor* or *volumetric stress tensor* or *mean normal stress tensor*, $p\delta_{ij}$, which tends to change the volume of the stressed body; and
2. a deviatoric component called the *stress deviator tensor*, $s_{ij}$, which tends to distort it.

So:

$$\sigma_{ij} = s_{ij} + p\delta_{ij},$$

where $p$ is the mean stress given by

$$p = \frac{\sigma_{kk}}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \tfrac{1}{3}I_1.$$

Note that convention in solid mechanics differs slightly from what is listed above. In solid mechanics, pressure is generally defined as negative one-third the trace of the stress tensor.

The deviatoric stress tensor can be obtained by subtracting the hydrostatic stress tensor from the stress tensor:

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},$$

$$\begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - p & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - p & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - p \end{bmatrix}.$$

## Invariants of the stress deviator tensor

As it is a second order tensor, the stress deviator tensor also has a set of invariants, which can be obtained using the same procedure used to calculate the invariants of the stress tensor. It can be shown that the principal directions of the stress deviator tensor $s_{ij}$ are the same as the principal directions of the stress tensor $\sigma_{ij}$. Thus, the characteristic equation is

$$|s_{ij} - \lambda\delta_{ij}| = \lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0,$$

where $J_1$, $J_2$ and $J_3$ are the first, second, and third *deviatoric stress invariants*, respectively. Their values are the same (invariant) regardless of the orientation of the coordinate system chosen. These deviatoric stress invariants can be expressed as a function of the components of $s_{ij}$ or its principal values $s_1$, $s_2$, and $s_3$, or alternatively, as a function of $\sigma_{ij}$ or its principal values $\sigma_1$, $\sigma_2$, and $\sigma_3$. Thus,

$$\begin{aligned} J_1 &= s_{kk} = 0, \\ J_2 &= \tfrac{1}{2}s_{ij}s_{ji} \\ &= -s_1s_2 - s_2s_3 - s_3s_1 \\ &= \tfrac{1}{6}\left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\right] + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \\ &= \tfrac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \\ &= \tfrac{1}{3}I_1^2 - I_2, \\ J_3 &= \det(s_{ij}) \\ &= \tfrac{1}{3}s_{ij}s_{jk}s_{ki} \\ &= s_1s_2s_3 \\ &= \tfrac{2}{27}I_1^3 - \tfrac{1}{3}I_1I_2 + I_3. \end{aligned}$$

Because $s_{kk} = 0$, the stress deviator tensor is in a state of pure shear.

A quantity called the equivalent stress or von Mises stress is commonly used in solid mechanics. The equivalent stress is defined as

$$\sigma_e = \sqrt{3\ J_2} = \sqrt{\tfrac{1}{2}\ \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]}\,.$$

## Octahedral stresses

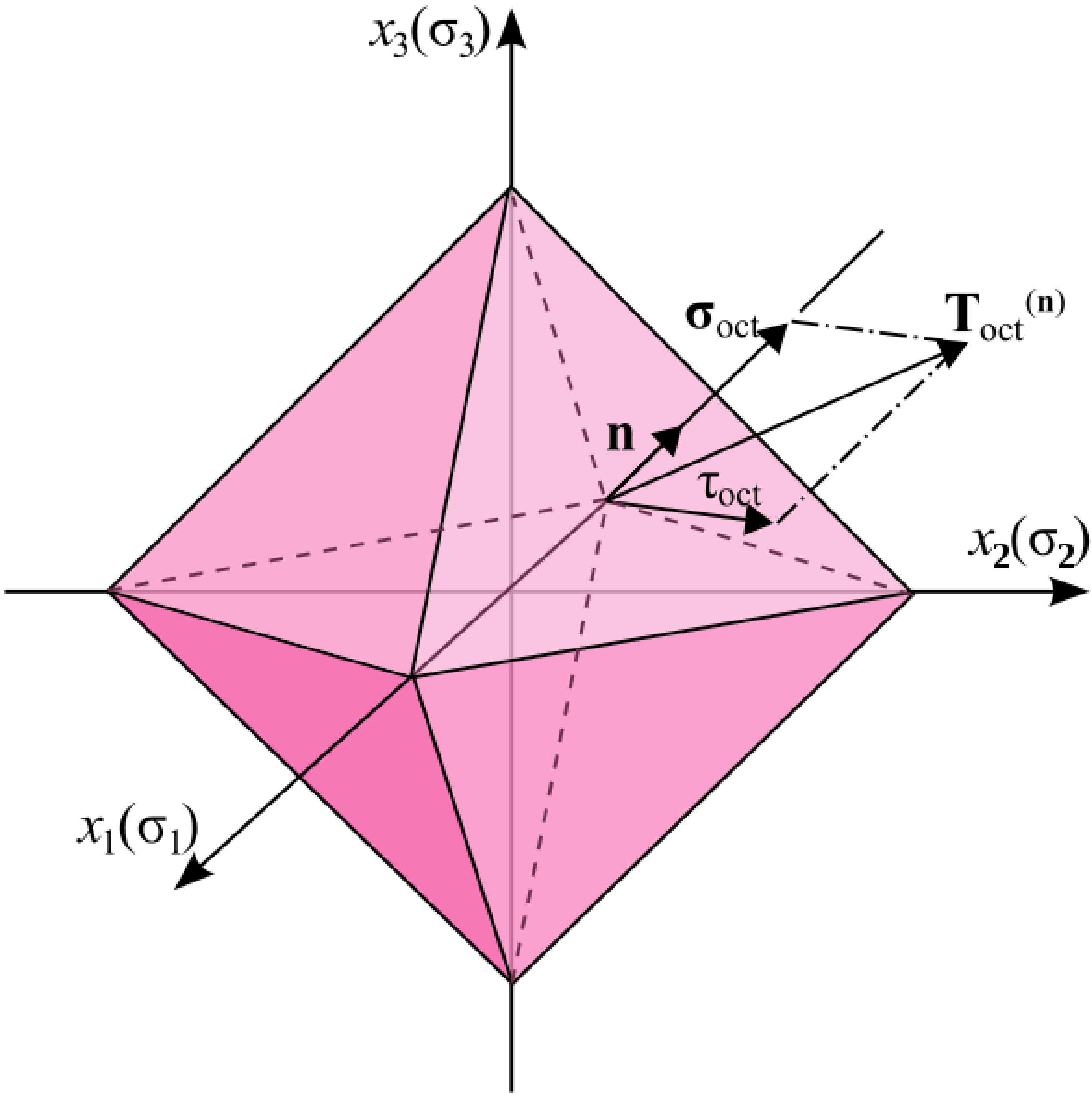



Figure 6. Octahedral stress planes

Considering the principal directions as the coordinate axes, a plane whose normal vector makes equal angles with each of the principal axes (i.e. having direction cosines equal to $|1/\sqrt{3}|$) is called an *octahedral plane*. There are a total of eight octahedral planes (Figure 6). The normal and shear components of the stress tensor on these planes are called *octahedral normal stress* $\sigma_{\mathrm{oct}}$and *octahedral shear stress* $\tau_{\mathrm{oct}}$, respectively.

Knowing that the stress tensor of point O (Figure 6) in the principal axes is

$$\sigma_{ij} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

the stress vector on an octahedral plane is then given by:

$$\begin{aligned} \mathbf{T}_{\mathrm{oct}}^{(\mathbf{n})} &= \sigma_{ij} n_i \mathbf{e}_j \\ &= \sigma_1 n_1 \mathbf{e}_1 + \sigma_2 n_2 \mathbf{e}_2 + \sigma_3 n_3 \mathbf{e}_3 \\ &= \tfrac{1}{\sqrt{3}}(\sigma_1 \mathbf{e}_1 + \sigma_2 \mathbf{e}_2 + \sigma_3 \mathbf{e}_3) \end{aligned}$$

The normal component of the stress vector at point O associated with the octahedral plane is

$$\begin{aligned} \sigma_{\mathrm{oct}} &= T_i^{(n)} n_i \\ &= \sigma_{ij} n_i n_j \\ &= \sigma_1 n_1 n_1 + \sigma_2 n_2 n_2 + \sigma_3 n_3 n_3 \\ &= \tfrac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \tfrac{1}{3} I_1 \end{aligned}$$

which is the mean normal stress or hydrostatic stress. This value is the same in all eight octahedral planes. The shear stress on the octahedral plane is then

$$\begin{aligned} \tau_{\mathrm{oct}} &= \sqrt{T_i^{(n)} T_i^{(n)} - \sigma_{\mathrm{n}}^2} \\ &= \left[\tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{1}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2\right]^{1/2} \\ &= \tfrac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} = \tfrac{1}{3}\sqrt{2I_1^2 - 6I_2} = \sqrt{\tfrac{2}{3} J_2} \end{aligned}$$

## Alternative measures of stress

The Cauchy stress tensor is not the only measure of stress that is used in practice. Other measures of stress include the first and second Piola–Kirchhoff stress tensors, the Biot stress tensor, and the Kirchhoff stress tensor.

### Piola–Kirchhoff stress tensor

In the case of finite deformations, the *Piola–Kirchhoff stress tensors* are used to express the stress relative to the reference configuration. This is in contrast to the Cauchy stress tensor which expresses the stress relative to the present configuration. For infinitesimal deformations or rotations, the Cauchy and Piola–Kirchhoff tensors are identical. These tensors take their names from Gabrio Piola and Gustav Kirchhoff.

Whereas the Cauchy stress tensor, $\boldsymbol{\sigma}$ relates stresses in the current configuration, the deformation gradient and strain tensors are described by relating the motion to the reference configuration; thus not all tensors describing the state of the material are in either the reference or current configuration. Having the stress, strain and deformation all described either in the reference or current configuration would make it easier to define constitutive models (for example, the Cauchy Stress tensor is variant to a pure rotation, while the deformation strain tensor is invariant; thus creating problems in defining a constitutive model that relates a varying tensor, in terms of an invariant one during pure rotation; as by definition constitutive models have to be invariant to pure rotations). The $1^{st}$ Piola–Kirchhoff stress tensor, $\boldsymbol{P}$ is one possible solution to this problem. It defines a family of tensors, which describe the configuration of the body in either the current or the reference state.

The $1^{st}$ Piola–Kirchhoff stress tensor, $\boldsymbol{P}$ relates forces in the *present* configuration with areas in the *reference* ("material") configuration.

$$\boldsymbol{P} = J\ \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}$$

where $\boldsymbol{F}$ is the deformation gradient and $J = \det \boldsymbol{F}$ is the Jacobian determinant.

In terms of components with respect to an orthonormal basis, the first Piola–Kirchhoff stress is given by

$$P_{iL} = J\ \sigma_{ik}\ F^{-1}_{Lk} = J\ \sigma_{ik}\ \frac{\partial X_L}{\partial x_k}$$

Because it relates different coordinate systems, the $1^{st}$ Piola–Kirchhoff stress is a two-point tensor. In general, it is not symmetric. The $1^{st}$ Piola–Kirchhoff stress is the 3D generalization of the 1D concept of engineering stress.

If the material rotates without a change in stress state (rigid rotation), the components of the 1st Piola–Kirchhoff stress tensor will vary with material orientation.

The 1st Piola–Kirchhoff stress is energy conjugate to the deformation gradient.

**2nd Piola–Kirchhoff stress tensor**

Whereas the 1st Piola–Kirchhoff stress relates forces in the current configuration to areas in the reference configuration, the 2nd Piola–Kirchhoff stress tensor $\boldsymbol{S}$ relates forces in the reference configuration to areas in the reference configuration. The force in the reference configuration is obtained via a mapping that preserves the relative relationship between the force direction and the area normal in the current configuration.

$$\boldsymbol{S} = J\ \boldsymbol{F}^{-1} \cdot \boldsymbol{\sigma} \cdot \boldsymbol{F}^{-T}\ .$$

In index notation with respect to an orthonormal basis,

$$S_{IL} = J\ F^{-1}_{Ik}\ F^{-1}_{Lm}\ \sigma_{km} = J\ \frac{\partial X_I}{\partial x_k}\ \frac{\partial X_L}{\partial x_m}\ \sigma_{km}$$

This tensor is symmetric.

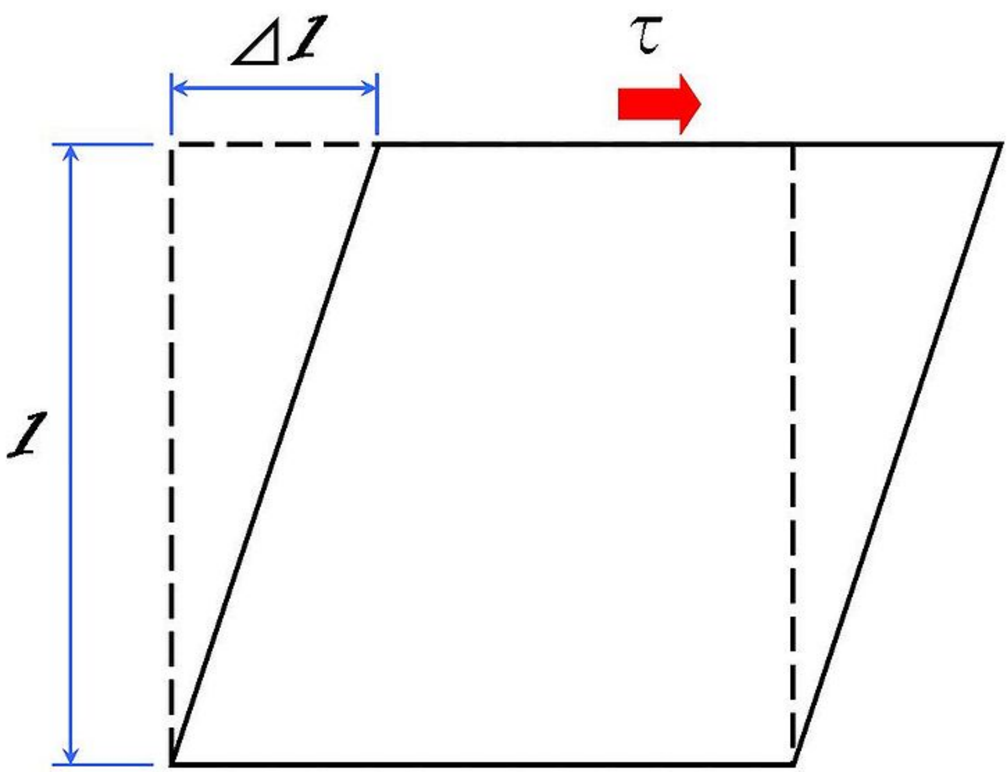
If the material rotates without a change in stress state (rigid rotation), the components of the 2nd Piola–Kirchhoff stress tensor will remain constant, irrespective of material orientation.

The 2nd Piola–Kirchhoff stress tensor is energy conjugate to the Green–Lagrange finite strain tensor.

## Chapter 4

# Shear Stress

| | *Shear stress* |
|---|---|
| **SI symbol:** | τ |
| **SI unit:** | pascal |
| **Derivations from other quantities:** | $\tau = F / A$ |




A shear stress, $\tau$ is applied to the top of the square while the bottom is held in place. This stress results in a strain, or deformation, changing the square into a parallelogram.

A **shear stress**, denoted $\tau$ (Greek: tau), is defined as a stress which is applied parallel or tangential to a face of a material, as opposed to a normal stress which is applied perpendicularly.

# General shear stress

The formula to calculate average shear stress is:

$$\tau = \frac{F}{A},$$

where

τ = the shear stress;
*F* = the force applied;
*A* = the cross sectional area.

# Other forms of shear stress

## Beam shear

Beam shear is defined as the internal shear stress of a beam caused by the shear force applied to the beam.

$$\tau = \frac{VQ}{It},$$

where

*V* = total shear force at the location in question;
*Q* = statical moment of area;
*t* = thickness in the material perpendicular to the shear;
*I* = Moment of Inertia of the entire cross sectional area.

This formula is also known as the Jourawski formula.

## Semi-monocoque shear

Shear stresses within a semi-monocoque structure may be calculated by idealizing the cross-section of the structure into a set of stringers (carrying only axial loads) and webs (carrying only shear flows). Dividing the shear flow by the thickness of a given portion of the semi-monocoque structure yields the shear stress. Thus, the maximum shear stress will occur either in the web of maximum shear flow or minimum thickness.

Also constructions in soil can fail due to shear; e.g., the weight of an earth-filled dam or dike may cause the subsoil to collapse, like a small landslide.

## Impact shear

The maximum shear stress created in a solid round bar subject to impact is given as the equation:

$$\tau = 2\left(\frac{UG}{V}\right)^{\frac{1}{2}},$$

where

$U$ = change in kinetic energy;
$G$ = shear modulus;
$V$ = volume of rod;

and

$$U = U_{rotating} + U_{applied};$$
$$U_{rotating} = \frac{1}{2}I\omega^2;$$
$$U_{applied} = T\theta_{displaced};$$
$I$= mass moment of inertia;
$\omega$= angular speed.

## Shear stress in fluids

Any real fluids (liquids and gases included) moving along solid boundary will incur a shear stress on that boundary. The no-slip condition dictates that the speed of the fluid at the boundary (relative to the boundary) is zero, but at some height from the boundary the flow speed must equal that of the fluid. The region between these two points is aptly named the boundary layer. For all Newtonian fluids in laminar flow the shear stress is proportional to the strain rate in the fluid where the viscosity is the constant of proportionality. However for Non Newtonian fluids, this is no longer the case as for these fluids the viscosity is not constant. The shear stress is imparted onto the boundary as a result of this loss of velocity. The shear stress, for a Newtonian fluid, at a surface element parallel to a flat plate, at the point y, is given by:

$$\tau(y) = \mu\frac{\partial u}{\partial y},$$

where

$\mu$ is the dynamic viscosity of the fluid;

$u$ is the velocity of the fluid along the boundary;
$y$ is the height above the boundary.

Specifically, the wall shear stress is defined as:

$$\tau_{\mathrm{w}} \equiv \tau(y=0) = \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} .$$

In case of wind, the shear stress at the boundary is called wind stress.

# Measurement by shear stress sensors

## Diverging fringe shear stress sensor

This relationship can be exploited to measure the wall shear stress. If a sensor could directly measure the gradient of the velocity profile at the wall, then multiplying by the dynamic viscosity would yield the shear stress. Such a sensor was demonstrated by A. A. Naqwi and W. C. Reynolds. The interference pattern generated by sending a beam of light through two parallel slits forms a network of linearly diverging fringes that seem to originate from the plane of the two slits. As a particle in a fluid passes through the fringes, a receiver detects the reflection of the fringe pattern. The signal can be processed, and knowing the fringe angle, the height and velocity of the particle can be extrapolated.

## Micro-pillar shear-stress sensor

A further technique recently proposed is that of slender wall-mounted micro-pillars made of the flexible polymer PDMS, which bend in reaction to the applying drag forces in the vicinity of the wall. The deflection of the pillar tips from a reference position is detected optically and serves as a representative of the wall-shear stress. It allows the instantaneous detection of the streamwise and spanwise wall-shear stress distribution in turbulent flow up to high Reynolds numbers.

## Chapter 5

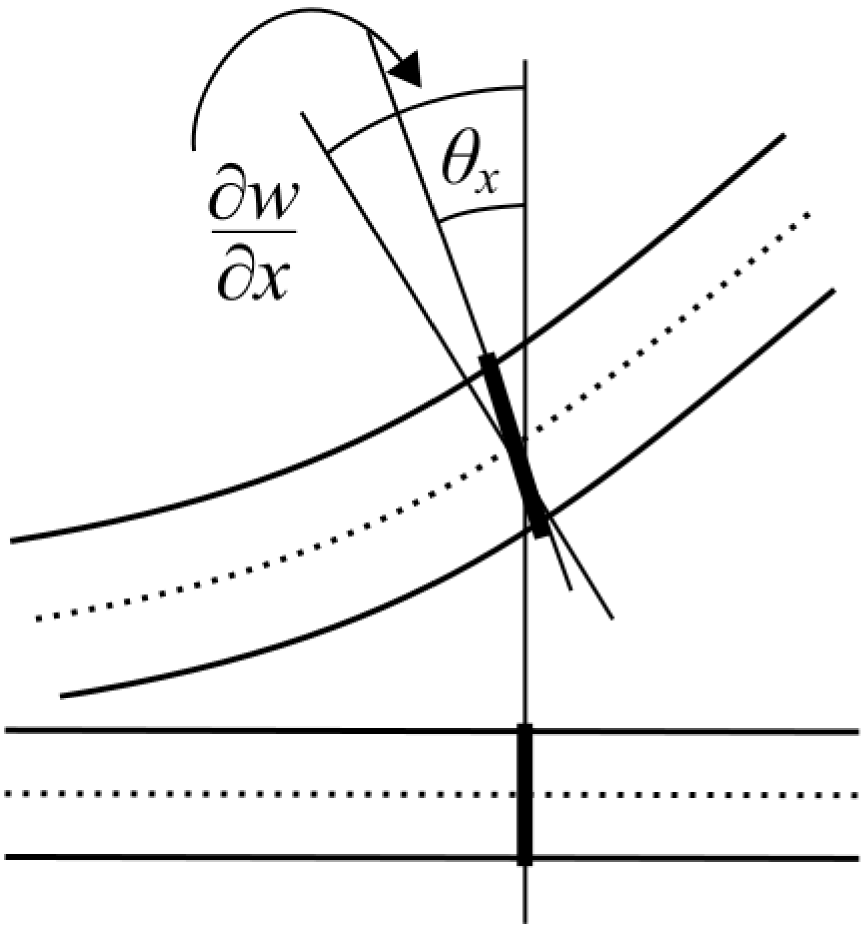
# Timoshenko Beam Theory

The **Timoshenko beam theory** was developed by Ukrainian-born scientist Stephen Timoshenko in the beginning of the 20th century. The model takes into account shear deformation and rotational inertia effects, making it suitable for describing the behaviour of short beams, sandwich composite beams or beams subject to high-frequency excitation when the wavelength approaches the thickness of the beam. The resulting equation is of $4^{th}$ order, but unlike ordinary beam theory - i.e. Bernoulli-Euler theory - there is also a second order spatial derivative present. Physically, taking into account the added mechanisms of deformation effectively lowers the stiffness of the beam, while the result is a larger deflection under a static load and lower predicted eigenfrequencies for a given set of boundary conditions. The latter effect is more noticeable for higher frequencies as the wavelength becomes shorter, and thus the distance between opposing shear forces decreases.

If the shear modulus of the beam material approaches infinity - and thus the beam becomes rigid in shear - and if rotational inertia effects are neglected, Timoshenko beam theory converges towards ordinary beam theory.

## Governing equations

### Quasistatic Timoshenko beam




Deformation of a Timoshenko beam. The normal rotates by an amount $\theta_x = \varphi(x)$ which is not equal to *dw* / *dx*.

In static Timoshenko beam theory without axial effects, the displacements of the beam are assumed to be given by

$$u_x(x,y,z) = -z\ \varphi(x)\ ;\quad u_y(x,y,z) = 0\ ;\quad u_z(x,y) = w(x)$$

where $(x,y,z)$ are the coordinates of a point in the beam, $u_x,u_y,u_z$ are the components of the displacement vector in the three coordinate directions, $\varphi$ is the angle of rotation of the normal to the mid-surface of the beam, and $w$ is the displacement of the mid-surface in the $z$-direction.

The governing equations are the following uncoupled system of ordinary differential equations:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(EI\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right) = q(x,t)$$
$$\frac{\mathrm{d}w}{\mathrm{d}x} = \varphi - \frac{1}{\kappa AG}\frac{\mathrm{d}}{\mathrm{d}x}\left(EI\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)$$

The Timoshenko beam theory for the static case is equivalent to the Euler-Bernoulli theory when the last term above is neglected, an approximation that is valid when

$$\frac{EI}{\kappa L^2 AG} \ll 1$$

where $L$ is the length of the beam.

Combining the two equations gives, for a homogeneous beam of constant cross-section,

$$EI~\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q(x) - \frac{EI}{\kappa AG}~\frac{\mathrm{d}^2 q}{\mathrm{d}x^2}$$

## Dynamic Timoshenko beam

In Timoshenko beam theory without axial effects, the displacements of the beam are assumed to be given by

$$u_x(x,y,z,t) = -z~\varphi(x,t)~;~~u_y(x,y,z,t) = 0~;~~u_z(x,y,z) = w(x,t)$$

where $(x,y,z)$ are the coordinates of a point in the beam, $u_x,u_y,u_z$ are the components of the displacement vector in the three coordinate directions, $\varphi$ is the angle of rotation of the normal to the mid-surface of the beam, and $w$ is the displacement of the mid-surface in the $z$-direction.

Starting from the above assumption, the Timoshenko beam theory, allowing for vibrations, may be described with the coupled linear partial differential equations :

$$\rho A\frac{\partial^2 w}{\partial t^2} - q(x,t) = \frac{\partial}{\partial x}\left[\kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right)\right]$$
$$\rho I\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right) + \kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right)$$

where the dependent variables are $w(x,t)$, the translational displacement of the beam, and $\varphi(x,t)$, the angular displacement. Note that unlike the Euler-Bernoulli theory, the angular deflection is another variable and not approximated by the slope of the deflection. Also,

- ρ is the density of the beam material (but not the linear density).
- *A* is the cross section area.
- *E* is the elastic modulus.
- *G* is the shear modulus.
- *I* is the second moment of area.
- κ, called the Timoshenko shear coefficient, depends on the geometry. Normally, κ = 5 / 6 for a rectangular section.
- $q(x,t)$ is a distributed load (force per length).

These parameters are not necessarily constants.

For a linear elastic, isotropic, homogeneous beam of constant cross-section these two equations can be combined to give

$$EI~\frac{\partial^4 w}{\partial x^4} + m~\frac{\partial^2 w}{\partial t^2} - \left(\rho I + \frac{EIm}{kAG}\right)\frac{\partial^4 w}{\partial x^2~\partial t^2} + \frac{Jm}{kAG}~\frac{\partial^4 w}{\partial t^4} = q(x,t) + \frac{\rho I}{kAG}~\frac{\partial^2 q}{\partial t^2} - \frac{EI}{kAG}~\frac{\partial^2 q}{\partial x^2}$$

**Axial effects**

If the displacements of the beam are given by

$$u_x(x,y,z,t) = u_0(x,t) - z~\varphi(x,t)~;~~u_y(x,y,z,t) = 0~;~~u_z(x,y,z) = w(x,t)$$

where $u_0$ is an additional displacement in the *x*-direction, then the governing equations of a Timoshenko beam take the form

$$\begin{aligned} m\frac{\partial^2 w}{\partial t^2} &= \frac{\partial}{\partial x}\left[\kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right)\right] + q(x,t) \\ J\frac{\partial^2 \varphi}{\partial t^2} &= N(x,t)~\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right) + \kappa AG\left(\frac{\partial w}{\partial x} - \varphi\right) \end{aligned}$$

where $J = \rho I$ and $N(x,t)$ is an externally applied axial force. Any external axial force is balanced by the stress resultant

$$N_{xx}(x,t) = \int_{-h}^{h} \sigma_{xx}~dz$$

where $\sigma_{xx}$ is the axial stress and the thickness of the beam has been assumed to be $2h$.

The combined beam equation with axial force effects included is

$$EI\,\frac{\partial^4 w}{\partial x^4}+N\,\frac{\partial^2 w}{\partial x^2}+m\,\frac{\partial^2 w}{\partial t^2}-\left(J+\frac{mEI}{\kappa AG}\right)\,\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{mJ}{\kappa AG}\,\frac{\partial^4 w}{\partial t^4}=q+\frac{J}{\kappa AG}\,\frac{\partial^2 q}{\partial t^2}-\frac{EI}{\kappa AG}\,\frac{\partial^2 q}{\partial x^2}$$

**Damping**

If, in addition to axial forces, we assume a damping force that is proportional to the velocity with the form

$$\eta(x)\,\frac{\partial w}{\partial x}$$

the coupled governing equations for a Timoshenko beam take the form

$$\begin{aligned} m\frac{\partial^2 w}{\partial t^2}+\eta(x)\,\frac{\partial w}{\partial x} &= \frac{\partial}{\partial x}\left[\kappa AG\left(\frac{\partial w}{\partial x}-\varphi\right)\right]+q(x,t) \\ J\frac{\partial^2 \varphi}{\partial t^2} &= N\frac{\partial w}{\partial x}+\frac{\partial}{\partial x}\left(EI\frac{\partial \varphi}{\partial x}\right)+\kappa AG\left(\frac{\partial w}{\partial x}-\varphi\right) \end{aligned}$$

and the combined equation becomes

$$\begin{aligned} EI\,\frac{\partial^4 w}{\partial x^4}+N\,\frac{\partial^2 w}{\partial x^2}+m\,\frac{\partial^2 w}{\partial t^2}-\left(J+\frac{mEI}{\kappa AG}\right)\,\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{mJ}{\kappa AG}\,\frac{\partial^4 w}{\partial t^4}-\frac{J\eta(x)}{\kappa AG}\,\frac{\partial^3 w}{\partial t^3} \\ +EI\,\frac{\partial^2}{\partial x^2}\left(\eta(x)\frac{\partial w}{\partial t}\right)-\eta(x)\frac{\partial w}{\partial t}=q+\frac{J}{\kappa AG}\,\frac{\partial^2 q}{\partial t^2}-\frac{EI}{\kappa AG}\,\frac{\partial^2 q}{\partial x^2} \end{aligned}$$

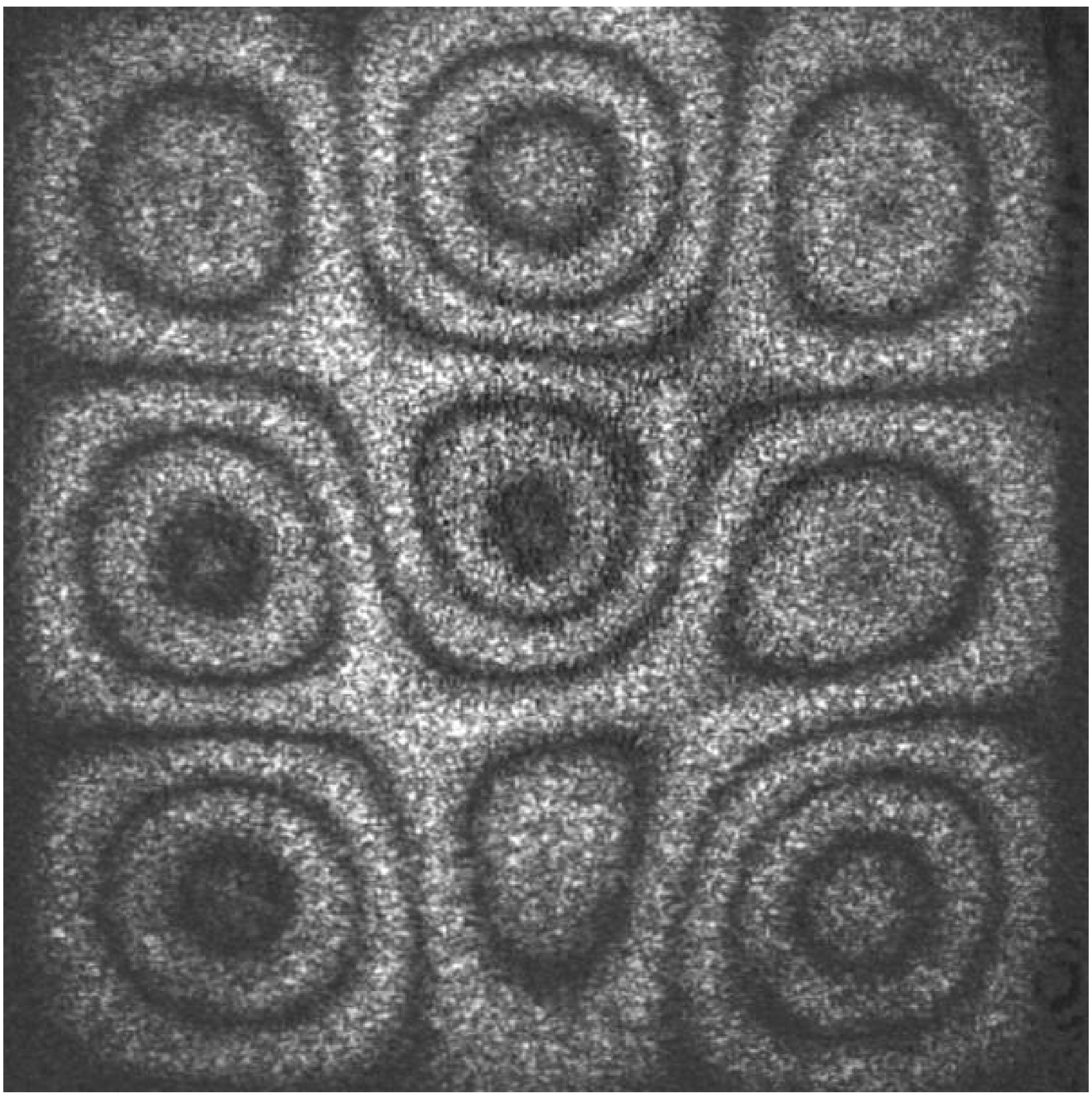
# Shear coefficient

Determining the shear coefficient is not straightforward (nor are the determined values widely accepted, ie there's more than one answer), generally it must satisfy:

$$\int_A \tau dA=\kappa AG\varphi$$

Chapter 6

# Plate Theory



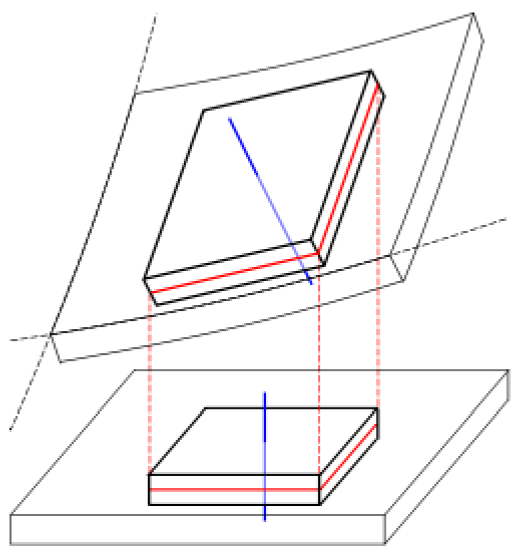
Vibration mode of a clamped square plate

In continuum mechanics, **plate theories** are mathematical descriptions of the mechanics of flat plates that draws on the theory of beams. Plates are defined as plane structural elements with a small thickness compared to the planar dimensions . The typical thickness to width ratio of a plate structure is less than 0.1. A plate theory takes advantage of this disparity in length scale to reduce the full three-dimensional solid mechanics problem to a two-dimensional problem. The aim of plate theory is to calculate the deformation and stresses in a plate subjected to loads.

Of the numerous plate theories that have been developed since the late 19th century, two are widely accepted and used in engineering. These are

- the Kirchhoff-Love theory of plates (classical plate theory)
- The Mindlin-Reissner theory of plates (first-order shear plate theory)

## Kirchhoff-Love theory for thin plates



Deformation of a thin plate highlighting the displacement, the mid-surface (red) and the normal to the mid-surface (blue)

The Kirchhoff-Love theory is an extension of Euler-Bernoulli beam theory to thin plates. The theory was developed in 1888 by Love. using assumptions proposed by Kirchhoff. It is assumed that there a mid-surface plane can be used to represent the three-dimensional plate in two dimensional form.

The following kinematic assumptions that are made in this theory:

- straight lines normal to the mid-surface remain straight after deformation

- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

### Displacement field

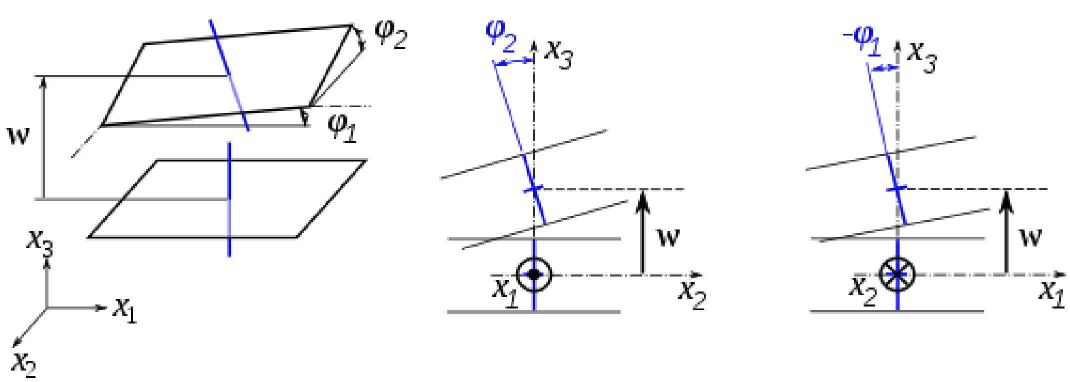
Let $\mathbf{x} = x_i\ \boldsymbol{E}_i$ be the position vector of a point in the undeformed plate where $\boldsymbol{E}_i$ is a Cartesian basis with origin on the mid-surface of the plate. Here $x_1$ and $x_2$ are the Cartesian coordinates on the mid-surface of the undeformed plate and $x_3$ is the coordinate for the thickness direction. Let $\mathbf{u}^0 = u^0_\alpha\ \boldsymbol{E}_\alpha, \alpha = 1, 2$ be the in-plane displacement of the mid-surface and let $w^0$ be the displacement of the mid-surface in the $x_3$ direction. Let $\mathbf{u} = u_i\ \boldsymbol{E}_i, i = 1, 2, 3$ be the displacement of a point in the plate. Then the Kirchhoff hypothesis implies that

$$u_\alpha(\mathbf{x}) = u^0_\alpha(x_1, x_2) - x_3\ \frac{\partial w^0}{\partial x_\alpha} = u^0_\alpha - x_3\ w^0_{,\alpha}\ ; \quad \alpha = 1, 2$$
$$u_3(\mathbf{x}) = w^0(x_1, x_2)$$

If $\varphi_\alpha$ are the angles of rotation of the normal to the mid-surface, then in the Kirchhoff-Love theory

$$\varphi_\alpha = w^0_{,\alpha}$$

Note that we can think of the expression for $u_\alpha$ as the first order Taylor series expansion of the displacement around the mid-surface.




Displacement of the mid-surface (left) and of a normal (right)

### Strain-displacement relations

**Small strains and small rotations**

For the situation where the strains in the plate are infinitesimal and the rotations of the mid-surface normals are less than 10° the strains-displacement relations are

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \\
\varepsilon_{\alpha 3} &= \frac{1}{2}\left(\frac{\partial u_\alpha}{\partial x_3} + \frac{\partial u_3}{\partial x_\alpha}\right) = \frac{1}{2}(u_{\alpha,3} + u_{3,\alpha}) \\
\varepsilon_{33} &= \frac{\partial u_3}{\partial x_3} = u_{3,3}
\end{aligned}
$$

Using the kinematic assumptions we have

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \tfrac{1}{2}(u^0_{\alpha,\beta} + u^0_{\beta,\alpha}) - x_3\ w^0_{,\alpha\beta} \\
\varepsilon_{\alpha 3} &= -w^0_{,\alpha} + w^0_{,\alpha} = 0 \\
\varepsilon_{33} &= 0
\end{aligned}
$$

Therefore the only non-zero strains are in the in-plane directions.

**Small strains and moderate rotations**

If the rotations of the normals to the mid-surface are in the range of 10° to 15°, the strain-displacement relations can be approximated as

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha} + u_{3,\alpha}\ u_{3,\beta}) \\
\varepsilon_{\alpha 3} &= \tfrac{1}{2}(u_{\alpha,3} + u_{3,\alpha}) \\
\varepsilon_{33} &= u_{3,3}
\end{aligned}
$$

Then the kinematic assumptions of Kirchhoff-Love theory lead to the classical plate theory with von Karman strains

$$
\begin{aligned}
\varepsilon_{\alpha\beta} &= \frac{1}{2}(u^0_{\alpha,\beta} + u^0_{\beta,\alpha} + w^0_{,\alpha}\ w^0_{,\beta}) - x_3\ w^0_{,\alpha\beta} \\
\varepsilon_{\alpha 3} &= -w^0_{,\alpha} + w^0_{,\alpha} = 0 \\
\varepsilon_{33} &= 0
\end{aligned}
$$

This theory is nonlinear because of the quadratic terms in the strain-displacement relations.

## Equilibrium equations

The equilibrium equations for the plate can be derived from the principle of virtual work.

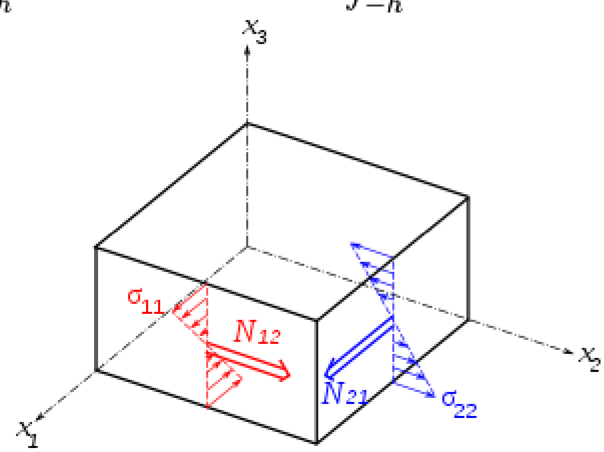
### Small strains and small rotations

For the situation where the strains and rotations of the plate are small the virtual internal energy is given by

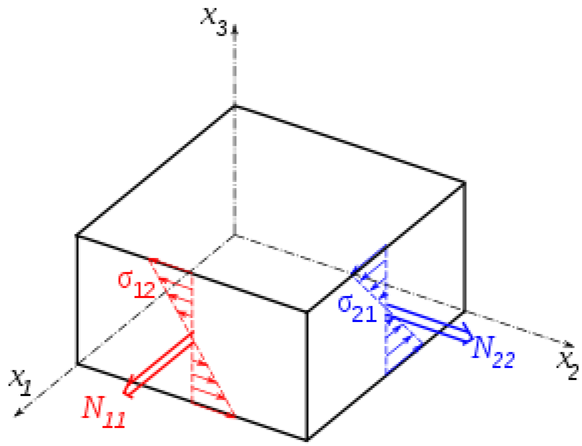
$$\begin{aligned}\delta U &= \int_{\Omega^0}\int_{-h}^{h} \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}~dx_3~d\Omega = \int_{\Omega^0}\int_{-h}^{h} \sigma_{\alpha\beta}~\delta\varepsilon_{\alpha\beta}~dx_3~d\Omega \\ &= \int_{\Omega^0}\int_{-h}^{h}\left[\frac{1}{2}~\sigma_{\alpha\beta}~(\delta u^0_{\alpha,\beta}+\delta u^0_{\beta,\alpha}) - x_3~\sigma_{\alpha\beta}~\delta w^0_{,\alpha\beta}\right]~dx_3~d\Omega \\ &= \int_{\Omega^0}\left[\frac{1}{2}~N_{\alpha\beta}~(\delta u^0_{\alpha,\beta}+\delta u^0_{\beta,\alpha}) - M_{\alpha\beta}~\delta w^0_{,\alpha\beta}\right]~d\Omega\end{aligned}$$

where the thickness of the plate is $2h$ and the stress resultants and stress moment resultants are defined as

$$N_{\alpha\beta} := \int_{-h}^{h} \sigma_{\alpha\beta}~dx_3~;~~M_{\alpha\beta} := \int_{-h}^{h} x_3~\sigma_{\alpha\beta}~dx_3$$




Bending moments and normal stresses




Torques and shear stresses

Integration by parts leads to

$$\delta U = \int_{\Omega^0} \left[-\frac{1}{2}\,(N_{\alpha\beta,\beta}\,\delta u^0_\alpha + N_{\alpha\beta,\alpha}\,\delta u^0_\beta) + M_{\alpha\beta,\beta}\,\delta w^0_{,\alpha}\right]\,d\Omega + \int_{\Gamma^0} \left[\frac{1}{2}\,(n_\beta\,N_{\alpha\beta}\,\delta u^0_\alpha + n_\alpha\,N_{\alpha\beta}\,\delta u^0_\beta) - n_\beta\,M_{\alpha\beta}\,\delta w^0_{,\alpha}\right]\,d\Gamma$$

The symmetry of the stress tensor implies that $N_{\alpha\beta} = N_{\beta\alpha}$. Hence,

$$\delta U = \int_{\Omega^0} \left[-N_{\alpha\beta,\alpha}\,\delta u^0_\beta + M_{\alpha\beta,\beta}\,\delta w^0_{,\alpha}\right]\,d\Omega + \int_{\Gamma^0} \left[n_\alpha\,N_{\alpha\beta}\,\delta u^0_\beta - n_\beta\,M_{\alpha\beta}\,\delta w^0_{,\alpha}\right]\,d\Gamma$$

Another integration by parts gives

$$\delta U = \int_{\Omega^0} \left[-N_{\alpha\beta,\alpha}\,\delta u^0_\beta - M_{\alpha\beta,\beta\alpha}\,\delta w^0\right]\,d\Omega + \int_{\Gamma^0} \left[n_\alpha\,N_{\alpha\beta}\,\delta u^0_\beta + n_\alpha\,M_{\alpha\beta,\beta}\,\delta w^0 - n_\beta\,M_{\alpha\beta}\,\delta w^0_{,\alpha}\right]\,d\Gamma$$

For the case where there are no prescribed external forces, the principle of virtual work implies that $\delta U = 0$. The equilibrium equations for the plate are then given by

$$\begin{aligned} N_{\alpha\beta,\alpha} &= 0 \\ M_{\alpha\beta,\alpha\beta} &= 0 \end{aligned}$$

If the plate is loaded by an external distributed load $q(x)$ that is normal to the mid-surface and directed in the positive $x_3$ direction, the external virtual work due to the load is

$$\delta V_{\text{ext}} = \int_{\Omega^0} q\,\delta w^0\,d\Omega$$

The principle of virtual work then leads to the equilibrium equations

$$N_{\alpha\beta,\alpha} = 0$$
$$M_{\alpha\beta,\alpha\beta} - q = 0$$

**Small strains and moderate rotations**

If the strain-displacement relations take the von Karman form, the equilibrium equations can be expressed as

$$N_{\alpha\beta,\alpha} = 0$$
$$M_{\alpha\beta,\alpha\beta} + [N_{\alpha\beta}\ w^0_{,\beta}]_{,\alpha} - q = 0$$

## Boundary conditions

The boundary conditions that are needed to solve the equilibrium equations of plate theory can be obtained from the boundary terms in the principle of virtual work.

**Small strains and small rotations**

In the absence of external forces on the boundary, the boundary conditions are

$$n_\alpha\ N_{\alpha\beta} \quad \text{or} \quad u^0_\beta$$
$$n_\alpha\ M_{\alpha\beta,\beta} \quad \text{or} \quad w^0$$
$$n_\beta\ M_{\alpha\beta} \quad \text{or} \quad w^0_{,\alpha}$$

Note that the quantity $n_\alpha\ M_{\alpha\beta,\beta}$ is an effective shear force.

## Stress-strain relations

The stress-strain relations for a linear elastic Kirchhoff plate are given by

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\theta}\ \varepsilon_{\gamma\theta}$$
$$\sigma_{\alpha 3} = C_{\alpha 3\gamma\theta}\ \varepsilon_{\gamma\theta}$$
$$\sigma_{33} = C_{33\gamma\theta}\ \varepsilon_{\gamma\theta}$$

Since $\sigma_{\alpha 3}$ and $\sigma_{33}$ do not appear in the equilibrium equations it is implicitly assumed that these quantities do not have any effect on the momentum balance and are neglected. The remaining stress-strain relations, in matrix form, can be written as

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix}$$

Then,

$$\begin{bmatrix} N_{11} \\ N_{22} \\ N_{12} \end{bmatrix} = \int_{-h}^{h} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} dx_3 = \left\{ \int_{-h}^{h} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} dx_3 \right\} \begin{bmatrix} u^0_{1,1} \\ u^0_{2,2} \\ \frac{1}{2}\left(u^0_{1,2} + u^0_{2,1}\right) \end{bmatrix}$$

and

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = \int_{-h}^{h} x_3 \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} dx_3 = -\left\{ \int_{-h}^{h} x_3^2 \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} dx_3 \right\} \begin{bmatrix} w^0_{,11} \\ w^0_{,22} \\ w^0_{,12} \end{bmatrix}$$

The **extensional stiffnesses** are the quantities

$$A_{\alpha\beta} := \int_{-h}^{h} C_{\alpha\beta} \; dx_3$$

The **bending stiffnesses** (also called **flexural rigidity**) are the quantities

$$D_{\alpha\beta} := \int_{-h}^{h} x_3^2 \; C_{\alpha\beta} \; dx_3$$

# Isotropic and homogeneous Kirchhoff plate

For an isotropic and homogeneous plate, the stress-strain relations are

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \cfrac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} .$$

The moments corresponding to these stresses are

$$\begin{bmatrix} M_{11} \\ M_{22} \\ M_{12} \end{bmatrix} = -\cfrac{2h^3E}{3(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \begin{bmatrix} w^0_{,11} \\ w^0_{,22} \\ w^0_{,12} \end{bmatrix}$$

### Pure bending

The displacements $u^0_1$ and $u^0_2$ are zero under pure bending conditions. For an isotropic, homogeneous plate under pure bending the only relevant governing equation is

$$M_{\alpha\beta,\alpha\beta} = 0 \implies M_{11,11} + 2M_{12,12} + M_{22,22} = 0$$

Differentiation of the relations between the moments and displacements leads to

$$M_{11,11} = -\frac{2h^3E}{3(1-\nu^2)}\left(w^0_{,1111} + \nu\, w^0_{,2211}\right) \;\;,\;\; M_{22,22} = -\frac{2h^3E}{3(1-\nu^2)}\left(\nu\, w^0_{,1122} + w^0_{,2222}\right)$$
$$M_{12,22} = -\frac{2h^3E}{3(1-\nu^2)}(1-\nu)\; w^0_{,1212}$$

Plugging these into the governing equation gives us

$$w^0_{,1111} + \nu\, w^0_{,2211} + 2(1-\nu)\; w^0_{,1212} + \nu\, w^0_{,1122} + w^0_{,2222} = 0\,.$$

Since the order of differentiation is irrelevant we have $w^0_{,2211} = w^0_{,1212} = w^0_{,1122}$ and therefore

$$w^0_{,1111} + 2\; w^0_{,1212} + w^0_{,2222} = 0$$

or

$$\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 w}{\partial x_2^4} = 0 \quad \text{where} \quad w := w^0\,.$$

In direct tensor notation, the governing equation of the plate is

$$\nabla^2\nabla^2 w = 0\,.$$

### Transverse loading

For a transversely loaded plate without axial deformations, the governing equation has the form

$$M_{\alpha\beta,\alpha\beta} = q \implies M_{11,11} + 2M_{12,12} + M_{22,22} = q$$

where *q* is a distributed transverse load (per unit area). Substitution of the expressions for the derivatives of $M_{\alpha\beta}$ into the governing equation gives

$$-\frac{2h^3E}{3(1-\nu^2)}\left[w^0_{,1111} + 2\,w^0_{,1212} + w^0_{,2222}\right] = q\,.$$

Noting that the bending stiffness is the quantity

$$D := \frac{2h^3E}{3(1-\nu^2)}$$

we can write the governing equation in the form

$$\nabla^2\nabla^2 w = -\frac{q}{D}\,.$$

In cylindrical coordinates ($r$,θ,$z$),

$$\nabla^2 w \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial^2 w}{\partial z^2}\,.$$

For symmetrically loaded circular plates, $w = w(r)$, and we have

$$\nabla^2 w \equiv \frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\,.$$

Therefore, the governing equation is

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\}\right] = -\frac{q}{D}\,.$$

# Dynamics of thin Kirchhoff plates

The dynamic theory of plates determines the propagation of waves in the plates, and the study of standing waves and vibration modes.

## Governing equations

The governing equations for the dynamics of a Kirchhoff-Love plate are

$$\begin{aligned} N_{\alpha\beta,\beta} &= J_1\;\ddot{u}^0_\alpha \\ M_{\alpha\beta,\alpha\beta} - q(x,t) &= J_1\;\ddot{w}^0 - J_3\;\ddot{w}^0_{,\alpha\alpha} \end{aligned}$$

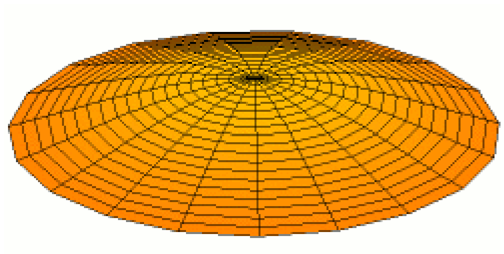
where, for a plate with density ρ = ρ(*x*),

$$J_1 := \int_{-h}^{h} \rho \, dx_3 = 2 \, \rho \, h \; ; \quad J_3 := \int_{-h}^{h} x_3^2 \, \rho \, dx_3 = \frac{2}{3} \, \rho \, h^3$$

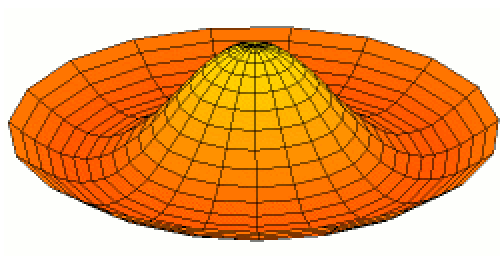
and

$$\dot{u}_i = \frac{\partial u_i}{\partial t} \; ; \quad \ddot{u}_i = \frac{\partial^2 u_i}{\partial t^2} \; ; \quad u_{i,\alpha} = \frac{\partial u_i}{\partial x_\alpha} \; ; \quad u_{i,\alpha\beta} = \frac{\partial^2 u_i}{\partial x_\alpha \partial x_\beta}$$

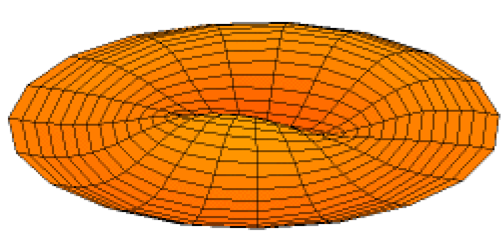
The figures below show some vibrational modes of a circular plate.



mode $k = 0$, $p = 1$



mode $k = 0$, $p = 2$



mode $k = 1$, $p = 2$

# Mindlin-Reissner theory for thick plates

In the theory of thick plates, or theory of Eric Reissner and Raymond Mindlin, the normal to the mid-surface remains straight but not necessarily perpendicular to the mid-surface. If $\varphi_1$and $\varphi_2$designate the angles which the mid-surface makes with the $x_3$ axis then

$$\varphi_1 \neq w_{,1} \; ; \quad \varphi_2 \neq w_{,2}$$

Then the Mindlin-Reissner hypothesis implies that

$$\begin{aligned} u_\alpha(\mathbf{x}) &= u^0_\alpha(x_1, x_2) - x_3 \; \varphi_\alpha \; ; \quad \alpha = 1, 2 \\ u_3(\mathbf{x}) &= w^0(x_1, x_2) \end{aligned}$$

## Strain-displacement relations

Depending on the amount of rotation of the plate normals two different approximations for the strains can be derived from the basic kinematic assumptions.

### Small strains and small rotations

For small strains and small rotations the strain-displacement relations for Mindlin-Reissner plates are

$$\begin{aligned} \varepsilon_{\alpha\beta} &= \frac{1}{2}(u^0_{\alpha,\beta} + u^0_{\beta,\alpha}) - x_3 \; \varphi_{\alpha,\beta} \\ \varepsilon_{\alpha 3} &= \frac{1}{2}\left(w^0_{,\alpha} - \varphi_\alpha\right) \\ \varepsilon_{33} &= 0 \end{aligned}$$

The shear strain, and hence the shear stress, across the thickness of the plate is not neglected in this theory. However, the shear strain is constant across the thickness of the plate. This cannot be accurate since the shear stress is known to be parabolic even for simple plate geometries. To account for the inaccuracy in the shear strain, a **shear correction factor** (κ) is applied so that the correct amount of internal energy is predicted by the theory. Then

$$\varepsilon_{\alpha 3} = \frac{1}{2} \, \kappa \; \left(w^0_{,\alpha} - \varphi_\alpha\right)$$

## Equilibrium equations

The equilibrium equations have slightly different forms depending on the amount of bending expected in the plate.

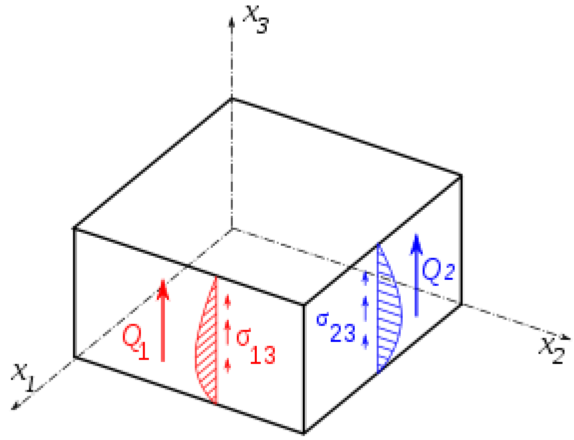
**Small strains and small rotations**

For the situation where the strains and rotations of the plate are small the virtual internal energy is given by

$$\begin{aligned}\delta U &= \int_{\Omega^0}\int_{-h}^{h} \boldsymbol{\sigma} : \delta\boldsymbol{\epsilon}\ dx_3\ d\Omega = \int_{\Omega^0}\int_{-h}^{h} \left[\sigma_{\alpha\beta}\ \delta\varepsilon_{\alpha\beta} + 2\ \kappa\ \sigma_{\alpha 3}\ \delta\varepsilon_{\alpha 3}\right]\ dx_3\ d\Omega \\ &= \int_{\Omega^0}\int_{-h}^{h} \left[\frac{1}{2}\ \sigma_{\alpha\beta}\ (\delta u^0_{\alpha,\beta} + \delta u^0_{\beta,\alpha}) - x_3\ \sigma_{\alpha\beta}\ \delta\varphi_{\alpha,\beta} + \kappa\ \sigma_{\alpha 3}\left(\delta w^0_{,\alpha} - \delta\varphi_\alpha\right)\right]\ dx_3\ d\Omega \\ &= \int_{\Omega^0} \left[\frac{1}{2}\ N_{\alpha\beta}\ (\delta u^0_{\alpha,\beta} + \delta u^0_{\beta,\alpha}) - M_{\alpha\beta}\ \delta\varphi_{\alpha,\beta} + Q_\alpha\left(\delta w^0_{,\alpha} - \delta\varphi_\alpha\right)\right]\ d\Omega\end{aligned}$$

where the stress resultants and stress moment resultants are defined in a way similar to that for Kirchhoff plates. The shear resultant is defined as

$$Q_\alpha := \kappa\ \int_{-h}^{h} \sigma_{\alpha 3}\ dx_3$$




Shear resultant and shear stresses

Integration by parts gives

$$\begin{aligned}\delta U &= \int_{\Omega^0} \left[-\frac{1}{2}\ (N_{\alpha\beta,\beta}\ \delta u^0_\alpha + N_{\alpha\beta,\alpha}\ \delta u^0_\beta) + M_{\alpha\beta,\beta}\ \delta\varphi_\alpha - Q_{\alpha,\alpha}\ \delta w^0 - Q_\alpha\ \delta\varphi_\alpha\right]\ d\Omega \\ &+ \int_{\Gamma^0} \left[\frac{1}{2}\ (n_\beta\ N_{\alpha\beta}\ \delta u^0_\alpha + n_\alpha\ N_{\alpha\beta}\ \delta u^0_\beta) - n_\beta\ M_{\alpha\beta}\ \delta\varphi_\alpha + n_\alpha\ Q_\alpha\ \delta w^0\right]\ d\Gamma\end{aligned}$$

The symmetry of the stress tensor implies that $N_{\alpha\beta} = N_{\beta\alpha}$. Hence,

$$\delta U = \int_{\Omega^0} \left[-N_{\alpha\beta,\alpha}\ \delta u^0_\beta + (M_{\alpha\beta,\beta} - Q_\alpha)\ \delta\varphi_\alpha - Q_{\alpha,\alpha}\ \delta w^0\right]\ d\Omega$$
$$+ \int_{\Gamma^0} \left[n_\alpha\ N_{\alpha\beta}\ \delta u^0_\beta - n_\beta\ M_{\alpha\beta}\ \delta\varphi_\alpha + n_\alpha\ Q_\alpha\ \delta w^0\right]\ d\Gamma$$

For the special case when the top surface of the plate is loaded by a force per unit area $q(\mathbf{x}^0)$, the virtual work done by the external forces is

$$\delta V_{\text{ext}} = \int_{\Omega^0} q\ \delta w^0\ \mathrm{d}\Omega$$

Then, from the principle of virtual work,

$$\int_{\Omega^0} \left[N_{\alpha\beta,\alpha}\ \delta u^0_\beta - (M_{\alpha\beta,\beta} - Q_\alpha)\ \delta\varphi_\alpha + (Q_{\alpha,\alpha} + q)\ \delta w^0\right]\ d\Omega$$
$$= \int_{\Gamma^0} \left[n_\alpha\ N_{\alpha\beta}\ \delta u^0_\beta - n_\beta\ M_{\alpha\beta}\ \delta\varphi_\alpha + n_\alpha\ Q_\alpha\ \delta w^0\right]\ d\Gamma$$

Using standard arguments from the calculus of variations, the equilibrium equations for a Mindlin-Reissner plate are

$$N_{\alpha\beta,\alpha} = 0$$
$$M_{\alpha\beta,\beta} - Q_\alpha = 0$$
$$Q_{\alpha,\alpha} + q = 0$$

## Boundary conditions

The boundary conditions are indicated by the boundary terms in the principle of virtual work.

### Small strains and small rotations

If the only external force is a vertical force on the top surface of the plate, the boundary conditions are

$$\begin{aligned} n_\alpha\ N_{\alpha\beta} &\quad \text{or} \quad u^0_\beta \\ n_\alpha\ M_{\alpha\beta} &\quad \text{or} \quad \varphi_\alpha \\ n_\alpha\ Q_\alpha &\quad \text{or} \quad w^0 \end{aligned}$$

## Stress-strain relations

The stress-strain relations for a linear elastic Mindlin-Reissner plate are given by

$$\begin{aligned}\sigma_{\alpha\beta} &= C_{\alpha\beta\gamma\theta}\ \varepsilon_{\gamma\theta}\\ \sigma_{\alpha 3} &= C_{\alpha 3\gamma\theta}\ \varepsilon_{\gamma\theta}\\ \sigma_{33} &= C_{33\gamma\theta}\ \varepsilon_{\gamma\theta}\end{aligned}$$

Since $\sigma_{33}$ does not appear in the equilibrium equations it is implicitly assumed that it do not have any effect on the momentum balance and is neglected. This assumption is also called the **plane stress** assumption. The remaining stress-strain relations for an orthotropic material, in matrix form, can be written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & 0 & 0 & 0\\ C_{12} & C_{22} & 0 & 0 & 0\\ 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{23}\\ \varepsilon_{31}\\ \varepsilon_{12}\end{bmatrix}$$

Then,

$$\begin{bmatrix}N_{11}\\ N_{22}\\ N_{12}\end{bmatrix} = \int_{-h}^{h}\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix} dx_3 = \left\{\int_{-h}^{h}\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix} dx_3\right\}\begin{bmatrix}u^0_{1,1}\\ u^0_{2,2}\\ \frac{1}{2}\,(u^0_{1,2}+u^0_{2,1})\end{bmatrix}$$

and

$$\begin{bmatrix}M_{11}\\ M_{22}\\ M_{12}\end{bmatrix} = \int_{-h}^{h} x_3\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{bmatrix} dx_3 = -\left\{\int_{-h}^{h} x_3^2\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{12} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix} dx_3\right\}\begin{bmatrix}\varphi_{1,1}\\ \varphi_{2,2}\\ \varphi_{1,2}\end{bmatrix}$$

For the shear terms

$$\begin{bmatrix}Q_1\\ Q_2\end{bmatrix} = \kappa\int_{-h}^{h}\begin{bmatrix}C_{55} & 0\\ 0 & C_{44}\end{bmatrix}\begin{bmatrix}\varepsilon_{31}\\ \varepsilon_{32}\end{bmatrix} dx_3 = \frac{\kappa}{2}\left\{\int_{-h}^{h}\begin{bmatrix}C_{55} & 0\\ 0 & C_{44}\end{bmatrix} dx_3\right\}\begin{bmatrix}w^0_{,1}-\varphi_1\\ w^0_{,2}-\varphi_2\end{bmatrix}$$

The **extensional stiffnesses** are the quantities

$$A_{\alpha\beta} := \int_{-h}^{h} C_{\alpha\beta}\ dx_3$$

The **bending stiffnesses** are the quantities

$$D_{\alpha\beta} := \int_{-h}^{h} x_3^2 \; C_{\alpha\beta} \; dx_3$$