Laws and Theories Used in Electrical Engineering

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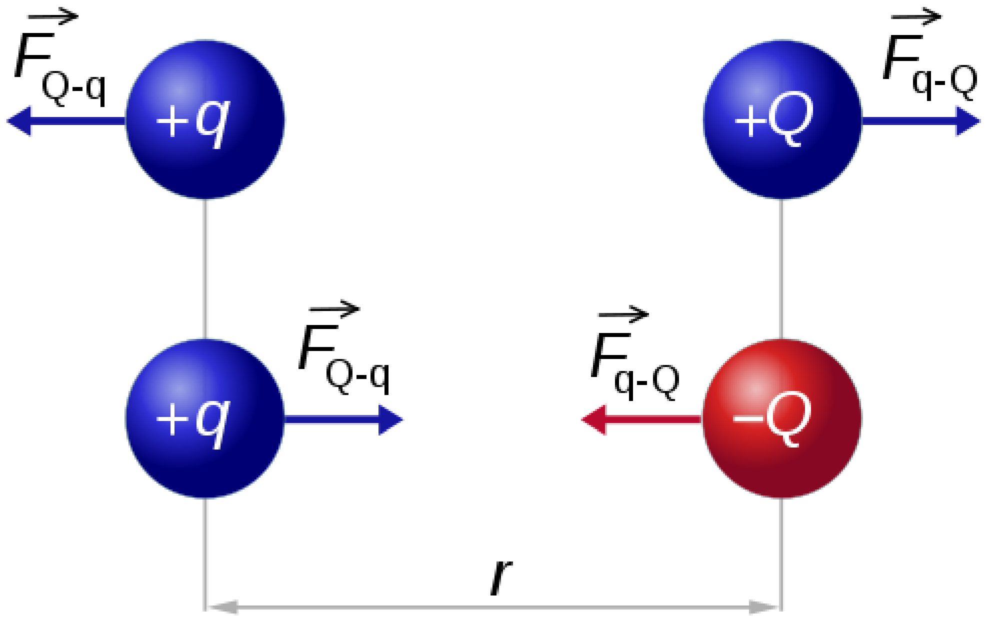
Chapter- 1

Coulomb's Law

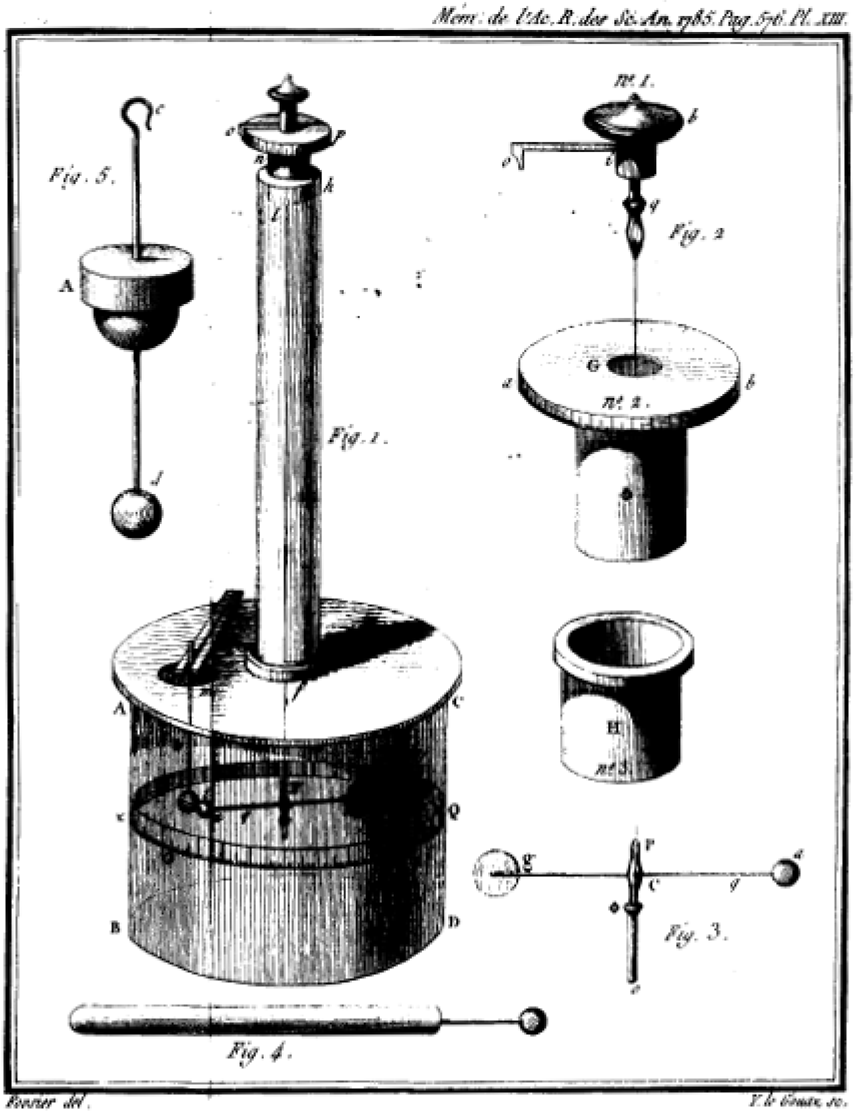
Coulomb's law, or *Coulomb's inverse square law*, is a law of physics describing the electrostatic interaction between electrically charged particles. It was studied and first published in 1783 by French physicist Charles Augustin de Coulomb and was essential to the development of the theory of electromagnetism. Nevertheless, the dependence of the electric force with distance had been proposed previously by Joseph Priestley and the dependence with both distance and charge had been discovered, but not published, by Henry Cavendish, prior to Coulomb's works.

Basic Equation



$$|\vec{F}_{Q\text{-}q}| = |\vec{F}_{q\text{-}Q}| = k\,\frac{|q \times Q|}{r^2}$$

Diagram describing the basic mechanism of Coulomb's law; like charges repel each other and opposite charges attract each other.



Coulomb's torsion balance

It states that : "The magnitude of the Electrostatics force of interaction between two point charges is directly proportional to the scalar multiplication of the magnitudes of charges and inversely proportional to the square of the distances between them."

The scalar form of Coulomb's law is an expression for the magnitude and sign of the electrostatic force between two idealized *point charges*, small in size compared to their separation. This force (F) acting simultaneously on point charges (q_1) and (q_2), is given by

$$F = k_e \frac{q_1 q_2}{r^2}$$

where r is the separation distance and k_e is a proportionality constant. A positive force implies it is repulsive, while a negative force implies it is attractive. The proportionality constant k_e, called the **Coulomb constant** (sometimes called the **Coulomb force constant**), is related to defined properties of space and can be calculated based on knowledge of empirical measurements of the speed of light:

$$k_e = \frac{1}{4\pi\varepsilon_0} = \frac{c^2\ \mu_0}{4\pi} = c^2 \cdot 10^{-7}\ \mathrm{H} \cdot \mathrm{m}^{-1}$$
$$= 8.987\ 551\ 787\ 368\ 176\ 4 \times 10^9\ \mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}^2$$

In SI units, the meter is defined such that the speed of light in vacuum (or electromagnetic waves, in general), denoted c, is exactly 299,792,458 $\mathrm{m{\cdot}s^{-1}}$, and the magnetic constant (μ_0) is set at $4\pi \times 10^{-7}$ $\mathrm{H{\cdot}m^{-1}}$. In agreement with electromagnetic theory, requiring that

$$\frac{1}{\mu_0\varepsilon_0} = c^2,$$

the value for the electric constant (ε_0) is derived to be $\varepsilon_0 = 1/(\mu_0 c^2) \approx 8.85418782\times10{-}12$ $\mathrm{F{\cdot}m^{-1}}$. In electrostatic units and Gaussian units, the unit charge (*esu* or statcoulomb) is defined in such a way that the Coulomb constant is 1 and dimensionless.

In the more useful vector-form statement, the force in the equation is a vector force acting on either point charge, so directed as to push it away from the other point charge; the right-hand side of the equation, in this case, must have an additional product term of a unit vector pointing in one of two opposite directions, e.g., from q_1 to q_2 if the force is acting on q_2; the charges may have either sign and the sign of their product determines the ultimate direction of that force. Thus, the vector force pushing the charges away from each other (pulling towards each other if negative) is directly proportional to the product of the charges and inversely proportional to the square of the distance between them. The square of the distance part arises from the fact that the force field due to an isolated point charge is uniform in all directions and gets "diluted" with distance as much as the area of a sphere centered on the point charge expands with its radius.

The law of superposition allows this law to be extended to include any number of point charges, to derive the force on any one point charge by a vector addition of these individual forces acting alone on that point charge. The resulting vector happens to be parallel to the electric field vector at that point, with that point charge (or "test charge") removed.

Coulomb's law can also be interpreted in terms of atomic units with the force expressed in Hartrees per Bohr radius, the charge in terms of the elementary charge, and the distances in terms of the *Bohr radius*.

Electric field

It follows from the Coulomb's Law that the magnitude of the electric field (E) created by a single point charge (q) at a certain distance (r) is given by:

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}.$$

For a positive charge, the direction of the electric field points along lines directed radially away from the location of the point charge, while the direction is the opposite for a negative charge. The SI units of electric field are volts per metre or newtons per coulomb.

Vector form

In order to obtain both the magnitude and direction of the force on a charge, q_1 at position $\mathbf{r}_1$, experiencing a field due to the presence of another charge, q_2 at position $\mathbf{r}_2$, the full vector form of Coulomb's law is required.

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{21},$$

where r is the separation of the two charges. This is simply the scalar definition of Coulomb's law with the direction given by the unit vector, $\hat{\mathbf{r}}_{21}$, parallel with the line *from* charge q_2 *to* charge q_1.

If both charges have the same sign (like charges) then the product $q_1 q_2$ is positive and the direction of the force on q_1 is given by $\hat{\mathbf{r}}_{21}$; the charges repel each other. If the charges have opposite signs then the product $q_1 q_2$ is negative and the direction of the force on q_1 is given by $-\hat{\mathbf{r}}_{21}$; the charges attract each other.

System of discrete charges

The principle of linear superposition may be used to calculate the force on a small test charge, q, due to a system of N discrete charges:

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} = \frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{R_i^2}\hat{\mathbf{R}}_i,$$

where q_i and $\mathbf{r}_i$are the magnitude and position respectively of the i^{th} charge, $\hat{\mathbf{R}}_i$is a unit vector in the direction of $\mathbf{R}_i = \mathbf{r} - \mathbf{r}_i$(a vector pointing from charge q_i to charge q), and R_i is the magnitude of $\mathbf{R}_i$(the separation between charges q_i and q).

Continuous charge distribution

For a charge distribution an integral over the region containing the charge is equivalent to an infinite summation, treating each infinitesimal element of space as a point charge dq.

For a linear charge distribution (a good approximation for charge in a wire) where $\lambda(\mathbf{r}')$gives the charge per unit length at position $\mathbf{r}'$, and $dl'$is an infinitesimal element of length,

$$dq = \lambda(\mathbf{r}')dl'.$$

For a surface charge distribution (a good approximation for charge on a plate in a parallel plate capacitor) where $\sigma(\mathbf{r}')$gives the charge per unit area at position $\mathbf{r}'$, and $dA'$is an infinitesimal element of area,

$$dq = \sigma(\mathbf{r}')\, dA'.$$

For a volume charge distribution (such as charge within a bulk metal) where $\rho(\mathbf{r}')$gives the charge per unit volume at position $\mathbf{r}'$, and $dV'$is an infinitesimal element of volume,

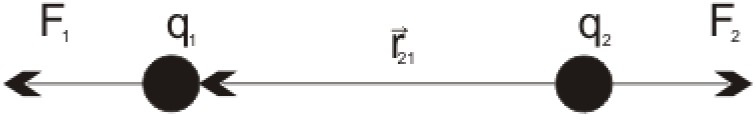
$$dq = \rho(\mathbf{r}')\, dV'.$$

The force on a small test charge q'at position $\mathbf{r}$is given by

$$\mathbf{F} = \frac{q'}{4\pi\varepsilon_0}\int dq \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Graphical representation

Below is a graphical representation of Coulomb's law, when $q_1 q_2 > 0$. The vector $\mathbf{F}_1$is the force experienced by q_1. The vector $\mathbf{F}_2$is the force experienced by q_2. Their magnitudes will always be equal. The vector $\mathbf{r}_{21}$is the displacement vector between two charges (q_1 and q_2).



A graphical representation of Coulomb's law.

Electrostatic approximation

In either formulation, Coulomb’s law is fully accurate only when the objects are stationary, and remains approximately correct only for slow movement. These conditions are collectively known as the electrostatic approximation. When movement takes place, magnetic fields are produced which alter the force on the two objects. The magnetic interaction between moving charges may be thought of as a manifestation of the force from the electrostatic field but with Einstein’s theory of relativity taken into consideration.

Table of derived quantities

	Particle property	Relationship	Field property
Vector quantity	*Force (on 1 by 2)* $\mathbf{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{21}$	$\mathbf{F}_{21} = q_1 \mathbf{E}_{21}$	*Electric field (at 1 by 2)* $\mathbf{E}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r^2}\hat{\mathbf{r}}_{21}$
Relationship	$\mathbf{F}_{21} = -\nabla U_{21}$		$\mathbf{E}_{21} = -\nabla V_{21}$
Scalar quantity	*Potential energy (at 1 by 2)* $U_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}$	$U_{21} = q_1 V_{21}$	*Potential (at 1 by 2)* $V_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{r}$

Chapter- 2

Electric Field

In physics, an **electric field** surrounds electrically charged particles and time-varying magnetic fields. This electric field exerts a force on other electrically charged objects. Michael Faraday introduced the concept of an electric field.

The electric field is a vector field with SI units of newtons per coulomb ($N\ C^{-1}$) or, equivalently, volts per metre ($V\ m^{-1}$). The SI base units of the electric field are $kg{\cdot}m{\cdot}s^{-3}{\cdot}A^{-1}$. The strength or magnitude of the field at a given point is defined as the force that would be exerted on a positive test charge of 1 coulomb placed at that point; the direction of the field is given by the direction of that force. Electric fields contain electrical energy with energy density proportional to the square of the field amplitude. The electric field is to charge as gravitational acceleration is to mass and force density is to volume.

An electric field that changes with time, such as due to the motion of charged particles in the field, influences the local magnetic field. That is, the electric and magnetic fields are not completely separate phenomena; what one observer perceives as an electric field, another observer in a different frame of reference perceives as a mixture of electric and magnetic fields. For this reason, one speaks of "electromagnetism" or "electromagnetic fields". In quantum mechanics, disturbances in the electromagnetic fields are called photons, and the energy of photons is quantized.

Definition

The electric field is defined as the force per unit charge that would be experienced by a stationary point charge at a given location in the field :

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

where

F is the **electric force** experienced by the particle
q is its charge
E is the electric field wherein the particle is located

Taken literally, this equation only defines the electric field at the places where there are stationary charges present to experience it. Furthermore, the force exerted by another charge q will alter the source distribution, which means the electric field in the presence of q differs from itself in the absence of q. However, the electric field of a given source distribution remains defined in the absence of any charges with which to interact. This is achieved by measuring the force exerted on successively smaller *test charges* placed in the vicinity of the source distribution. By this process, the electric field created by a given source distribution is defined as the limit as the test charge approaches zero of the force per unit charge exerted thereupon.

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q}$$

This allows the electric field to be dependent on the source distribution alone.

As is clear from the definition, the direction of the electric field is the same as the direction of the force it would exert on a positively-charged particle, and opposite the direction of the force on a negatively-charged particle. Since like charges repel and opposites attract (as quantified below), the electric field tends to point away from positive charges and towards negative charges.

Based on Coulomb's Law for interacting point charges, the contribution to the E-field at a point in space due to a single, discrete charge located at another point in space is given by the following :

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$$

where

q is the charge of the particle creating the electric force,
r is the distance from the particle with charge q to the E-field evaluation point,
$\hat{\mathbf{r}}$is the unit vector pointing from the particle with charge q to the E-field evaluation point,
ε_0 is the electric constant.

The total E-field due to a quantity of point charges, n_q, is simply the superposition of the contribution of each individual point charge :

$$\mathbf{E} = \sum_{i=1}^{n_q} \mathbf{E}_i = \sum_{i=1}^{n_q} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i.$$

Alternately, Gauss's Law allows the E-field to be calculated in terms of a continuous distribution of charge density in space, ρ:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$$

Coulomb's law is actually a special case of Gauss's Law, a more fundamental description of the relationship between the distribution of electric charge in space and the resulting electric field. Gauss's law is one of Maxwell's equations, a set of four laws governing electromagnetics.

Time-varying fields

An electric field can be produced, not only by a static charge, but also by a changing magnetic field. The combined electric field is expressed as,

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

where,

$$\mathbf{B} = \nabla \times \mathbf{A}$$

The vector $\mathbf{B}$is the magnetic flux density and the vector $\mathbf{A}$is the magnetic vector potential. Taking the curl of the electric field equation we obtain,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

which is one of Maxwell's equations, referred to as Faraday's law of induction.

Where electrostatics is the study of the fields surrounding static charges, the study of the electric fields induced by changing magnetic field comes under the domain of electrodynamics or electromagnetics.

Properties (in electrostatics)

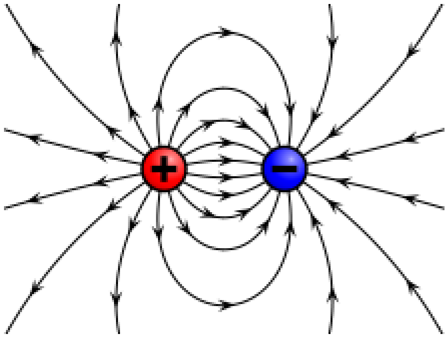


Illustration of the electric field surrounding a positive (red) and a negative (blue) charge.

According to equation (1) above, electric field is dependent on position. The electric field due to any single charge falls off as the square of the distance from that charge.

Electric fields follow the superposition principle. If more than one charge is present, the total electric field at any point is equal to the vector sum of the respective electric fields that each object would create in the absence of the others.

$$\mathbf{E}_{\rm total} = \sum_i \mathbf{E}_i = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 \ldots$$

If this principle is extended to an infinite number of infinitesimally small elements of charge, the following formula results:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{r^2} \hat{\mathbf{r}}\, \mathrm{d}V$$

where

ρ is the charge density, or the amount of charge per unit volume.

The electric field at a point is equal to the negative gradient of the electric potential there. In symbols,

$$\mathbf{E} = -\nabla \Phi$$

where

$\Phi(x,y,z)$ is the scalar field representing the electric potential at a given point.

If several spatially distributed charges generate such an electric potential, e.g. in a solid, an electric field gradient may also be defined.

Considering the permittivity ε of a linear material, which may differ from the permittivity of free space ε_0, the electric displacement field is:

$$\mathbf{D} = \varepsilon \mathbf{E}.$$

Energy in the electric field

The electric field stores energy. The energy density of the electric field is given by

$$u = \frac{1}{2}\varepsilon |\mathbf{E}|^2 ,$$

where *ε* is the permittivity of the medium in which the field exists, and **E** is the electric field vector.

The total energy stored in the electric field in a given volume *V* is therefore

$$\frac{1}{2}\varepsilon \int_V |\mathbf{E}|^2 \mathrm{d}V ,$$

where d*V* is the differential volume element.

Parallels between electrostatics and gravity

Coulomb's law, which describes the interaction of electric charges:

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\hat{\mathbf{r}} = q\mathbf{E}$$

is similar to Newton's law of universal gravitation:

$$\mathbf{F} = G\frac{Mm}{r^2}\hat{\mathbf{r}} = m\mathbf{g}.$$

This suggests similarities between the electric field **E** and the gravitational field **g**, so sometimes mass is called "gravitational charge".

Similarities between electrostatic and gravitational forces:

1. Both act in a vacuum.
2. Both are central and conservative.
3. Both obey an inverse-square law (both are inversely proportional to square of r).
4. Both propagate with finite speed c, the speed of light.

5. Electric charge and relativistic mass are conserved; note, though, that rest mass is not conserved.

Differences between electrostatic and gravitational forces:

1. Electrostatic forces are much greater than gravitational forces (by about 10^{36} times).
2. Gravitational forces are attractive for like charges, whereas electrostatic forces are repulsive for like charges.
3. There are no negative gravitational charges (no negative mass) while there are both positive and negative electric charges. This difference combined with previous implies that gravitational forces are always attractive, while electrostatic forces may be either attractive or repulsive.

Chapter- 3

Faraday's Law of Induction

Faraday's law of induction is a basic law of electromagnetism relating to the operating principles of transformers, inductors, and many types of electrical motors and generators. The law states that:

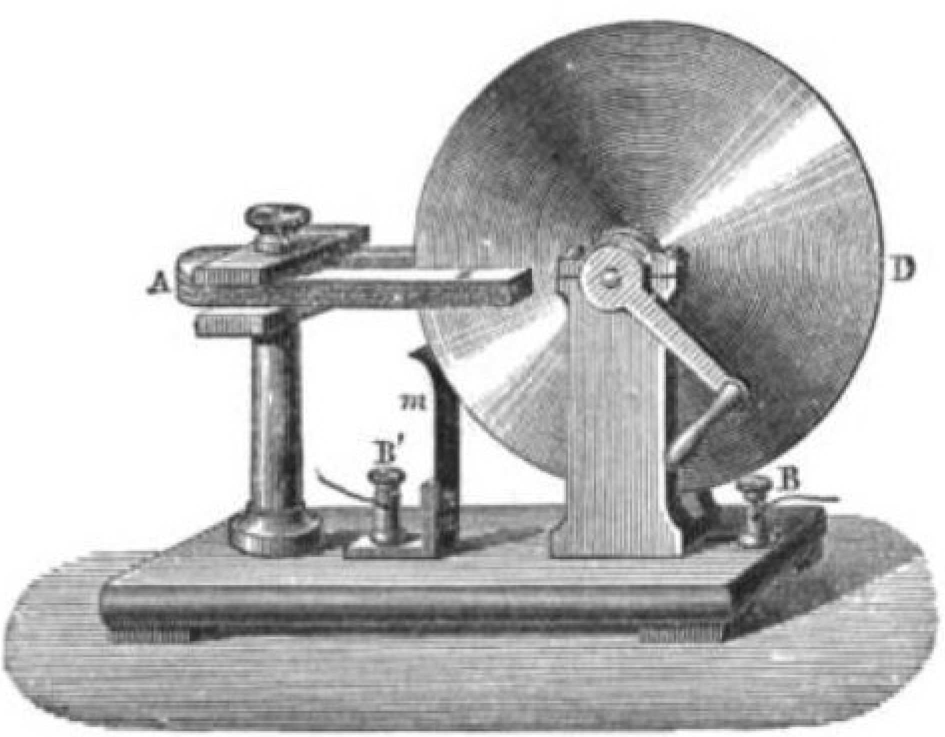
The induced electromotive force (EMF) in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit.

Or alternatively:

The EMF generated is proportional to the rate of change of the magnetic flux.

History

Electromagnetic induction was discovered independently by Michael Faraday and Joseph Henry in 1831; however, Faraday was the first to publish the results of his experiments.

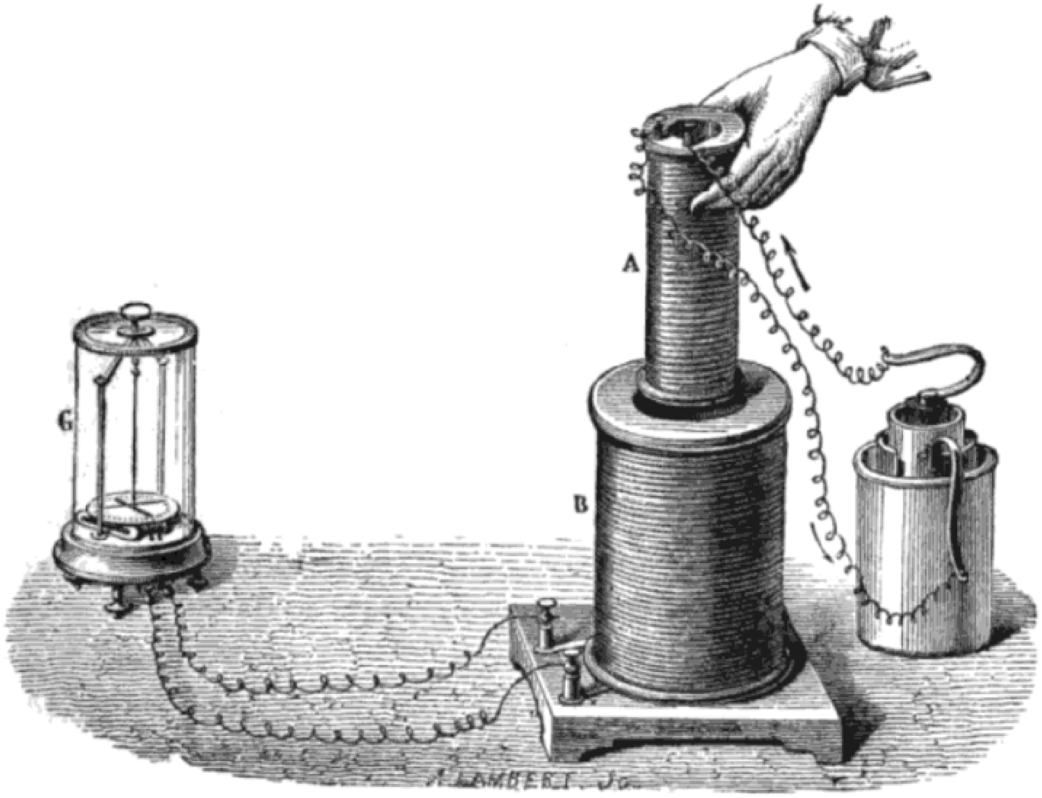


Faraday's disk

In Faraday's first experimental demonstration of electromagnetic induction (August 1831), he wrapped two wires around opposite sides of an iron torus (an arrangement similar to a modern transformer). Based on his assessment of recently-discovered properties of electromagnets, he expected that when current started to flow in one wire, a sort of wave would travel through the ring and cause some electrical effect on the opposite side. He plugged one wire into a galvanometer, and watched it as he connected the other wire to a battery. Indeed, he saw a transient current (which he called a "wave of electricity") when he connected the wire to the battery, and another when he disconnected it. Within two months, Faraday had found several other manifestations of electromagnetic induction. For example, he saw transient currents when he quickly slid a bar magnet in and out of a coil of wires, and he generated a steady (DC) current by rotating a copper disk near a bar magnet with a sliding electrical lead ("Faraday's disk").

Faraday explained electromagnetic induction using a concept he called lines of force. However, scientists at the time widely rejected his theoretical ideas, mainly because they were not formulated mathematically. An exception was Maxwell, who used Faraday's ideas as the basis of his quantitative electromagnetic theory. In Maxwell's papers, the time varying aspect of electromagnetic induction is expressed as a differential equation which Oliver Heaviside referred to as Faraday's law even though it is slightly different in form from the original version of Faraday's law, and doesn't cater for motionally induced EMF. Heaviside's version is the form recognized today in the group of equations known as Maxwell's equations.

Lenz's law, formulated by Heinrich Lenz in 1834, describes "flux through the circuit", and gives the direction of the induced electromotive force and current resulting from electromagnetic induction (elaborated upon in the examples below).



Faraday's experiment showing induction between coils of wire: The liquid battery *(right)* provides a current which flows through the small coil *(A)*, creating a magnetic field. When the coils are stationary, no current is induced. But when the small coil is moved in or out of the large coil *(B)*, the magnetic flux through the large coil changes, inducing a current which is detected by the galvanometer *(G)*.

Faraday's law as two different phenomena

Some physicists have remarked that Faraday's law is a single equation describing two different phenomena: The **motional EMF** generated by a magnetic force on a moving wire, and the **transformer EMF** generated by an electric force due to a changing magnetic field. James Clerk Maxwell drew attention to this fact in his 1861 paper *On Physical Lines of Force*. In the latter half of part II of that paper, Maxwell gives a separate physical explanation for each of the two phenomena. A reference to these two aspects of electromagnetic induction is made in some modern textbooks. As Richard Feynman states:

So the "flux rule" that the emf in a circuit is equal to the rate of change of the magnetic flux through the circuit applies whether the flux changes because the field changes or because the circuit moves (or both).... Yet in our explanation of the rule we have used two completely – distinct laws for the two cases $\mathbf{v}\times\mathbf{B}$ for "circuit moves" and $\nabla \times \mathbf{E} = -\partial_t\mathbf{B}$ for "field changes".
We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of *two different phenomena*.

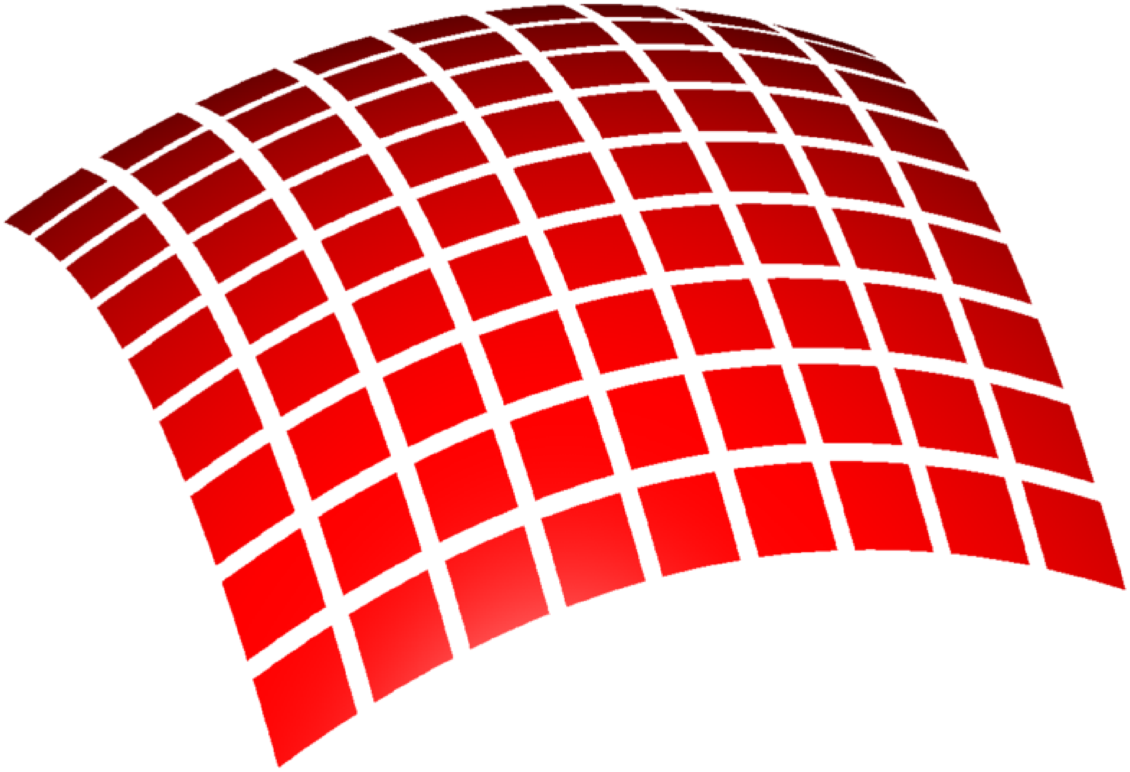
– ***Richard P. Feynman*** *, The Feynman Lectures on Physics*

Reflection on this apparent dichotomy was one of the principal paths that led Einstein to develop special relativity:

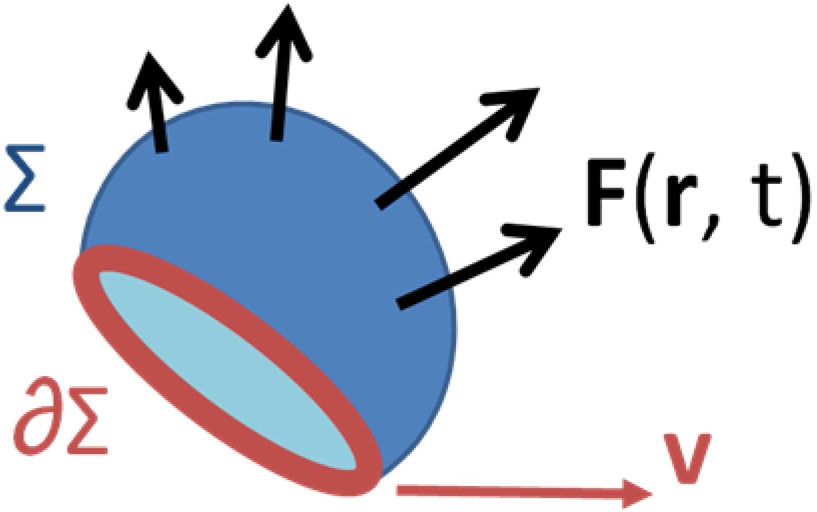
It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

– ***Albert Einstein****, On the Electrodynamics of Moving Bodies*

Flux through a surface and EMF around a loop



The definition of surface integral relies on splitting the surface Σ into small surface elements. Each element is associated with a vector $d\mathbf{A}$ of magnitude equal to the area of the element and with direction normal to the element and pointing outward.



A vector field $\mathbf{F}(\mathbf{r}, t)$ defined throughout space, and a surface Σ bounded by curve ∂Σ moving with velocity $\mathbf{v}$ over which the field is integrated.

Faraday's law of induction makes use of the magnetic flux Φ_B through a surface Σ, defined by an integral over a surface:

$$\Phi_B = \iint\limits_{\Sigma(t)} \mathbf{B}(\mathbf{r}, t) \cdot d\mathbf{A} \ ,$$

where $d\mathbf{A}$ is an element of surface area of the moving surface $\Sigma(t)$, $\mathbf{B}$ is the magnetic field, and $\mathbf{B}\cdot d\mathbf{A}$ is a vector dot product. The surface is considered to have a "mouth" outlined by a closed curve denoted $\partial\Sigma(t)$. When the flux changes, Faraday's law of induction says that the work $\mathcal{E}$ done (per unit charge) moving a test charge around the closed curve $\partial\Sigma(t)$, called the electromotive force (EMF), is given by:

$$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| \ ,$$

where $|\mathcal{E}|$ is the magnitude of the electromotive force (EMF) in volts and Φ_B is the magnetic flux in webers. The direction of the electromotive force is given by Lenz's law.

For a tightly-wound coil of wire, composed of N identical loops, each with the same Φ_B, Faraday's law of induction states that

$$|\mathcal{E}| = N \left|\frac{d\Phi_B}{dt}\right|$$

where N is the number of turns of wire and Φ_B is the magnetic flux in webers through a *single* loop.

In choosing a path $\partial\Sigma(t)$ to find EMF, the path must satisfy the basic requirements that (i) it is a closed path, and (ii) the path must capture the relative motion of the parts of the circuit (the origin of the t-dependence in $\partial\Sigma(t)$). It is *not* a requirement that the path follow a line of current flow, but of course the EMF that is found using the flux law will be the EMF around the chosen path. If a current path is not followed, the EMF might not be the EMF driving the current.

Example: Spatially varying Magnetic field

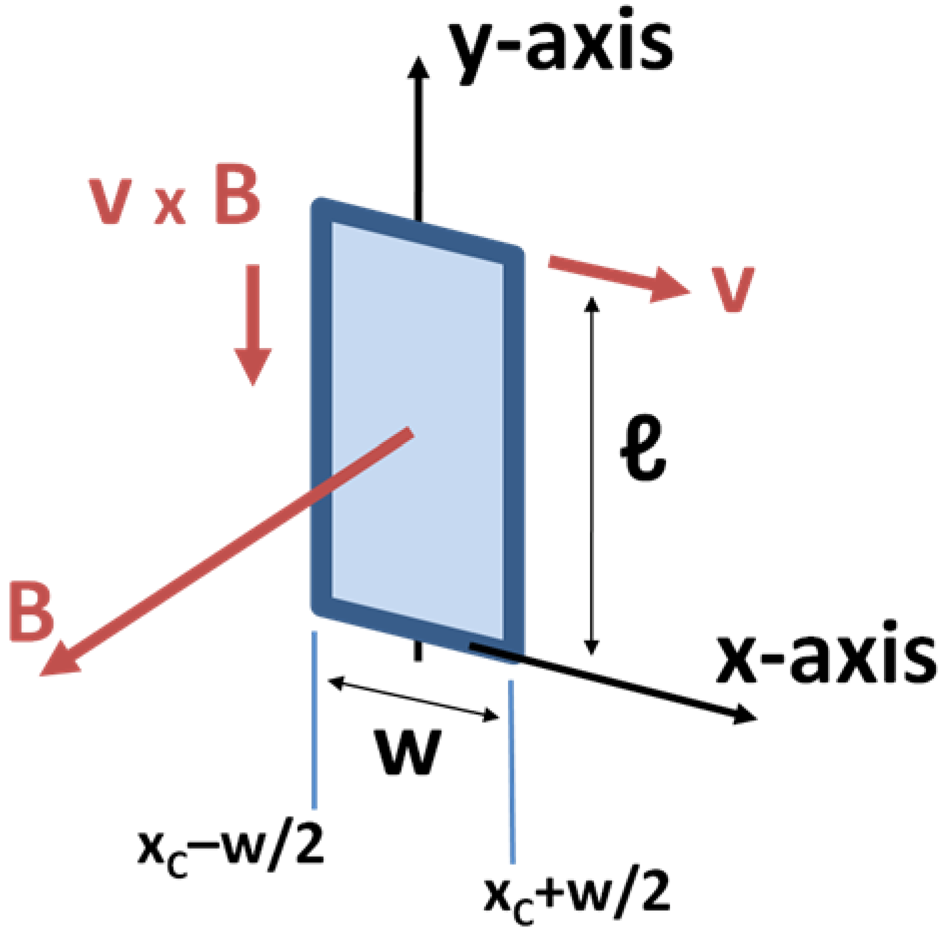


Figure 3: Closed rectangular wire loop moving along *x*-axis at velocity **v** in magnetic field **B** that varies with position *x*.

Consider the case in Figure 3 of a closed rectangular loop of wire in the *xy*-plane translated in the *x*-direction at velocity *v*. Thus, the center of the loop at x_C satisfies $v = dx_C / dt$. The loop has length ℓ in the *y*-direction and width *w* in the *x*-direction. A time-independent but spatially varying magnetic field $B(x)$ points in the *z*-direction. The magnetic field on the left side is $B(x_C - w / 2)$, and on the right side is $B(x_C + w / 2)$. The electromotive force is to be found by using either the Lorentz force law or equivalently by using Faraday's induction law above.

Lorentz force law method

A charge *q* in the wire on the left side of the loop experiences a Lorentz force $q\,\mathbf{v} \times B\,\mathbf{k} = -q\,v\,B(x_C - w / 2)\,\mathbf{j}$ (**j, k** unit vectors in the *y*- and *z*-directions), leading to an EMF (work per unit charge) of $v\,\ell\,B(x_C - w / 2)$ along the length of the left side of the loop. On the right side of the loop the same argument shows the EMF to be $v\,\ell\,B(x_C + w / 2)$. The two EMF's oppose each other, both pushing positive charge toward the bottom of the loop. In the case where the **B**-field increases with increase in *x*, the force on the right side is largest, and the current will be clockwise: using the right-hand rule, the *B*-field generated by the current opposes the impressed field. The EMF driving the current must increase as we move counterclockwise (opposite to the current). Adding the EMF's in a counterclockwise tour of the loop we find

$$\mathcal{E} = v\ell[B(x_C + w/2) - B(x_C - w/2)] \ .$$

Faraday's law method

At any position of the loop the magnetic flux through the loop is

$$\Phi_B = \pm \int_0^\ell dy \int_{x_C - w/2}^{x_C + w/2} B(x)dx$$
$$= \pm \ell \int_{x_C - w/2}^{x_C + w/2} B(x)dx \ .$$

The sign choice is decided by whether the normal to the surface points in the same direction as **B**, or in the opposite direction. If we take the normal to the surface as pointing in the same direction as the *B*-field of the induced current, this sign is negative. The time derivative of the flux is then (using the chain rule of differentiation or the general form of Leibniz rule for differentiation of an integral):

$$\frac{d\Phi_B}{dt} = (-)\frac{d}{dx_C}\left[\int_0^\ell dy \ \int_{x_C - w/2}^{x_C + w/2} dx B(x)\right]\frac{dx_C}{dt} \ ,$$
$$= (-)v\ell[B(x_C + w/2) - B(x_C - w/2)] \ ,$$

(where $v = \mathrm{d}x_C / \mathrm{d}t$ is the rate of motion of the loop in the *x*-direction) leading to:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = v\ell[B(x_C + w/2) - B(x_C - w/2)] \ ,$$

as before.

The equivalence of these two approaches is general and, depending on the example, one or the other method may prove more practical.

Example: Moving loop in uniform Magnetic field

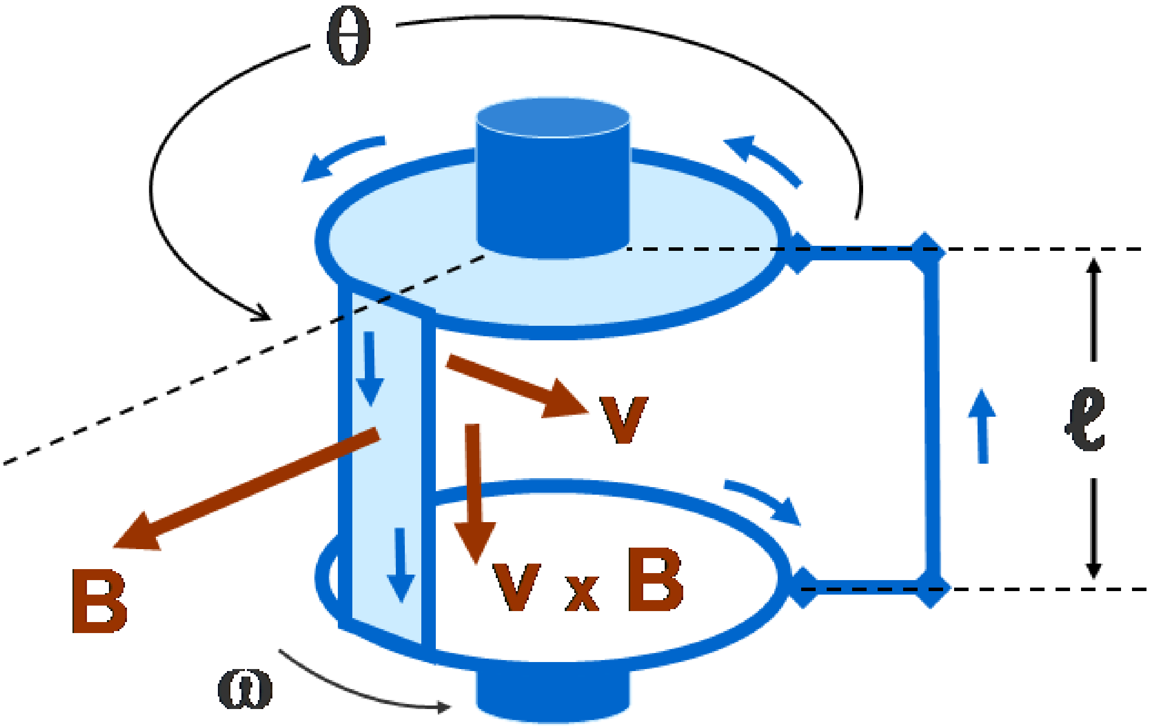


Figure 4: Rectangular wire loop rotating at angular velocity ω in radially outward pointing magnetic field **B** of fixed magnitude. Current is collected by brushes attached to top and bottom discs, which have conducting rims.

Figure 4 shows a spindle formed of two discs with conducting rims and a conducting loop attached vertically between these rims. The entire assembly spins in a magnetic field that points radially outward, but is the same magnitude regardless of its direction. A radially oriented collecting return loop picks up current from the conducting rims. At the location of the collecting return loop, the radial **B**-field lies in the plane of the collecting loop, so the collecting loop contributes no flux to the circuit. The electromotive force is to be found directly and by using Faraday's law above.

Lorentz force law method

In this case the Lorentz force drives the current in the two vertical arms of the moving loop downward, so current flows from the top disc to the bottom disc. In the conducting rims of the discs, the Lorentz force is perpendicular to the rim, so no EMF is generated in the rims, nor in the horizontal portions of the moving loop. Current is transmitted from the bottom rim to the top rim through the external return loop, which is oriented so the **B**-field is in its plane. Thus, the Lorentz force in the return loop is perpendicular to the loop, and no EMF is generated in this return loop. Traversing the current path in the direction opposite to the current flow, work is done against the Lorentz force only in the vertical arms of the moving loop, where

$$F = qBv \; .$$

where v= velocity of moving charge

Consequently, the EMF is

$$\mathcal{E} = Bv\ell = Br\ell\omega \; ,$$

where v = velocity of conductor or magnet and l = vertical length of the loop. In this case the velocity is related to the angular rate of rotation by $v = r\,\omega$, with r = radius of cylinder. Notice that the *same work* is done on *any* path that rotates with the loop and connects the upper and lower rim.

Faraday's law method

An intuitively appealing but mistaken approach to using the flux rule would say the flux through the circuit was just $\Phi_B = B\,w\,\ell$, where w = width of the moving loop. This number is time-independent, so the approach predicts incorrectly that no EMF is generated. The flaw in this argument is that it fails to consider the entire current path, which is a closed loop.

To use the flux rule, we have to look at the entire current path, which includes the path through the rims in the top and bottom discs. We can choose an arbitrary closed path through the rims and the rotating loop, and the flux law will find the EMF around the chosen path. Any path that has a segment attached to the rotating loop captures the relative motion of the parts of the circuit.

As an example path, let's traverse the circuit in the direction of rotation in the top disc, and in the direction opposite to the direction of rotation in the bottom disc (shown by arrows in Figure 4). In this case, for the moving loop at an angle θ from the collecting loop, a portion of the cylinder of area $A = r\,\ell\,\theta$ is part of the circuit. This area is perpendicular to the **B**-field, and so contributes to the flux an amount:

$$\Phi_B = -Br\theta\ell \; ,$$

where the sign is *negative* because the right-hand rule suggests the **B**-field generated by the current loop is opposite in direction to the applied **B** field. As this is the only time-dependent portion of the flux, the flux law predicts an EMF of

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = Br\ell\frac{d\theta}{dt}$$
$$= Br\ell\omega \; ,$$

in agreement with the Lorentz force law calculation.

Now let's try a different path. Follow a path traversing the rims *via* the opposite choice of segments. Then the coupled flux would *decrease* as θ increased, but the right-hand rule would suggest the current loop *added* to the applied **B**-field, so the EMF around this path is the same as for the first path. Any mixture of return paths leads to the same result for EMF, so it is actually immaterial which path is followed.

Direct evaluation of the change in flux

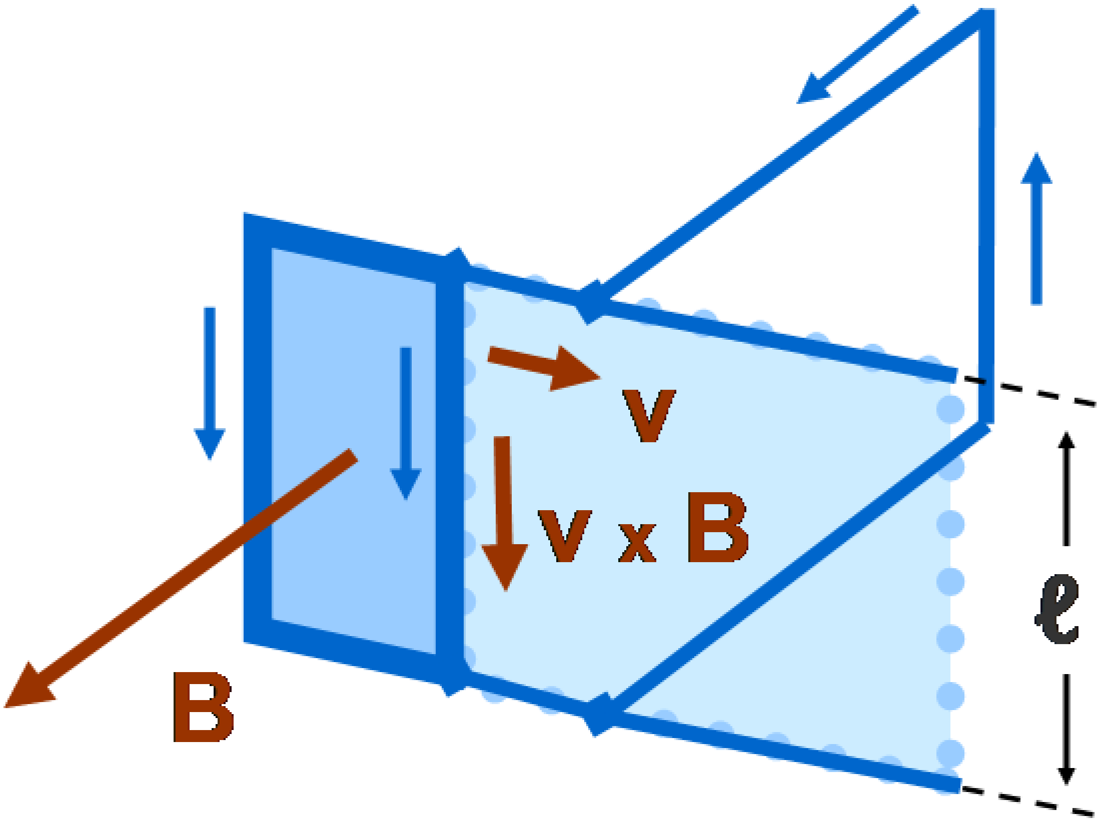


Figure 5: A simplified version of Figure 4. The loop slides with velocity **v** in a stationary, homogeneous **B**-field.

The use of a closed path to find EMF as done above appears to depend upon details of the path geometry. In contrast, the Lorentz-law approach is independent of such restrictions. The following discussion is intended to provide a better understanding of the equivalence of paths and escape the particulars of path selection when using the flux law.

Figure 5 is an idealization of Figure 4 with the cylinder unwrapped onto a plane. The same path-related analysis works, but a simplification is suggested. The time-independent aspects of the circuit cannot affect the time-rate-of-change of flux. For example, at a constant velocity of sliding the loop, the details of current flow through the loop are not time dependent. Instead of concern over details of the closed loop selected to find the EMF, one can focus on the *area of **B**-field swept out* by the moving loop. This suggestion amounts to finding the rate at which flux is cut by the circuit. That notion provides direct evaluation of the rate of change of flux, without concern over the time-independent details of various path choices around the circuit. Just as with the Lorentz law approach, it is clear that any two paths attached to the sliding loop, but differing in how they cross the loop, produce the same rate-of-change of flux.

In Figure 5 the area swept out in unit time is simply $dA / dt = v \ell$, regardless of the details of the selected closed path, so Faraday's law of induction provides the EMF as:

$$\mathcal{E} = \frac{d\Phi_B}{dt} = Bv\ell \ .$$

This path independence of EMF shows that if the sliding loop is replaced by a solid conducting plate, or even some complex warped surface, the analysis is the same: find the flux in the area swept out by the moving portion of the circuit. In a similar way, if the sliding loop in the drum generator of Figure 4 is replaced by a 360° solid conducting cylinder, the swept area calculation is exactly the same as for the case with only a loop. That is, the EMF predicted by Faraday's law is exactly the same for the case with a cylinder with solid conducting walls or, for that matter, a cylinder with a cheese grater for walls. Notice, though, that the current that flows as a result of this EMF will *not* be the same because the resistance of the circuit determines the current.

The Maxwell-Faraday equation

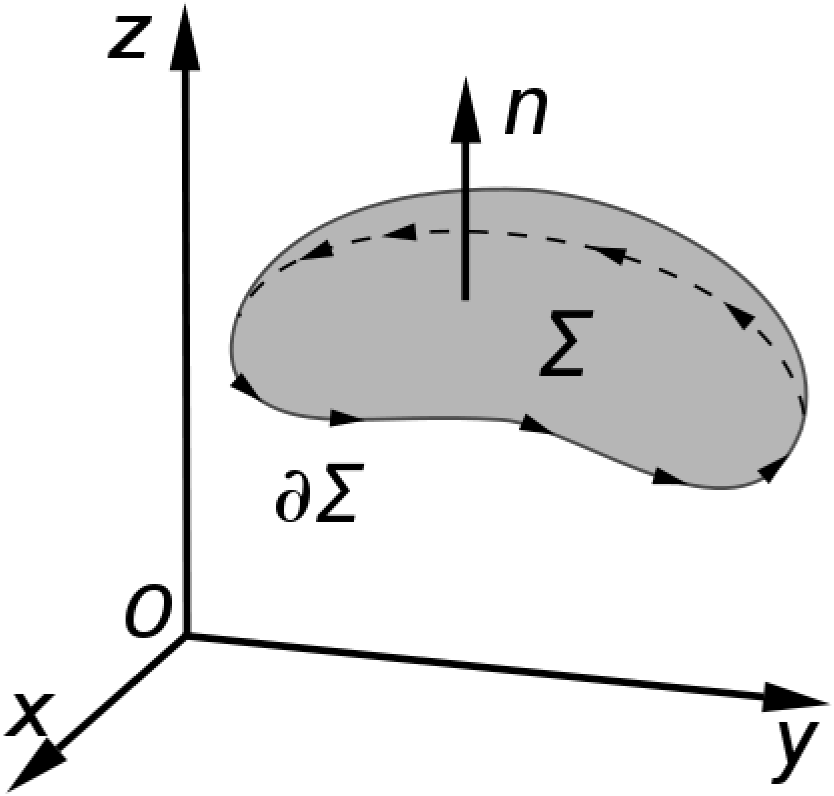


Figure 6: An illustration of Kelvin-Stokes theorem with surface **Σ** its boundary ***∂*Σ** and orientation ***n*** set by the right-hand rule.

A changing magnetic field creates an electric field; this phenomenon is described by the Maxwell-Faraday equation:

$$\nabla \times \mathbf{E}(\mathbf{r},\ t) = -\frac{\partial \mathbf{B}(\mathbf{r},\ t)}{\partial t}$$

where:

$\nabla \times$denotes curl
E is the electric field
B is the magnetic field

This equation appears in modern sets of Maxwell's equations and is often referred to as Faraday's law. However, because it contains only partial time derivatives, its application is restricted to situations where the test charge is stationary in a time varying magnetic field. It does not account for electromagnetic induction in situations where a charged particle is moving in a magnetic field.

It can also be written in an **integral form** by the Kelvin-Stokes theorem:

$$\oint_{\partial\Sigma} \mathbf{E} \cdot d\boldsymbol{\ell} = -\ \iint_{\Sigma} \frac{\partial}{\partial t}\mathbf{B} \cdot d\mathbf{A}$$
$$= -\frac{\partial}{\partial t} \iint_{\Sigma} \mathbf{B} \cdot d\mathbf{A}$$

where the movement of the derivative before the integration requires a time-independent surface **Σ** (considered in this context to be part of the interpretation of the partial derivative), and as indicated in Figure 6:

Σ is a surface bounded by the closed contour ***∂*Σ**; both **Σ** and ***∂*Σ** are fixed, independent of time
E is the electric field,
d**ℓ** is an infinitesimal vector element of the contour ***∂*Σ**,
B is the magnetic field.
d**A** is an infinitesimal vector element of surface **Σ** , whose magnitude is the area of an infinitesimal patch of surface, and whose direction is orthogonal to that surface patch.

Both d**ℓ** and d**A** have a sign ambiguity; to get the correct sign, the right-hand rule is used, as explained in the article Kelvin-Stokes theorem. For a planar surface Σ, a positive path element *d***ℓ** of curve ∂Σ is defined by the right-hand rule as one that points with the fingers of the right hand when the thumb points in the direction of the normal **n** to the surface Σ.

The integral around **∂Σ** is called a *path integral* or *line integral*. The surface integral at the right-hand side of the Maxwell-Faraday equation is the explicit expression for the magnetic flux Φ_B through **Σ**. Notice that a nonzero path integral for **E** is different from the behavior of the electric field generated by charges. A charge-generated **E**-field can be expressed as the gradient of a scalar field that is a solution to Poisson's equation, and has a zero path integral.

The integral equation is true for *any* path **∂Σ** through space, and any surface **Σ** for which that path is a boundary. Note, however, that **∂Σ** and **Σ** are understood *not* to vary in time in this formula. This integral form cannot treat **motional** EMF because **Σ** is time-independent. Notice as well that this equation makes no reference to EMF $\mathcal{E}$, and indeed cannot do so without introduction of the Lorentz force law to enable a calculation of work.

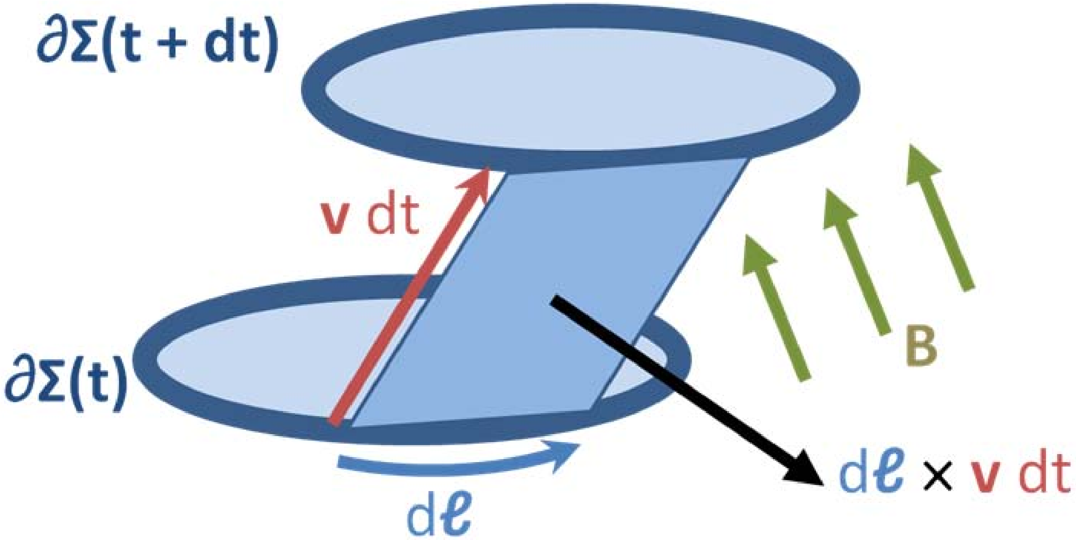


Figure 7: Area swept out by vector element *d*ℓ of curve **∂Σ** in time *dt* when moving with velocity **v**.

Using the complete Lorentz force to calculate the EMF,

$$\mathcal{E} = \oint_{\partial\Sigma(t)} (\mathbf{E}(\mathbf{r},\ t) + \mathbf{v} \times \mathbf{B}(\mathbf{r},\ t)) \cdot d\boldsymbol{\ell}\ ,$$

a statement of Faraday's law of induction more general than the integral form of the Maxwell-Faraday equation is:

$$\oint_{\partial\Sigma(t)} (\mathbf{E}(\mathbf{r},\ t) + \mathbf{v} \times \mathbf{B}(\mathbf{r},\ t)) \cdot d\boldsymbol{\ell} = -\frac{d}{dt} \iint_{\Sigma(t)} d\boldsymbol{A} \cdot \mathbf{B}(\mathbf{r},\ t)\ ,$$

where $\partial\boldsymbol{\Sigma}(t)$ is the moving closed path bounding the moving surface $\boldsymbol{\Sigma}(t)$, and **v** is the velocity of movement. See Figure 2. Notice that the *ordinary* time derivative is used, not a *partial* time derivative, implying the time variation of $\boldsymbol{\Sigma}(t)$ must be included in the differentiation. In the integrand the element of the curve $d\boldsymbol{\ell}$ moves with velocity **v**.

Figure 7 provides an interpretation of the magnetic force contribution to the EMF on the left side of the above equation. The area swept out by segment $d\boldsymbol{\ell}$ of curve $\partial\boldsymbol{\Sigma}$ in time dt when moving with velocity **v** is:

$$d\mathbf{A} = -d\boldsymbol{\ell} \times \boldsymbol{v} dt\ ,$$

so the change in magnetic flux $\Delta\Phi_B$ through the portion of the surface enclosed by $\partial\boldsymbol{\Sigma}$ in time dt is:

$$\frac{d\Delta\Phi_B}{dt} = -\mathbf{B} \cdot\ d\boldsymbol{\ell} \times \boldsymbol{v}\ = -\mathbf{v} \times \mathbf{B} \cdot\ d\boldsymbol{\ell}\ ,$$

and if we add these $\Delta\Phi_B$-contributions around the loop for all segments $d\boldsymbol{\ell}$, we obtain the magnetic force contribution to Faraday's law. That is, this term is related to *motional* EMF.

Example: viewpoint of a moving observer

Revisiting the example of Figure 3 in a moving frame of reference brings out the close connection between *E*- and *B*-fields, and between *motional* and *induced* EMF's. Imagine an observer of the loop moving with the loop. The observer calculates the EMF around the loop using both the Lorentz force law and Faraday's law of induction. Because this observer moves with the loop, the observer sees no movement of the loop, and zero **v × B**. However, because the *B*-field varies with position *x*, the moving observer sees a time-varying magnetic field, namely:

$$\mathbf{B} = \mathbf{k} B(x + vt)\ ,$$

where ***k*** is a unit vector pointing in the *z*-direction.

Lorentz force law version

The Maxwell-Faraday equation says the moving observer sees an electric field E_y in the *y*-direction given by:

$$\nabla \times \mathbf{E} = \mathbf{k}\ \frac{dE_y}{dx}$$

$$= -\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{k}\frac{dB(x+vt)}{dt} = -\mathbf{k}\frac{dB}{dx}v \quad ,$$

Here the chain rule is used:

$$\frac{dB}{dt} = \frac{dB}{d(x+vt)}\frac{d(x+vt)}{dt} = \frac{dB}{dx}v \ .$$

Solving for E_y, to within a constant that contributes nothing to an integral around the loop,

$$E_y(x,\ t) = -B(x+vt)\ v\ .$$

Using the Lorentz force law, which has only an electric field component, the observer finds the EMF around the loop at a time t to be:

$$\begin{aligned}&\mathcal{E} = -\ell[E_y(x_C + w/2,\ t) - E_y(x_C - w/2,\ t)]\\ &= v\ell[B(x_C + w/2 + vt) - B(x_C - w/2 + vt)]\ ,\end{aligned}$$

which is exactly the same result found by the stationary observer, who sees the centroid x_C has advanced to a position $x_C + v\,t$. However, the moving observer obtained the result under the impression that the Lorentz force had only an ***electric*** component, while the stationary observer thought the force had only a ***magnetic*** component.

Faraday's law of induction

Using Faraday's law of induction, the observer moving with x_C sees a changing magnetic flux, but the loop does not appear to move: the center of the loop x_C is fixed because the moving observer is moving with the loop. The flux is then:

$$\Phi_B = -\int_0^{\ell} dy \int_{x_C - w/2}^{x_C + w/2} B(x+vt)dx\ ,$$

where the minus sign comes from the normal to the surface pointing oppositely to the applied B-field. The EMF from Faraday's law of induction is now:

$$\begin{aligned}&\mathcal{E} = -\frac{d\Phi_B}{dt} = \int_0^{\ell} dy \int_{x_C - w/2}^{x_C + w/2} \frac{d}{dt}B(x+vt)dx\\ &= \int_0^{\ell} dy \int_{x_C - w/2}^{x_C + w/2} \frac{d}{dx}B(x+vt)\ v\ dx\\ &= v\ell\ [B(x_C + w/2 + vt) - B(x_C - w/2 + vt)]\ ,\end{aligned}$$

the same result. The time derivative passes through the integration because the limits of integration have no time dependence. Again, the chain rule was used to convert the time derivative to an x-derivative.

The stationary observer thought the EMF was a ***motional*** EMF, while the moving observer thought it was an ***induced*** EMF.

Electrical generator

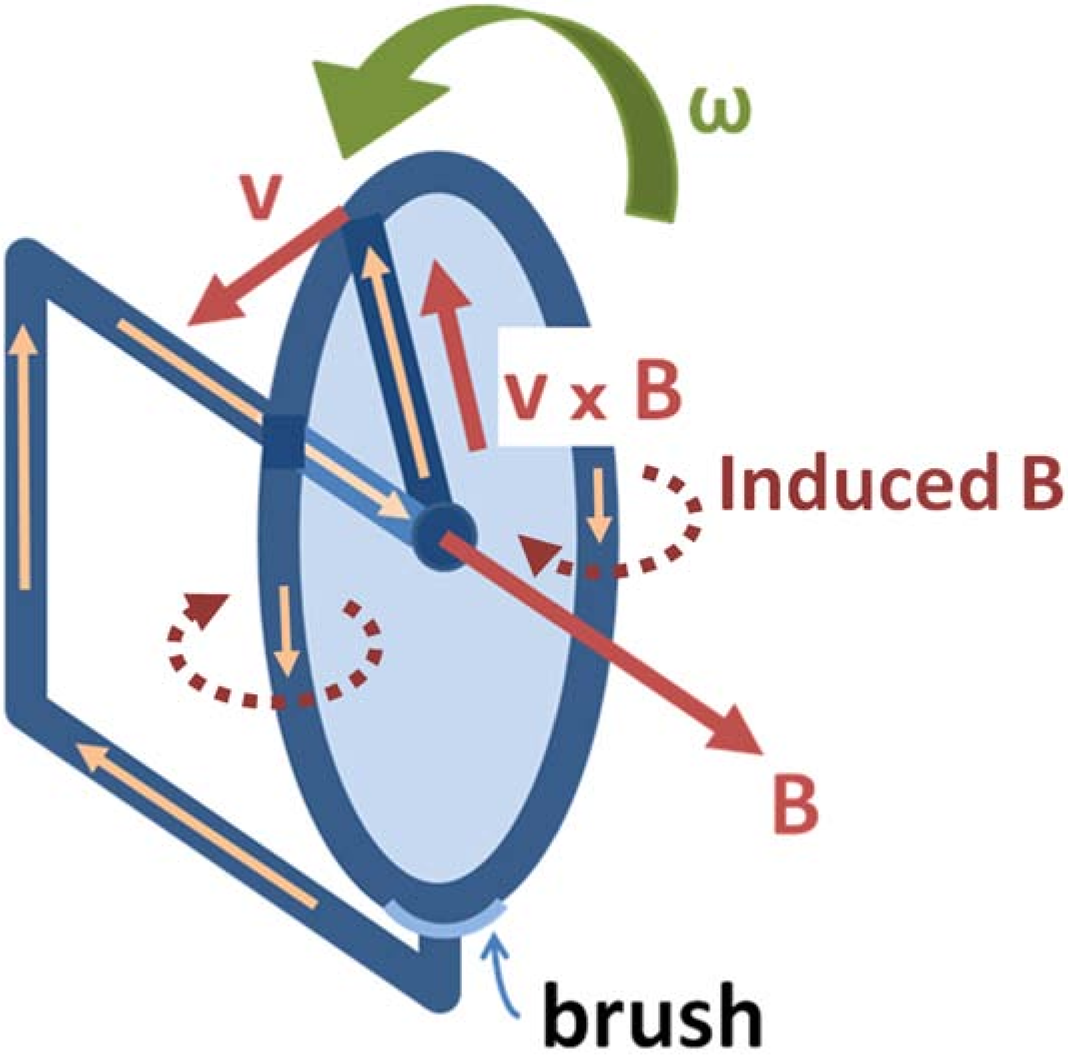


Figure 8: Faraday's disc electric generator. The disc rotates with angular rate ω, sweeping the conducting radius circularly in the static magnetic field **B**. The magnetic Lorentz force **v** × **B** drives the current along the conducting radius to the conducting rim, and from there the circuit completes through the lower brush and the axle supporting the disc. Thus, current is generated from mechanical motion.

The EMF generated by Faraday's law of induction due to relative movement of a circuit and a magnetic field is the phenomenon underlying electrical generators. When a permanent magnet is moved relative to a conductor, or vice versa, an electromotive force is created. If the wire is connected through an electrical load, current will flow, and thus electrical energy is generated, converting the mechanical energy of motion to electrical energy. For example, the *drum generator* is based upon Figure 4. A different implementation of this idea is the Faraday's disc, shown in simplified form in Figure 8. Note that either the analysis of Figure 5, or direct application of the Lorentz force law, shows that a *solid* conducting disc works the same way.

In the Faraday's disc example, the disc is rotated in a uniform magnetic field perpendicular to the disc, causing a current to flow in the radial arm due to the Lorentz force. It is interesting to understand how it arises that mechanical work is necessary to drive this current. When the generated current flows through the conducting rim, a magnetic field is generated by this current through Ampere's circuital law (labeled "induced B" in Figure 8). The rim thus becomes an electromagnet that resists rotation of the disc (an example of Lenz's law). On the far side of the figure, the return current flows from the rotating arm through the far side of the rim to the bottom brush. The B-field induced by this return current opposes the applied B-field, tending to *decrease* the flux through that side of the circuit, opposing the *increase* in flux due to rotation. On the near side of the figure, the return current flows from the rotating arm through the near side of the rim to the bottom brush. The induced B-field *increases* the flux on this side of the circuit, opposing the *decrease* in flux due to rotation. Thus, both sides of the circuit generate an emf opposing the rotation. The energy required to keep the disc moving, despite this reactive force, is exactly equal to the electrical energy generated (plus energy wasted due to friction, Joule heating, and other inefficiencies). This behavior is common to all generators converting mechanical energy to electrical energy.

Although Faraday's law always describes the working of electrical generators, the detailed mechanism can differ in different cases. When the magnet is rotated around a stationary conductor, the changing magnetic field creates an electric field, as described by the Maxwell-Faraday equation, and that electric field pushes the charges through the wire. This case is called an ***induced*** EMF. On the other hand, when the magnet is stationary and the conductor is rotated, the moving charges experience a magnetic force (as described by the Lorentz force law), and this magnetic force pushes the charges through the wire. This case is called ***motional*** EMF.

Electrical motor

An electrical generator can be run "backwards" to become a motor. For example, with the Faraday disc, suppose a DC current is driven through the conducting radial arm by a voltage. Then by the Lorentz force law, this traveling charge experiences a force in the magnetic field B that will turn the disc in a direction given by Fleming's left hand rule. In the absence of irreversible effects, like friction or Joule heating, the disc turns at the rate necessary to make $d\,\Phi_B\,/\,dt$ equal to the voltage driving the current.

Electrical transformer

The EMF predicted by Faraday's law is also responsible for electrical transformers. When the electric current in a loop of wire changes, the changing current creates a changing magnetic field. A second wire in reach of this magnetic field will experience this change in magnetic field as a change in its coupled magnetic flux, a $d\,\Phi_B\,/\,d\,t$. Therefore, an electromotive force is set up in the second loop called the **induced EMF** or **transformer EMF**. If the two ends of this loop are connected through an electrical load, current will flow.

Magnetic flow meter

Faraday's law is used for measuring the flow of electrically conductive liquids and slurries. Such instruments are called magnetic flow meters. The induced voltage $\mathcal{E}$ generated in the magnetic field B due to a conductive liquid moving at velocity v is thus given by:

$$\mathcal{E} = B\ell v,$$

where ℓ is the distance between electrodes in the magnetic flow meter.

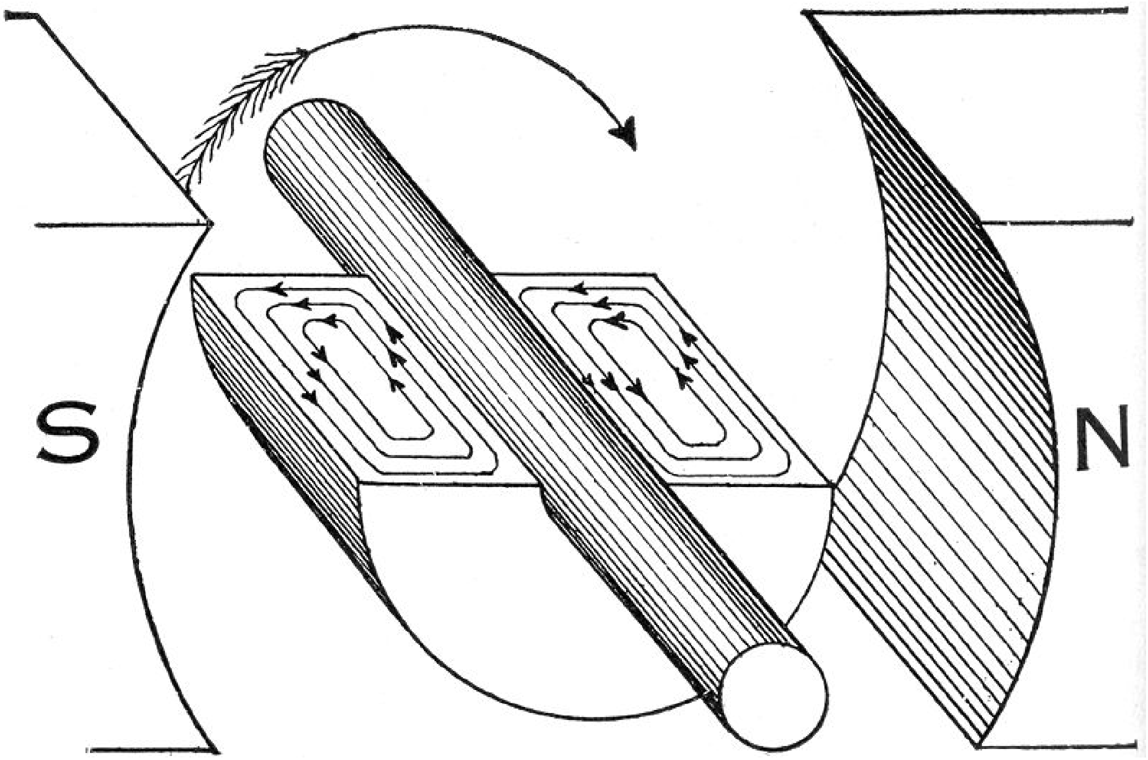
Parasitic induction and waste heating

All metal objects moving in relation to a static magnetic field will experience inductive power flow, as do all stationary metal objects in relation to a moving magnetic field. These power flows are occasionally undesirable, resulting in flowing electric current at very low voltage and heating of the metal.

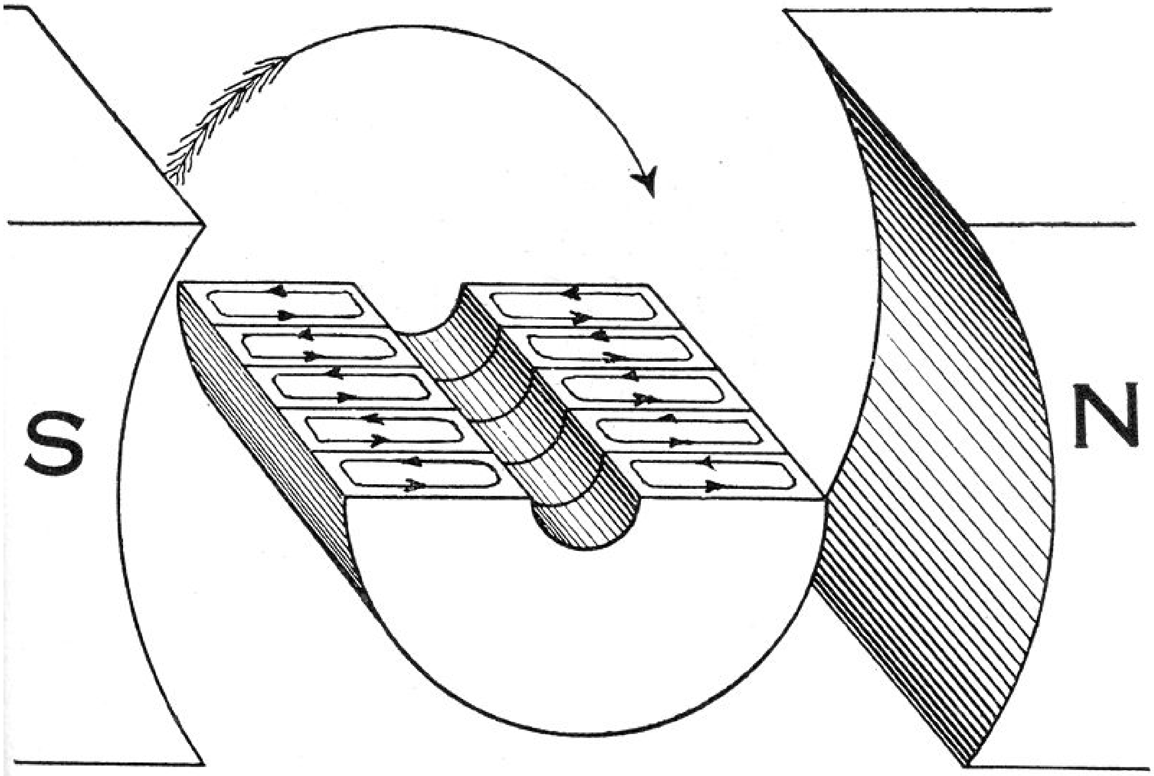
There are a number of methods employed to control these undesirable inductive effects.

- Electromagnets in electric motors, generators, and transformers do not use solid metal, but instead use thin sheets of metal plate, called *laminations*. These thin plates reduce the parasitic eddy currents, as described below.
- Inductive coils in electronics typically use magnetic cores to minimize parasitic current flow. They are a mixture of metal powder plus a resin binder that can hold any shape. The binder prevents parasitic current flow through the powdered metal.

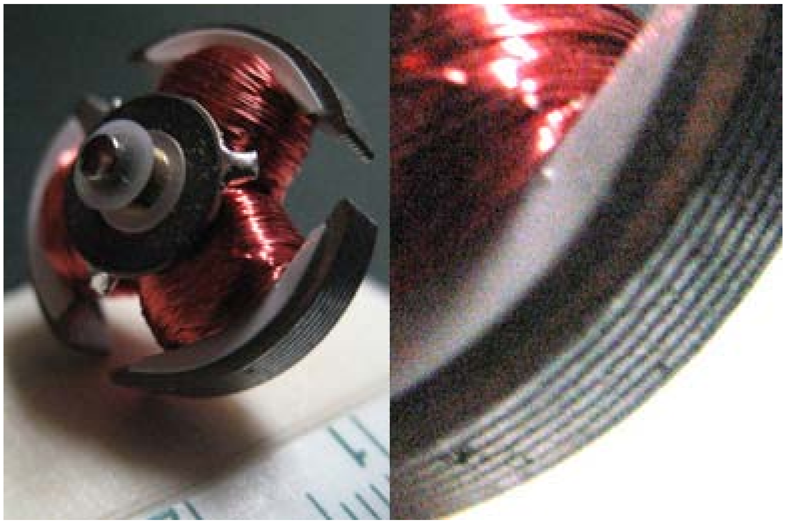
Electromagnet laminations



Eddy currents occur when a solid metallic mass is rotated in a magnetic field, because the outer portion of the metal cuts more lines of force than the inner portion, hence the induced electromotive force not being uniform, tends to set up currents between the points of greatest and least potential. Eddy currents consume a considerable amount of energy and often cause a harmful rise in temperature.

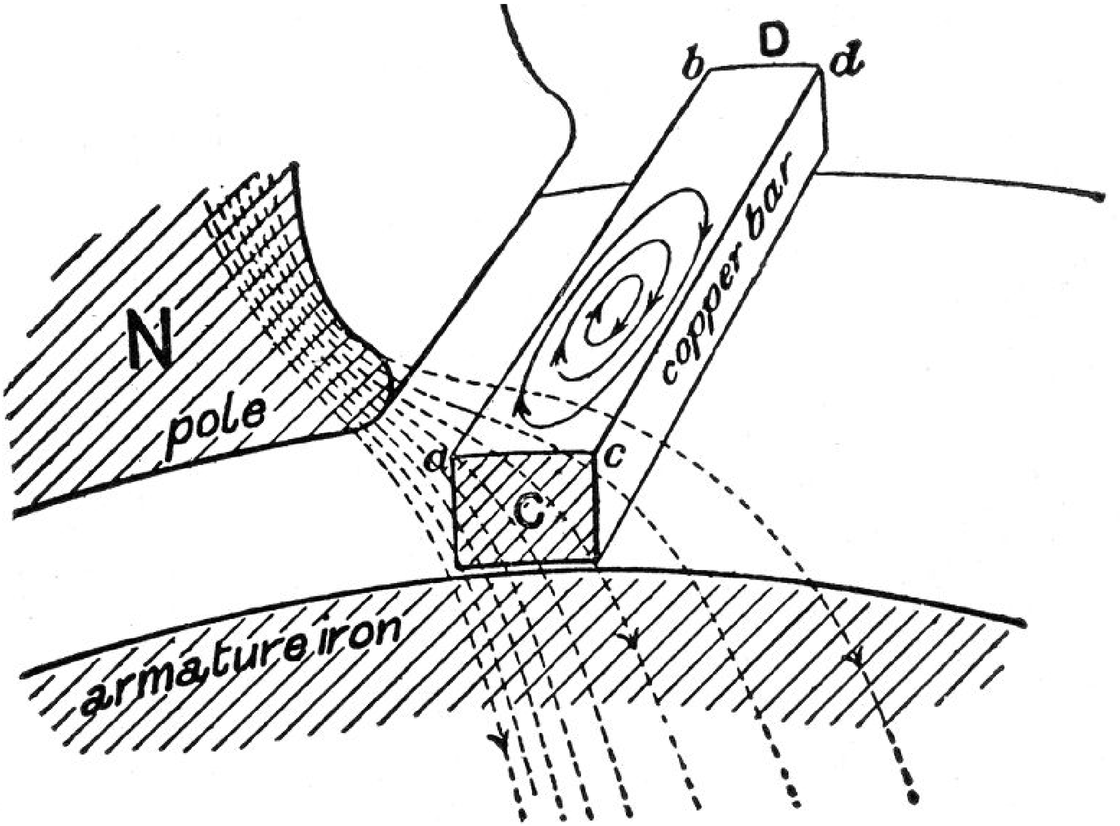


Only five laminations or plates are shown in this example, so as to show the subdivision of the eddy currents. In practical use, the number of laminations or punchings ranges from 40 to 66 per inch, and brings the eddy current loss down to about one percent. While the plates can be separated by insulation, the voltage is so low that the natural rust/oxide coating of the plates is enough to prevent current flow across the laminations.



This is a rotor approximately 20mm in diameter from a DC motor used in a CD player. Note the laminations of the electromagnet pole pieces, used to limit parasitic inductive losses.

Parasitic induction within inductors



In this illustration, a solid copper bar inductor on a rotating armature is just passing under the tip of the pole piece N of the field magnet. Note the uneven distribution of the lines of force across the bar inductor. The magnetic field is more concentrated and thus stronger on the left edge of the copper bar (a,b) while the field is weaker on the right edge (c,d). Since the two edges of the bar move with the same velocity, this difference in field strength across the bar creates whirls or current eddies within the copper bar.

This is one reason high voltage devices tend to be more efficient than low voltage devices. High voltage devices use many turns of small-gauge wire in motors, generators, and transformers. These many small turns of inductor wire in the electromagnet break up the eddy flows that can form within the large, thick inductors of low voltage, high current devices.

Chapter- 4

Faraday Paradox

The **Faraday paradox** (or **Faraday's paradox**) is an experiment that illustrates Michael Faraday's law of electromagnetic induction. Faraday deduced this law in 1831, after inventing the first electromagnetic generator or dynamo, but was never satisfied with his own explanation of the paradox.

The equipment

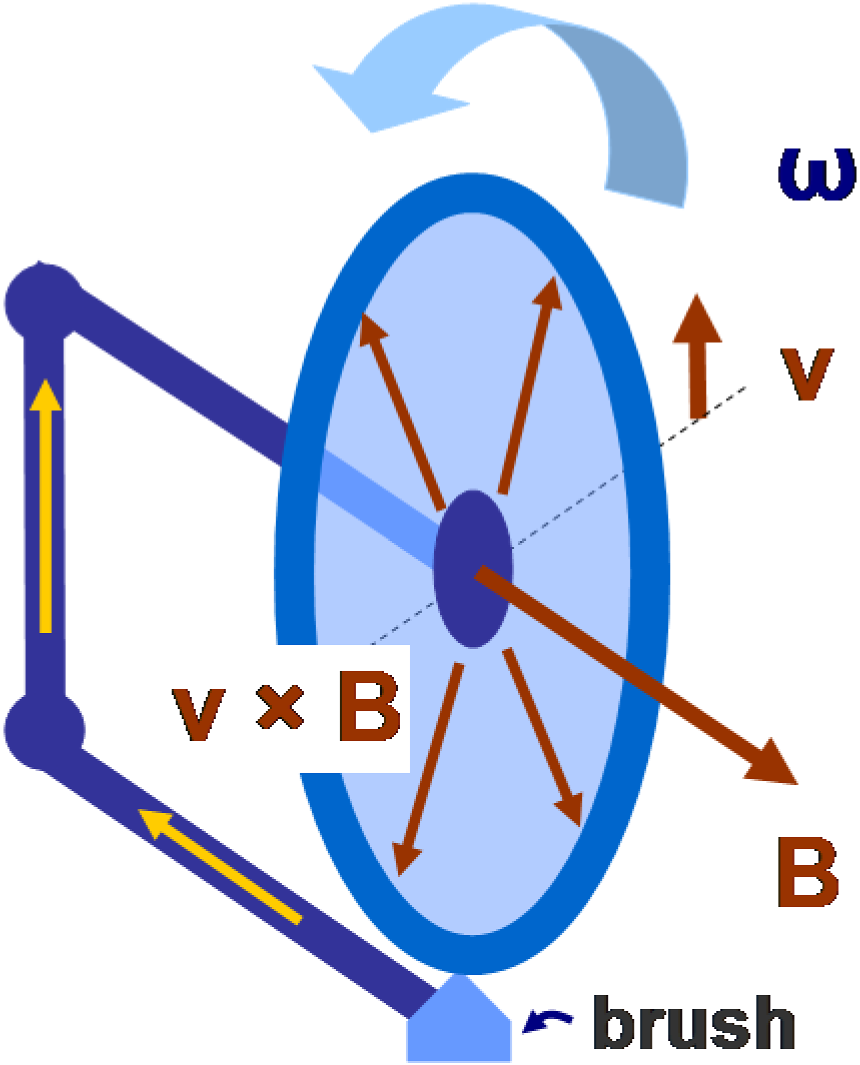


Figure 1: Faraday's disc electric generator. The disc rotates with angular rate ω, sweeping the conducting disc circularly in the static magnetic field **B** due to a permanent magnet. The magnetic Lorentz force **v** × **B** drives the current radially across the conducting disc to the conducting rim, and from there the circuit path completes through the lower brush and the axle supporting the disc. Thus, current is generated from mechanical motion.

The experiment requires a few simple components (see Figure 1): a cylindrical magnet, a conducting disc with a conducting rim, a conducting axle, some wiring, and a galvanometer. The disc and the magnet are fitted a short distance apart on the axle, on which they are free to rotate about their own axes of symmetry. An electrical circuit is formed by connecting sliding contacts: one to the axle of the disc, the other to its rim. A galvanometer can be inserted in the circuit to measure the current.

The procedure

The experiment proceeds in three steps:

1. The magnet is held to prevent it from rotating, while the disc is spun on its axis. The result is that the galvanometer registers a direct current. The apparatus therefore acts as a generator, variously called the Faraday generator, the Faraday disc, or the homopolar (or unipolar) generator.
2. The disc is held stationary while the magnet is spun on its axis. The result is that the galvanometer registers no current.
3. The disc and magnet are spun together. The galvanometer registers a current, as it did in step 1.

Why is this paradoxical?

The experiment is described by some as a "paradox" as it seems, at first sight, to violate Faraday's law of electromagnetic induction, because the flux through the disc appears to be the same no matter what is rotating. Hence, the EMF is predicted to be zero in all three cases of rotation. The discussion below shows this viewpoint stems from an incorrect choice of surface over which to calculate the flux.

The paradox appears a bit different from the lines of flux viewpoint: in Faraday's model of electromagnetic induction, a magnetic field consisted of imaginary lines of magnetic flux, similar to the lines that appear when iron filings are sprinkled on paper and held near a magnet. The EMF is proposed to be proportional to the rate of cutting lines of flux. If the lines of flux are imagined to originate in the magnet, then they would be stationary in the frame of the magnet, and rotating the disc relative to the magnet, whether by rotating the magnet or the disc, should produce an EMF, but rotating both of them together should not.

Faraday's explanation

In Faraday's model of electromagnetic induction, a circuit received an induced current when it cut lines of magnetic flux. According to this model, the Faraday disc should have worked when either the disc or the magnet was rotated, but not both. Faraday attempted to explain the disagreement with observation by assuming that the magnet's field, complete with its lines of flux, remained stationary as the magnet rotated (a completely accurate picture, but maybe not intuitive in the lines-of-flux model). In other words, the lines of flux have their own frame of reference. As we shall see in the next section, modern physics (since the discovery of the electron) does not need the lines-of-flux picture and dispels the paradox.

Modern explanations

Using the Lorentz force

After the discovery of the electron and the forces that affect it, a microscopic resolution of the paradox became possible. See Figure 1. The metal portions of the apparatus are conducting, and confine a current due to electronic motion to within the metal boundaries. All electrons that move in a magnetic field experience a Lorentz force of $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, where $\boldsymbol{v}$ is the velocity of the electrons and $\boldsymbol{q}$ is the charge on an electron. This force is perpendicular to both the velocity of the electrons, which is in the plane of the disc, and to the magnetic field, which is normal (surface normal) to the disc. An electron at rest in the frame of the disc moves circularly with the disc relative to the B-field, and so experiences a radial Lorentz force. In Figure 1 this force (on a *positive* charge, not an electron) is outward toward the rim according to the right-hand rule.

Of course, this radial force, which is the cause of the current, creates a radial component of electron velocity, generating in turn its own Lorentz force component that opposes the circular motion of the electrons, tending to slow the disc's rotation, but the electrons retain a component of circular motion that continues to drive the current via the radial Lorentz force.

This mechanism agrees with the observations: an EMF is generated whenever the disc moves relative to the magnetic field, regardless of how that field is generated.

The use of the Lorentz equation to explain the Faraday Paradox has led to a debate in the literature as to whether or not a magnetic field rotates with a magnet. Since the force on charges expressed by the Lorentz equation depends upon the relative motion of the magnetic field to the conductor where the EMF is located it was speculated that in the case when the magnet rotates with the disk but a voltage still develops, that the magnetic field must therefore not rotate with the magnetic material as it turns with no relative motion with respect to the conductive disk.

However, careful thought showed if the magnetic field was assumed to rotate with the magnet and the magnet rotated with the disk that a current should still be produced, not by EMF in the disk (there is no relative motion between the disk and magnet) but in the external circuit linking the brushes which is in fact in relative motion with respect to the rotating magnet. In fact it was shown that so long as a current loop was used to measure induced EMFs from the motion of the disk and magnet it is not possible to tell if the magnetic field does or does not rotate with the magnet.

Several experiments have been proposed using electrostatic measurements or electron beams to resolve the issue but apparently none has been successfully performed to date.

Relation to Faraday's law of induction

The flux through the portion of the path from the brush at the rim, through the outside loop and the axle to the center of the disc is always zero because the magnetic field is in the plane of this path (not perpendicular to it), no matter what is rotating, so the integrated emf around this part of the path is always zero. Therefore, attention is focused on the portion of the path from the axle across the disc to the brush at the rim.

Faraday's law of induction can be stated in words as:

The induced electromotive force or EMF in any closed circuit is equal to the time rate of change of the magnetic flux through the circuit.

Mathematically, the law is stated:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\Sigma(t)} d\boldsymbol{A} \cdot \mathbf{B}(\mathbf{r},\ t)\ ,$$

where Φ_B is the flux, and $d\,\mathbf{A}$ is a vector element of area of a moving surface $\mathbf{\Sigma}(t)$ bounded by the loop around which the EMF is to be found.

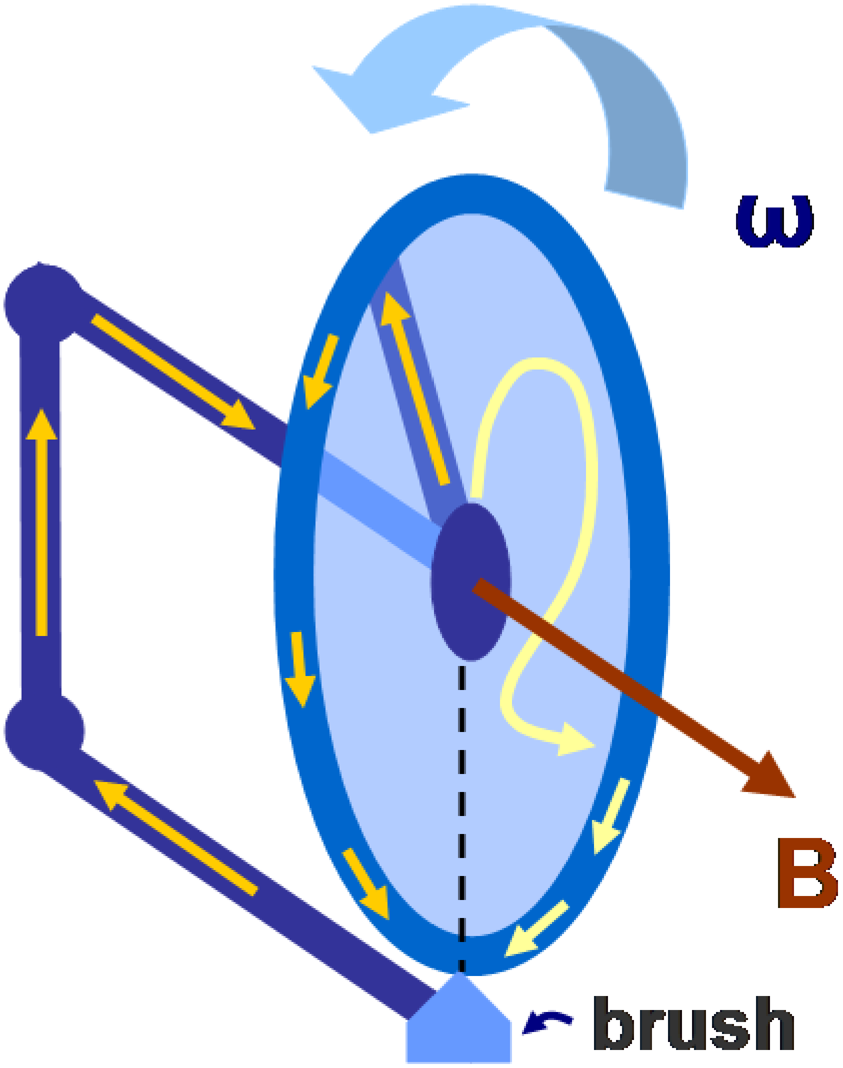


Figure 2: Two possible loops for finding EMF: the geometrically simple path is easy to use, but the other provides the same EMF. Neither is intended to imitate any line of physical current flow.

How can this law be connected to the Faraday disc generator, where the flux linkage appears to be just the *B*-field multiplied by the area of the disc?

One approach is to define the notion of "rate of change of flux linkage" by drawing a hypothetical line across the disc from the brush to the axle and asking how much flux linkage is swept past this line per unit time. See Figure 2. Assuming a radius *R* for the disc, a sector of disc with central angle θ has an area:

$$A = \frac{\theta}{2\pi}\pi R^2 \ ,$$

so the rate that flux sweeps past the imaginary line is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = B\frac{dA}{dt} = B\ \frac{R^2}{2}\ \frac{d\theta}{dt} = B\ \frac{R^2}{2}\omega\ ,$$

with $\omega = d\,\theta / dt$ the angular rate of rotation. The sign is chosen based upon Lenz's law: the field generated by the motion must oppose the change in flux caused by the rotation.

This flux-cutting result for EMF can be compared to calculating the work done per unit charge making an infinitesimal test charge traverse the hypothetical line using the Lorentz force / unit charge at radius r, namely $|\mathbf{v} \times \mathbf{B}| = B\,v = B\,r\,\omega$:

$$\mathcal{E} = \int_0^R dr B r \omega = \frac{R^2}{2} B\omega\ ,$$

which is the same result.

The above methodology for finding the flux cut by the circuit is formalized in the flux law by properly treating the time derivative of the bounding surface $\Sigma(t)$. Of course, the time derivative of an integral with time dependent limits is *not* simply the time derivative of the integrand alone, a point often forgotten.

In choosing the surface $\Sigma(t)$, the restrictions are that (i) it be bounded by a closed curve around which the EMF is to be found, and (ii) it capture the relative motion of all moving parts of the circuit. It is emphatically *not* required that the bounding curve correspond to a physical line of flow of the current. On the other hand, induction is all about relative motion, and the path emphatically *must* capture any relative motion. In a case like Figure 1 where a portion of the current path is distributed over a region in space, the EMF driving the current can be found using a variety of paths. Figure 2 shows two possibilities. All paths include the obvious return loop, but in the disc two paths are shown: one is a geometrically simple path, the other a tortuous one. We are free to choose whatever path we like, but a portion of any acceptable path is *fixed in the disc itself* and turns with the disc. The flux is calculated though the entire path, return loop *plus* disc segment, and its rate-of change found.

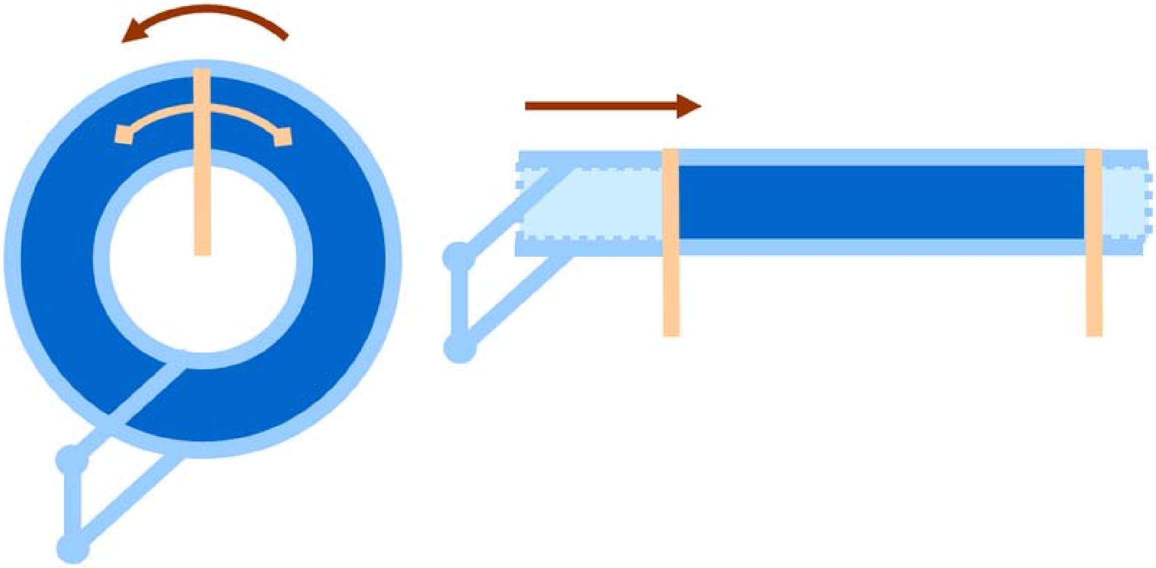


Figure 3: Mapping of the Faraday disc into a sliding conducting rectangle example. The disc is viewed as an annulus; it is cut along a radius and bent open to become a rectangle.

In this example, all these paths lead to the same rate of change of flux, and hence the same EMF. To provide some intuition about this path independence, in Figure 3 the Faraday disc is unwrapped onto a strip, making it resemble a sliding rectangle problem. In the sliding rectangle case, it becomes obvious that the pattern of current flow inside the rectangle is time-independent and therefore irrelevant to the rate of change of flux linking the circuit. There is no need to consider exactly how the current traverses the rectangle (or the disc). Any choice of path connecting the top and bottom of the rectangle (axle- to-brush in the disc) and moving with the rectangle (rotating with the disc) sweeps out the same rate-of-change of flux, and predicts the same EMF. For the disc, this rate-of-change of flux estimation is the same as that done above based upon rotation of the disc past a line joining the brush to the axle.

Some observations

Whether the magnet is "moving" is irrelevant in this analysis, as it does not appear in Faraday's law. In fact, rotating the magnet does not alter the *B*-field. Likewise, rotation of the magnet *and* the disc is the same as rotating the disc and keeping the magnet stationary. The crucial relative motion is that of the disk and the return path, not of the disk and the magnet.

This becomes clearer if a modified Faraday disk is used in which the return path is not a wire but another disk. That is, mount two conducting disks just next to each other on the same axle and let them have sliding electrical contact at the center and at the circumference. The current will be proportional to the relative rotation of the two disks and independent of any rotation of the magnet.

Configuration without a return path

A Faraday disk can also be operated with neither a galvanometer nor a return path. When the disk spins, the electrons collect along the rim and leave a deficit near the axis (or the other way around). It is possible in principle to measure the distribution of charge, for example, through the electromotive force generated between the rim and the axle (though not necessarily easy). This charge separation will be proportional to the magnetic field and the rotational velocity of the disk. The magnetic field will be independent of any rotation of the magnet. In this configuration, the polarisation is determined by the absolute rotation of the disk, that is, the rotation relative to an inertial frame. The relative rotation of the disk and the magnet plays no role.

Inapplicability of Faraday's law

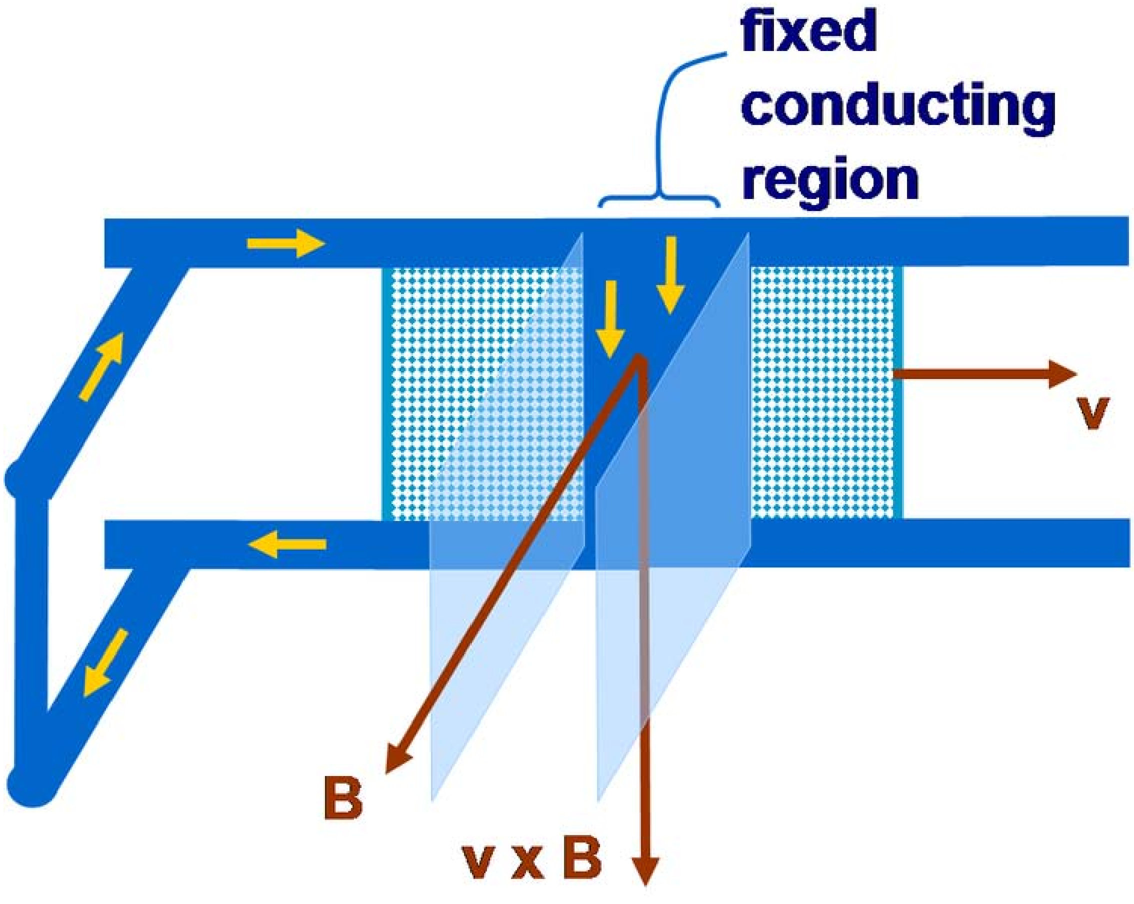


Figure 4: An example, based on one of Feynman's examples, where Faraday's law does not work. A rectangle of photoconductive material slides along a pair of wires. At a fixed location a strong light and a strong magnetic field create a narrow immovable strip of conducting material subject to a Lorentz force.

Figure 4 shows a translating rectangle of material with a narrow conducting strip subject to a magnetic field. This strip of material is rendered conducting at a fixed location by, for example, a strong light beam at this location. The magnetic field also is confined to the same strip. The Lorentz force drives a current from the top rail to the bottom rail through this strip, and the circuit is completed through leads attached to the top and bottom conducting rails. In this example, the circuit does not move, and the magnetic flux through the circuit is not changing, so Faraday's law suggests no current flows. However, the Lorentz force law suggests a current does flow. This example is based upon one devised by Richard Feynman to illustrate the inapplicability of Faraday's law of induction to certain situations (that is, the version of Faraday's law of induction which relates EMF to magnetic flux, which he terms the "flux rule"). Referring to his example, Feynman said:

> The "flux rule" does not work in this case. It must be applied to circuits in which the *material* of the circuit remains the same. When the material of the circuit is changing, we must return to the basic laws. The *correct* physics is always given by the two basic laws
>
> $$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$
> $$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \ .$$
>
> – *Richard P Feynman The Feynman Lectures on Physics*

Accordingly, he explains the phenomenon using the Lorentz force law, as described above. The point is that the flux law applies only to some situations, albeit some very practical ones.

There is no paradox or difficulty if one invokes the special theory of relativity. It tells us that the observer sitting on the area being subjected to a high intensity laser beam to make it conductive is moving through the magnetic induction field, **B** with a velocity **V**. He sees an electric field in his frame of reference of **E** = **V**×**B** whilst the observer resting on the stationary conductors see **E** = 0. Thus there is an EMF of VB times the path length of the so-called stationary conductor.

An additional rule

Now that it has been proven that the magnetic field rotates with the magnet as discussed in the observation section, what is really causing the paradox? Before addressing this question, we will discuss when Faraday's Law is valid and when it breaks down as in the disk experiment. In the case when the disk alone spins there is no change in flux through the circuit, however, there is an electromotive force induced contrary to Faraday's law. We can also show an example when there is a change in flux, but no induced voltage. Figure 5 shows the setup used in Tilley's experiment. It is circuit with two loops or meshes. There is a galvanometer connected in the righthand loop, a magnet in the center of the lefthand loop, a switch in the lefthand loop, and a switch between the loops. We start with the switch on the left open and that on the right closed. When the switch on the left is closed and the switch on the right is open there is no change in the field of the magnet, but there is a change in the area of the galvanometer circuit. This means that there is a change in flux. However the galvanometer did not move meaning there was no induced voltage, and Faraday's law does not work in this case. According to A. G. Kelly this suggests that an induced voltage in Faraday's experiment is due to the "cutting" of the circuit by the flux lines, and not by "flux linking" or the actual change in flux. This follows from the Tilley experiment because there is no movement of the lines of force across the circuit and therefore no current induced althouogh there is a change in flux through the circuit. Nussbaum suggests that for Faraday's law to be valid work must be done in producing the change in flux.

To understand this idea, we will step through the argument given by Nussbaum. We start by calculating the force between two current carrying wires. The force on wire 1 due to wire 2 is given by:

$$\mathbf{F}_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l_1} \times (d\mathbf{l_2} \times \hat{\mathbf{r}}_{21})}{r_{21}^2}$$

The magnetic field from the second wire is given by:

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi} I_2 \oint_{C_2} \frac{(d\mathbf{l_2} \times \hat{\mathbf{r}}_{21})}{r_{21}^2}$$

So we can rewrite the force on wire 1 as:

$$\mathbf{F}_{21} = I_1 \oint_{C_1} d\mathbf{l_1} \times \mathbf{B}_2$$

Now consider a segment $d\mathbf{l}$of a conductor displaced $d\mathbf{r}$in a constant magnetic field. The work done is found from:

$$d\mathbf{W} = d\mathbf{F} \cdot d\mathbf{r}$$

If we plug in what we previously found for $d\mathbf{F}$we get:

$$d\mathbf{W} = (I d\mathbf{l} \times \mathbf{B}) \cdot d\mathbf{r}$$

The area covered by the displacement of the conductor is:

$$d\mathbf{S} = d\mathbf{r} \times d\mathbf{l}$$

Therefore:

$$d\mathbf{W} = I\mathbf{B} \cdot d\mathbf{s} = I d\Phi$$

The differential work can also be given in terms of charge $d\mathbf{q}$and potential difference V:

$$d\mathbf{W} = V d\mathbf{q} = V I d\mathbf{t}$$

By setting the two equations for differential work equal to each other we arrive at Faraday's Law.

$$d\mathbf{\Phi} = V d\mathbf{t}$$

Furthermore, we now see that this is only true if $d\mathbf{W}$ is nonvanishing. Meaning, Faraday's Law is only valid if work is performed in bringing about the change in flux.

Chapter- 5

Maxwell's Equations

Maxwell's equations are a set of partial differential equations that, together with the Lorentz force law, form the foundation of classical electrodynamics, classical optics, and electric circuits. These in turn underlie modern electrical and communications technologies.

Maxwell's equations have two major variants. The "microscopic" set of Maxwell's equations uses total charge and total current including the difficult-to-calculate atomic level charges and currents in materials. The "macroscopic" set of Maxwell's equations defines two new auxiliary fields that can sidestep having to know these 'atomic' sized charges and currents.

Maxwell's equations are named after the Scottish physicist and mathematician James Clerk Maxwell, since in an early form they are all found in a four-part paper, "On Physical Lines of Force," which he published between 1861 and 1862. The mathematical form of the Lorentz force law also appeared in this paper.

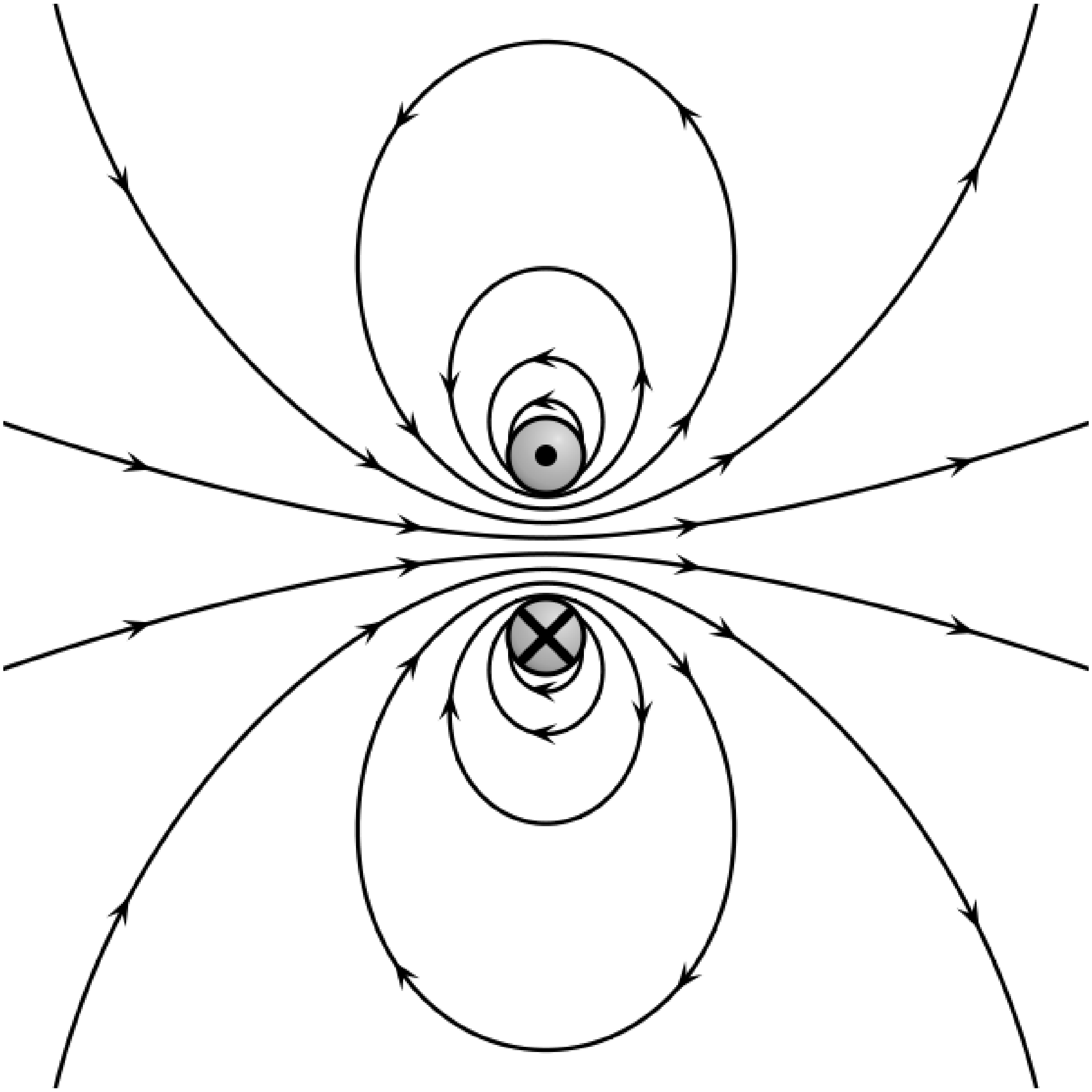
It is often useful to write Maxwell's equations in other forms; these representations are still formally termed "Maxwell's equations". A relativistic formulation in terms of covariant field tensors is used in special relativity, while, in quantum mechanics, a version based on the electric and magnetic potentials is preferred.

Conceptual description

Conceptually, Maxwell's equations describe how electric charges and electric currents act as sources for the electric and magnetic fields. Further, it describes how a time varying electric field generates a time varying magnetic field and vice versa. Of the four equations, two of them, Gauss's law and Gauss's law for magnetism, describe how the fields emanate from charges. (For the magnetic field there is no magnetic charge and therefore magnetic fields lines neither begin nor end anywhere.) The other two equations describe how the fields 'circulate' around their respective sources; the magnetic field 'circulates' around electric currents and time varying electric field in Ampère's law with Maxwell's correction, while the electric field 'circulates' around time varying magnetic fields in Faraday's law.

Gauss's law

Gauss's law describes the relationship between an electric field and the generating electric charges: The electric field points away from positive charges and towards negative charges. In the field line description, electric field lines begin only at positive electric charges and end only at negative electric charges. 'Counting' the number of field lines in a closed surface, therefore, yields the total charge enclosed by that surface. More technically, it relates the electric flux through any hypothetical closed "Gaussian surface" to the electric charge within the surface.

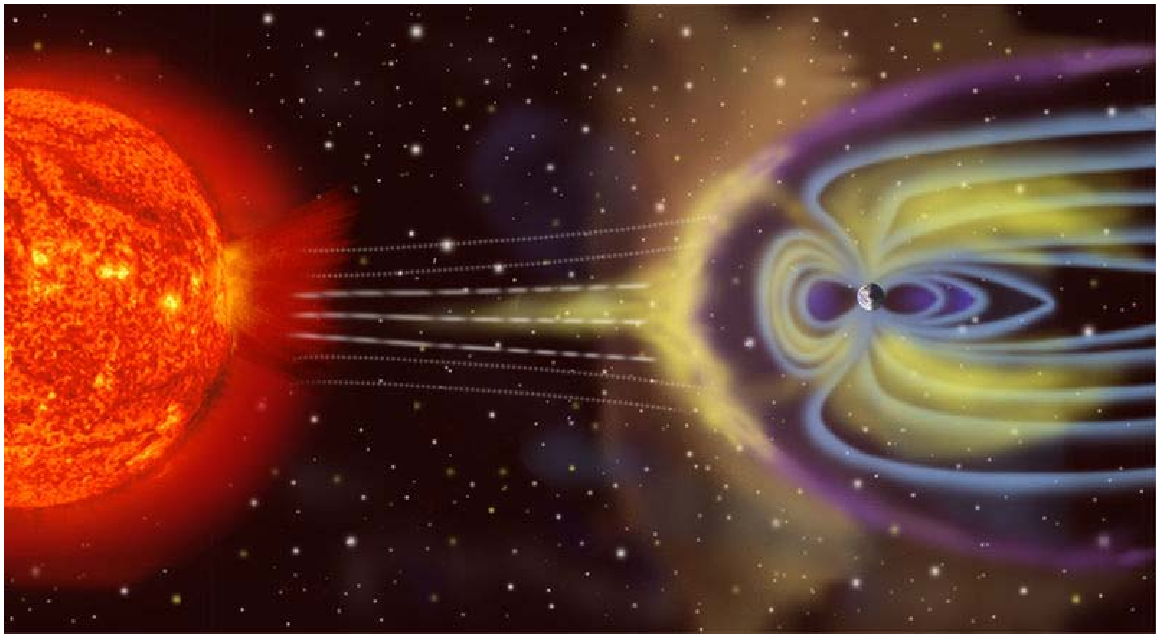


Gauss's law for magnetism: magnetic field lines never begin nor end but form loops or extend to infinity as shown here with the magnetic field due to a ring of current.

Gauss's law for magnetism

Gauss's law for magnetism states that there are no "magnetic charges" (also called magnetic monopoles), analogous to electric charges. Instead, the magnetic field due to materials is generated by a configuration called a dipole. Magnetic dipoles are best represented as loops of current but resemble positive and negative 'magnetic charges', inseparably bound together, having no net 'magnetic charge'. In terms of field lines, this equation states that magnetic field lines neither begin nor end but make loops or extend to infinity and back. In other words, any magnetic field line that enters a given volume must somewhere exit that volume. Equivalent technical statements are that the total magnetic flux through any Gaussian surface is zero, or that the magnetic field is a solenoidal vector field.

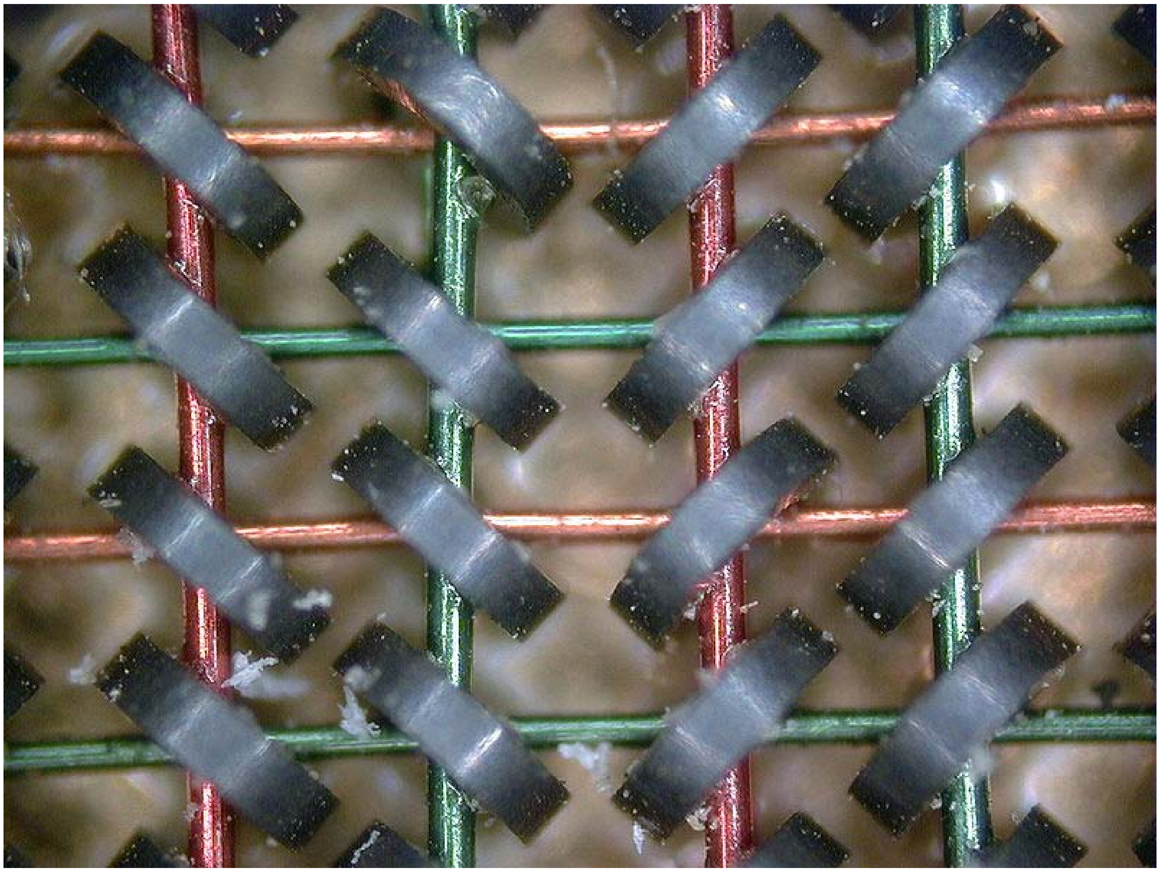
Faraday's law



In a geomagnetic storm, a surge in the flux of charged particles temporarily alters Earth's magnetic field, which induces electric fields in Earth's atmosphere, thus causing surges in our electrical power grids.

Faraday's law describes how a time varying magnetic field creates ("induces") an electric field. This aspect of electromagnetic induction is the operating principle behind many electric generators: for example a rotating bar magnet creates a changing magnetic field, which in turn generates an electric field in a nearby wire. (Note: there are two closely related equations which are called Faraday's law. The form used in Maxwell's equations is always valid but more restrictive than that originally formulated by Michael Faraday.)

Ampère's law with Maxwell's correction



An Wang's magnetic core memory (1954) is an application of Ampere's law. Each core stores one bit of data.

Ampère's law with Maxwell's correction states that magnetic fields can be generated in two ways: by electrical current (this was the original "Ampère's law") and by changing electric fields (this was "Maxwell's correction").

Maxwell's correction to Ampère's law is particularly important: It means that a changing magnetic field creates an electric field, *and* a changing electric field creates a magnetic field. Therefore, these equations allow self-sustaining "electromagnetic waves" to travel through empty space.

The speed calculated for electromagnetic waves, which could be predicted from experiments on charges and currents, exactly matches the speed of light; indeed, light *is* one form of electromagnetic radiation (as are X-rays, radio waves, and others). Maxwell understood the connection between electromagnetic waves and light in 1861, thereby unifying the previously-separate fields of electromagnetism and optics.

Units and summary of equations

Maxwell's equations vary with the unit system used. Though the general form remains the same, various definitions get changed and different constants appear at different places. The equations in this section are given in SI units. Other units commonly used are Gaussian units (based on the cgs system), Lorentz-Heaviside units (used mainly in particle physics) and Planck units (used in theoretical physics).

In the equations given below, symbols in **bold** represent vector quantities, and symbols in *italics* represent scalar quantities. The definitions of terms used in the two tables of equations are given in another table immediately following.

Table of 'microscopic' equations

Formulation in terms of *total* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$

Table of 'macroscopic' equations

Formulation in terms of *free* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_f$	$\oiint_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q_f(V)$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$

Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t}$

Table of terms used in Maxwell's equations

The following table provides the meaning of each symbol and the SI unit of measure:

Definitions and units

Symbol	Meaning (first term is the most common)	SI Unit of Measure
$\mathbf{E}$	electric field also called the electric field intensity	volt per meter or, equivalently, newton per coulomb
$\mathbf{B}$	magnetic field also called the magnetic induction also called the magnetic field density also called the magnetic flux density	tesla, or equivalently, weber per square meter, volt-second per square meter
$\mathbf{D}$	electric displacement field also called the electric induction also called the electric flux density	coulombs per square meter or equivalently, newton per volt-meter
$\mathbf{H}$	magnetizing field also called auxiliary magnetic field also called magnetic field intensity also called magnetic field	ampere per meter
$\nabla\cdot$	the divergence operator	per meter (factor contributed by applying either operator)
$\nabla\times$	the curl operator	
$\frac{\partial}{\partial t}$	partial derivative with respect to time	per second (factor contributed by applying the operator)
$\mathrm{d}\mathbf{A}$	differential vector element of surface area *A*, with infinitesimally small magnitude and direction normal to surface *S*	square meters
$\mathrm{d}\mathbf{l}$	differential vector element of *path*	meters

	length tangential to the path/curve	
ε_0	permittivity of free space, also called the electric constant, a universal constant	farads per meter
μ_0	permeability of free space, also called the magnetic constant, a universal constant	henries per meter, or newtons per ampere squared
ρ_f	free charge density (not including bound charge)	coulombs per cubic meter
ρ	total charge density (including both free and bound charge)	coulombs per cubic meter
$\mathbf{J}_f$	free current density (not including bound current)	amperes per square meter
$\mathbf{J}$	total current density (including both free and bound current)	amperes per square meter
$Q_f(V)$	net free electric charge within the three-dimensional volume *V* (not including bound charge)	coulombs
$Q(V)$	net electric charge within the three-dimensional volume *V* (including both free and bound charge)	coulombs
$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l}$	line integral of the electric field along the boundary ∂S of a surface S (∂S is always a closed curve).	joules per coulomb
$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l}$	line integral of the magnetic field over the closed boundary ∂S of the surface S	tesla-meters
$\oiint_{\partial V} \mathbf{E} \ \cdot \mathrm{d}\mathbf{A}$	the electric flux (surface integral of the electric field) through the (closed) surface ∂V (the boundary of the volume *V*)	joule-meter per coulomb
$\oiint_{\partial V} \mathbf{B} \ \cdot \mathrm{d}\mathbf{A}$	the magnetic flux (surface integral of the magnetic B-field) through the (closed) surface ∂V (the boundary of the volume *V*)	tesla meters-squared or webers
$\iint_S \mathbf{B} \cdot \mathrm{d}\mathbf{A} = \Phi_{B,S}$	magnetic flux through any surface S, not necessarily closed	webers or equivalently, volt-seconds
$\iint_S \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \Phi_{E,S}$	electric flux through any surface S, not necessarily closed	joule-meters per coulomb
	flux of electric displacement field	coulombs

$\iint_S \mathbf{D} \cdot \mathrm{d}\mathbf{A} = \Phi_{D,S}$	through any surface S, not necessarily closed	
$\iint_S \mathbf{J}_f \cdot \mathrm{d}\mathbf{A} = I_{f,s}$	net free electrical current passing through the surface S (not including bound current)	amperes
$\iint_S \mathbf{J} \cdot \mathrm{d}\mathbf{A} = I_S$	net electrical current passing through the surface S (including both free and bound current)	amperes

Proof that the two general formulations are equivalent

The two alternate general formulations of Maxwell's equations given above are mathematically equivalent and related by the following relations:

$$\begin{aligned}
&\rho_b = -\nabla \cdot \mathbf{P}, \\
&\mathbf{J}_b = \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}, \\
&\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \\
&\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}), \\
&\rho = \rho_b + \rho_f, \\
&\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f,
\end{aligned}$$

where **P** and **M** are polarization and magnetization, and ρ_b and $\mathbf{J}_b$ are bound charge and current, respectively. Substituting these equations into the 'macroscopic' Maxwell's equations gives identically the microscopic equations.

Maxwell's 'microscopic' equations

The *microscopic* variant of Maxwell's equation expresses the electric **E** field and the magnetic **B** field in terms of the *total charge* and total *current* present including the charges and currents at the atomic level. It is sometimes called the general form of Maxwell's equations or "Maxwell's equations in a vacuum". Both variants of Maxwell's equations are equally general, though, as they are mathematically equivalent. The microscopic equations are most useful in waveguides for example, when there are no dielectric or magnetic materials nearby.

Formulation in terms of *total* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\oiint_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{A} = \frac{Q(V)}{\varepsilon_0}$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$

Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$	$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I_S + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t}$

With neither charges nor currents

In a region with no charges ($\rho = 0$) and no currents ($\mathbf{J} = 0$), such as in a vacuum, Maxwell's equations reduce to:

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} &= \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

These equations lead directly to **E** and **B** satisfying the wave equation for which the solutions are linear combinations of plane waves traveling at the speed of light,

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}.$$

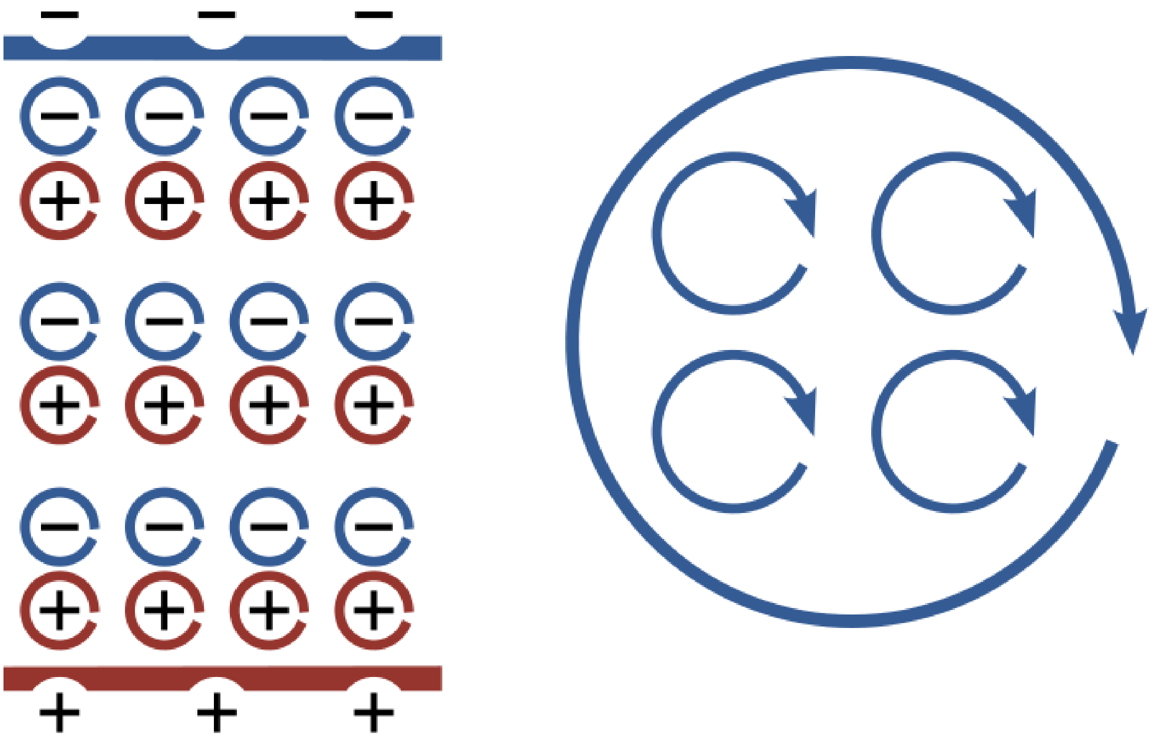
In addition, **E** and **B** are mutually perpendicular to each other and the direction of motion and are in phase with each other. A sinusoidal plane wave is one special solution of these equations.

In fact, Maxwell's equations explain how these waves can physically propagate through space. The changing magnetic field creates a changing electric field through Faraday's law. In turn, that electric field creates a changing magnetic field through Maxwell's correction to Ampère's law. This perpetual cycle allows these waves, now known as electromagnetic radiation, to move through space at velocity *c*.

Maxwell's 'macroscopic' equations

Unlike the 'microscopic' equations, "Maxwell's macroscopic equations", also known as **Maxwell's equations in matter**, factor out the bound charge and current to obtain equations that depend only on the free charges and currents. These equations are more similar to those that Maxwell himself introduced. The cost of this factorization is that additional fields need to be defined: the displacement field **D** which is defined in terms of the electric field **E** and the polarization **P** of the material, and the magnetic-**H** field, which is defined in terms of the magnetic-**B** field and the magnetization **M** of the material.

Bound charge and current



Left: A schematic view of how an assembly of microscopic dipoles produces opposite surface charges as shown at top and bottom. *Right:* How an assembly of microscopic current loops add together to produce a macroscopically circulating current loop. Inside the boundaries, the individual contributions tend to cancel, but at the boundaries no cancellation occurs.

When an electric field is applied to a dielectric material its molecules respond by forming microscopic electric dipoles—their atomic nuclei move a tiny distance in the direction of the field, while their electrons move a tiny distance in the opposite direction. This produces a *macroscopic bound charge* in the material even though all of the charges involved are bound to individual molecules. For example, if every molecule responds the same, similar to that shown in the figure, these tiny movements of charge combine to produce a layer of positive bound charge on one side of the material and a layer of negative charge on the other side. The bound charge is most conveniently described in terms of a polarization, **P**, in the material. If **P** is uniform, a macroscopic separation of charge is produced only at the surfaces where **P** enter and leave the material. For non-uniform **P**, a charge is also produced in the bulk.

Somewhat similarly, in all materials the constituent atoms exhibit magnetic moments that are intrinsically linked to the angular momentum of the atoms' components, most notably their electrons. The connection to angular momentum suggests the picture of an assembly of microscopic current loops. Outside the material, an assembly of such microscopic current loops is not different from a macroscopic current circulating around the material's surface, despite the fact that no individual magnetic moment is traveling a large distance. These *bound currents* can be described using the magnetization **M**.

The very complicated and granular bound charges and bound currents, therefore can be represented on the macroscopic scale in terms of **P** and **M** which average these charges and currents on a sufficiently large scale so as not to see the granularity of individual atoms, but also sufficiently small that they vary with location in the material. As such, the *Maxwell's macroscopic equations* ignores many details on a fine scale that may be unimportant to understanding matters on a grosser scale by calculating fields that are averaged over some suitably sized volume.

Equations

Formulation in terms of *free* charge and current

Name	Differential form	Integral form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_f$	$\oiint_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{A} = Q_f(V)$
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{A} = 0$
Maxwell–Faraday equation (Faraday's law of induction)	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\partial \Phi_{B,S}}{\partial t}$
Ampère's circuital law (with Maxwell's correction)	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_{\partial S} \mathbf{H} \cdot \mathrm{d}\mathbf{l} = I_{f,S} + \frac{\partial \Phi_{D,S}}{\partial t}$

Constitutive relations

In order to apply 'Maxwell's macroscopic equations', it is necessary to specify the relations between displacement field **D** and **E**, and the magnetic H-field **H** and **B**. These equations specify the response of bound charge and current to the applied fields and are called constitutive relations.

Determining the constitutive relationship between the auxiliary fields **D** and **H** and the **E** and **B** fields starts with the definition of the auxiliary fields themselves:

$$\mathbf{D}(\mathbf{r},t) = \epsilon_0\mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$
$$\mathbf{H}(\mathbf{r},t) = \frac{1}{\mu_0}\mathbf{B}(\mathbf{r},t) - \mathbf{M}(\mathbf{r},t),$$

where **P** is the polarization field and **M** is the magnetization field which are defined in terms of microscopic bound charges and bound current respectively. Before getting to how to calculate **M** and **P** it is useful to examine some special cases, though.

Without magnetic or dielectric materials

In the absence of magnetic or dielectric materials, the constitutive relations are simple:

$$\mathbf{D} = \epsilon_0\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu_0$$

where ε_0 and μ_0 are two universal constants, called the permittivity of free space and permeability of free space, respectively. Substituting these back into Maxwell's macroscopic equations lead directly to Maxwell's microscopic equations, except that the currents and charges are replaced with free currents and free charges. This is expected since there are no bound charges nor currents.

Isotropic Linear materials

In an (isotropic) linear material, where **P** is proportional to **E** and **M** is proportional to **B** the constitutive relations are also straightforward. In terms of the polarizaton **P** and the magnetization **M** they are:

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E}, \quad \mathbf{M} = \chi_m\mathbf{H},$$

where χ_e and χ_m are the electric and magnetic susceptibilities of a given material respectively. In terms of **D** and **H** the constitutive relations are:

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu,$$

where ε and μ are constants (which depend on the material), called the permittivity and permeability, respectively, of the material. These are related to the susceptibilities by:

$$\epsilon = \epsilon_0(1+\chi_e) \quad \mu = \mu_0(1+\chi_m)$$

Substituting in the constitutive relations above into Maxwell's equations in linear, dispersionless, time-invariant materials (differential form only) are:

$$\nabla\cdot(\epsilon\mathbf{E}) = \rho_f$$

$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times (\mathbf{B}/\mu) = \mathbf{J}_f + \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

These are formally identical to the *general* formulation in terms of **E** and **B** (given above), except that the permittivity of free space was replaced with the permittivity of the material, the permeability of free space was replaced with the permeability of the material, and only free charges and currents are included (instead of all charges and currents). Unless that material is homogeneous in space, ε and μ cannot be factored out of the derivative expressions on the left sides.

General case

For real-world materials, the constitutive relations are not linear, except approximately. Calculating the constitutive relations from first principles involves determining how **P** and **M** are created from a given **E** and **B**. These relations may be empirical (based directly upon measurements), or theoretical (based upon statistical mechanics, transport theory or other tools of condensed matter physics). The detail employed may be macroscopic or microscopic, depending upon the level necessary to the problem under scrutiny.

In general, though the constitutive relations can usually still be written:

$$\mathbf{D} = \epsilon \mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$$

but ε and μ are not, in general, simple constants, but rather functions. Examples are:

- *Dispersion and absorption* where ε and μ are functions of frequency. (Causality does not permit materials to be nondispersive; see, for example, Kramers–Kronig relations). Neither do the fields need to be in phase which leads to ε and μ being complex. This also leads to absorption.
- Bi-(an)isotropy where **H** and **D** depend on both **B** and **E**:

$$D = \epsilon E + \xi H \quad B = \mu H + \zeta E.$$

- *Nonlinearity* where ε and μ are functions of **E** and **B**.
- *Anisotropy* (such as *birefringence* or *dichroism*) which occurs when ε and μ are second-rank tensors,

$$D_j = \epsilon_{ij} E_i \quad B_j = \mu_{ij} H_i.$$

- Dependence of **P** and **M** on **E** and **B** at other locations and times. This could be due to *spatial inhomogeneity*; for example in a domained structure, heterostructure or a liquid crystal, or most commonly in the situation where there are simply multiple materials occupying different regions of space). Or it could be due to a time varying medium or due to hysteresis. In such cases **P** and **M** can be calculated as:

$$\mathbf{P}(\mathbf{r},t) = \epsilon_0 \int \mathrm{d}^3\mathbf{r}'\mathrm{d}t'\ \hat{\chi}_{\mathrm{elec}}(\mathbf{r},\mathbf{r}',t,t';\mathbf{E})\,\mathbf{E}(\mathbf{r}',t')$$
$$\mathbf{M}(\mathbf{r},t) = \frac{1}{\mu_0} \int \mathrm{d}^3\mathbf{r}'\mathrm{d}t'\ \hat{\chi}_{\mathrm{magn}}(\mathbf{r},\mathbf{r}',t,t';\mathbf{B})\,\mathbf{B}(\mathbf{r}',t'),$$

in which the permittivity and permeability functions are replaced by integrals over the more general electric and magnetic susceptibilities.

In practice, some materials properties have a negligible impact in particular circumstances, permitting neglect of small effects. For example: optical nonlinearities can be neglected for low field strengths; material dispersion is unimportant when frequency is limited to a narrow bandwidth; material absorption can be neglected for wavelengths for which a material is transparent; and metals with finite conductivity often are approximated at microwave or longer wavelengths as perfect metals with infinite conductivity (forming hard barriers with zero skin depth of field penetration).

It may be noted that man-made materials can be designed to have customized permittivity and permeability, such as metamaterials and photonic crystals.

Calculation of constitutive relations

In general, the constitutive equations are theoretically determined by calculating how a molecule responds to the local fields through the Lorentz force. Other forces may need to be modeled as well such as lattice vibrations in crystals or bond forces. Including all of the forces leads to changes in the molecule which are used to calculate **P** and **M** as a function of the local fields.

The local fields differ from the applied fields due to the fields produced by the polarization and magnetization of nearby material; an effect which also needs to be modeled. Further, real materials are not continuous media; the local fields of real materials vary wildly on the atomic scale. The fields need to be averaged over a suitable volume to form a continuum approximation.

These continuum approximations often require some type of quantum mechanical analysis such as quantum field theory as applied to condensed matter physics. See, for example, density functional theory, Green–Kubo relations and Green's function. Various approximate transport equations have evolved, for example, the Boltzmann equation or the Fokker–Planck equation or the Navier–Stokes equations. Some examples where these equations are applied are magnetohydrodynamics, fluid dynamics, electrohydrodynamics, superconductivity, plasma modeling. An entire physical apparatus for dealing with these matters has developed. A different set of *homogenization methods* (evolving from a tradition in treating materials such as conglomerates and laminates) are based upon approximation of an inhomogeneous material by a homogeneous *effective medium* (valid for excitations with wavelengths much larger than the scale of the inhomogeneity).

The theoretical modeling of the continuum-approximation properties of many real materials often rely upon measurement as well, for example, ellipsometry measurements.

History

Relation between electricity, magnetism, and the speed of light

The relation between electricity, magnetism, and the speed of light can be summarized by the modern equation:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \ .$$

The left-hand side is the speed of light, and the right-hand side is a quantity related to the equations governing electricity and magnetism. Although the right-hand side has units of velocity, it can be inferred from measurements of electric and magnetic forces, which involve no physical velocities. Therefore, establishing this relationship provided convincing evidence that light is an electromagnetic phenomenon.

The discovery of this relationship started in 1855, when Wilhelm Eduard Weber and Rudolf Kohlrausch determined that there was a quantity related to electricity and magnetism, "the ratio of the absolute electromagnetic unit of charge to the absolute electrostatic unit of charge" (in modern language, the value $1/\sqrt{\mu_0\varepsilon_0}$), and determined that it should have units of velocity. They then measured this ratio by an experiment which involved charging and discharging a Leyden jar and measuring the magnetic force from the discharge current, and found a value 3.107×10^8 m/s, remarkably close to the speed of light, which had recently been measured at 3.14×10^8 m/s by Hippolyte Fizeau in 1848 and at 2.98×10^8 m/s by Léon Foucault in 1850. However, Weber and Kohlrausch did not make the connection to the speed of light. Towards the end of 1861 while working on part III of his paper *On Physical Lines of Force*, Maxwell travelled from Scotland to London and looked up Weber and Kohlrausch's results. He converted them into a format which was compatible with his own writings, and in doing so he established the connection to the speed of light and concluded that light is a form of electromagnetic radiation.

The term *Maxwell's equations*

The four modern Maxwell's equations can be found individually throughout his 1861 paper, derived theoretically using a molecular vortex model of Michael Faraday's "lines of force" and in conjunction with the experimental result of Weber and Kohlrausch. But it wasn't until 1884 that Oliver Heaviside, concurrently with similar work by Willard Gibbs and Heinrich Hertz, grouped the four together into a distinct set. This group of four equations was known variously as the Hertz-Heaviside equations and the Maxwell-Hertz equations, and are sometimes still known as the Maxwell–Heaviside equations.

Maxwell's contribution to science in producing these equations lies in the correction he made to Ampère's circuital law in his 1861 paper *On Physical Lines of Force*. He added the displacement current term to Ampère's circuital law and this enabled him to derive the electromagnetic wave equation in his later 1865 paper *A Dynamical Theory of the Electromagnetic Field* and demonstrate the fact that light is an electromagnetic wave. This fact was then later confirmed experimentally by Heinrich Hertz in 1887. The physicist Richard Feynman predicted that, "The American Civil War will pale into provincial insignificance in comparison with this important scientific event of the same decade."

The concept of fields was introduced by, among others, Faraday. Albert Einstein wrote:

The precise formulation of the time-space laws was the work of Maxwell. Imagine his feelings when the differential equations he had formulated proved to him that electromagnetic fields spread in the form of polarised waves, and at the speed of light! To few men in the world has such an experience been vouchsafed ... it took physicists some decades to grasp the full significance of Maxwell's discovery, so bold was the leap that his genius forced upon the conceptions of his fellow-workers
—(*Science*, May 24, 1940)

Heaviside worked to eliminate the potentials (electric potential and magnetic potential) that Maxwell had used as the central concepts in his equations; this effort was somewhat controversial, though it was understood by 1884 that the potentials must propagate at the speed of light like the fields, unlike the concept of instantaneous action-at-a-distance like the then conception of gravitational potential. Modern analysis of, for example, radio antennas, makes full use of Maxwell's vector and scalar potentials to separate the variables, a common technique used in formulating the solutions of differential equations. However the potentials can be introduced by algebraic manipulation of the four fundamental equations.

On Physical Lines of Force

The four modern day Maxwell's equations appeared throughout Maxwell's 1861 paper *On Physical Lines of Force*:

1. Equation (56) in Maxwell's 1861 paper is $\nabla \cdot \mathbf{B} = 0$.
2. Equation (112) is Ampère's circuital law with Maxwell's displacement current added. It is the addition of displacement current that is the most significant aspect of Maxwell's work in electromagnetism, as it enabled him to later derive the electromagnetic wave equation in his 1865 paper A Dynamical Theory of the Electromagnetic Field, and hence show that light is an electromagnetic wave. It is

therefore this aspect of Maxwell's work which gives the equations their full significance. (Interestingly, Kirchhoff derived the telegrapher's equations in 1857 without using displacement current. But he did use Poisson's equation and the equation of continuity which are the mathematical ingredients of the displacement current. Nevertheless, Kirchhoff believed his equations to be applicable only inside an electric wire and so he is not credited with having discovered that light is an electromagnetic wave).

3. Equation (115) is Gauss's law.
4. Equation (54) is an equation that Oliver Heaviside referred to as 'Faraday's law'. This equation caters for the time varying aspect of electromagnetic induction, but not for the motionally induced aspect, whereas Faraday's original flux law caters for both aspects. Maxwell deals with the motionally dependent aspect of electromagnetic induction, **v** × **B**, at equation (77). Equation (77) which is the same as equation (D) in the original eight Maxwell's equations listed below, corresponds to all intents and purposes to the modern day force law **F** = *q*(**E** + **v** × **B**) which sits adjacent to Maxwell's equations and bears the name Lorentz force, even though Maxwell derived it when Lorentz was still a young boy.

The difference between the **B** and the **H** vectors can be traced back to Maxwell's 1855 paper entitled *On Faraday's Lines of Force* which was read to the Cambridge Philosophical Society. The paper presented a simplified model of Faraday's work, and how the two phenomena were related. He reduced all of the current knowledge into a linked set of differential equations.

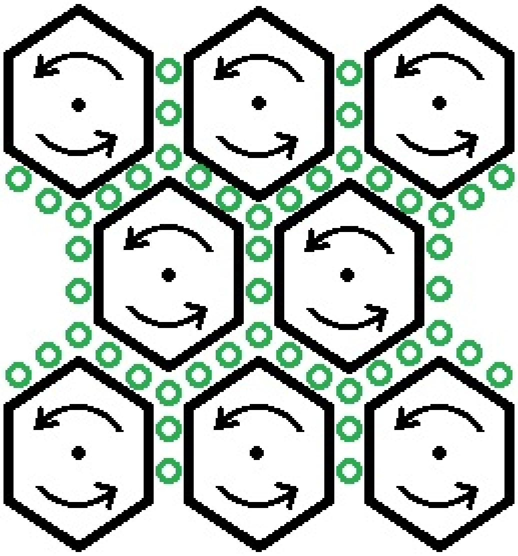


Figure of Maxwell's molecular vortex model. For a uniform magnetic field, the field lines point outward from the display screen, as can be observed from the black dots in the middle of the hexagons. The vortex of each hexagonal molecule rotates counter-clockwise. The small green circles are clockwise rotating particles sandwiching between the molecular vortices.

It is later clarified in his concept of a sea of molecular vortices that appears in his 1861 paper *On Physical Lines of Force*. Within that context, **H** represented pure vorticity (spin), whereas **B** was a weighted vorticity that was weighted for the density of the vortex sea. Maxwell considered magnetic permeability μ to be a measure of the density of the vortex sea. Hence the relationship,

1. **Magnetic induction current** causes a magnetic current density

$$\mathbf{B} = \mu\mathbf{H}$$

was essentially a rotational analogy to the linear electric current relationship,

1. **Electric convection current**

$$\mathbf{J} = \rho\mathbf{v}$$

where ρ is electric charge density. **B** was seen as a kind of magnetic current of vortices aligned in their axial planes, with **H** being the circumferential velocity of the vortices. With μ representing vortex density, it follows that the product of μ with vorticity **H** leads to the magnetic field denoted as **B**.

The electric current equation can be viewed as a convective current of electric charge that involves linear motion. By analogy, the magnetic equation is an inductive current involving spin. There is no linear motion in the inductive current along the direction of the **B** vector. The magnetic inductive current represents lines of force. In particular, it represents lines of inverse square law force.

The extension of the above considerations confirms that where **B** is to **H**, and where **J** is to ρ, then it necessarily follows from Gauss's law and from the equation of continuity of charge that **E** is to **D**. i.e. **B** parallels with **E**, whereas **H** parallels with **D**.

A Dynamical Theory of the Electromagnetic Field

In 1864 Maxwell published *A Dynamical Theory of the Electromagnetic Field* in which he showed that light was an electromagnetic phenomenon. Confusion over the term "Maxwell's equations" is exacerbated because it is also sometimes used for a set of eight equations that appeared in Part III of Maxwell's 1864 paper A Dynamical Theory of the Electromagnetic Field, entitled "General Equations of the Electromagnetic Field," a confusion compounded by the writing of six of those eight equations as three separate equations (one for each of the Cartesian axes), resulting in twenty equations and twenty unknowns. (As noted above, this terminology is not common: Modern references to the term "Maxwell's equations" refer to the Heaviside restatements.)

The eight original Maxwell's equations can be written in modern vector notation as follows:

(A) The law of total currents

$$\mathbf{J}_{tot} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

(B) The equation of magnetic force

$$\mu \mathbf{H} = \nabla \times \mathbf{A}$$

(C) Ampère's circuital law

$$\nabla \times \mathbf{H} = \mathbf{J}_{\mathrm{tot}}$$

(D) Electromotive force created by convection, induction, and by static electricity. (This is in effect the Lorentz force)

$$\mathbf{E} = \mu \mathbf{v} \times \mathbf{H} - \frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$$

(E) The electric elasticity equation

$$\mathbf{E} = \frac{1}{\epsilon} \mathbf{D}$$

(F) Ohm's law

$$\mathbf{E} = \frac{1}{\sigma} \mathbf{J}$$

(G) Gauss's law

$$\nabla \cdot \mathbf{D} = \rho$$

(H) Equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

or

$$\nabla \cdot \mathbf{J}_{\mathrm{tot}} = 0$$

Notation

H is the magnetizing field, which Maxwell called the *magnetic intensity*.
J is the current density (with**J**$_{\mathrm{tot}}$ being the total current including displacement current).
D is the displacement field (called the *electric displacement* by Maxwell).
ρ is the free charge density (called the *quantity of free electricity* by Maxwell).
A is the magnetic potential (called the *angular impulse* by Maxwell).
E is called the *electromotive force* by Maxwell. The term electromotive force is nowadays used for voltage, but it is clear from the context that Maxwell's meaning corresponded more to the modern term electric field.
φ is the electric potential (which Maxwell also called *electric potential*).
σ is the electrical conductivity (Maxwell called the inverse of conductivity the *specific resistance*, what is now called the resistivity).

It is interesting to note the $\mu \mathbf{v} \times \mathbf{H}$ term that appears in equation D. Equation D is therefore effectively the Lorentz force, similarly to equation (77) of his 1861 paper.

When Maxwell derives the electromagnetic wave equation in his 1865 paper, he uses equation D to cater for electromagnetic induction rather than Faraday's law of induction which is used in modern textbooks. (Faraday's law itself does not appear among his equations.) However, Maxwell drops the $\mu\mathbf{v} \times \mathbf{H}$ term from equation D when he is deriving the electromagnetic wave equation, as he considers the situation only from the rest frame.

A Treatise on Electricity and Magnetism

In *A Treatise on Electricity and Magnetism*, an 1873 treatise on electromagnetism written by James Clerk Maxwell, eleven general equations of the electromagnetic field are listed and these include the eight that are listed in the 1865 paper.

Maxwell's equations and relativity

Maxwell's original equations are based on the idea that light travels through a sea of molecular vortices known as the 'luminiferous aether', and that the speed of light has to be respective to the reference frame of this aether. Measurements designed to measure the speed of the Earth through the aether conflicted, though.

A more theoretical approach was suggested by Hendrik Lorentz along with George FitzGerald and Joseph Larmor. Both Larmor (1897) and Lorentz (1899, 1904) derived the Lorentz transformation (so named by Henri Poincaré) as one under which Maxwell's equations were invariant. Poincaré (1900) analyzed the coordination of moving clocks by exchanging light signals. He also established mathematically the group property of the Lorentz transformation (Poincaré 1905).

Einstein dismissed the aether as unnecessary and concluded that Maxwell's equations predict the existence of a fixed speed of light, independent of the speed of the observer, and as such he used Maxwell's equations as the starting point for his special theory of relativity. In doing so, he established the Lorentz transformation as being valid for all matter and not just Maxwell's equations. Maxwell's equations played a key role in Einstein's famous paper on special relativity; for example, in the opening paragraph of the paper, he motivated his theory by noting that a description of a conductor moving with respect to a magnet must generate a consistent set of fields irrespective of whether the force is calculated in the rest frame of the magnet or that of the conductor.

General relativity has also had a close relationship with Maxwell's equations. For example, Theodor Kaluza and Oskar Klein showed in the 1920s that Maxwell's equations can be derived by extending general relativity into five dimensions. This strategy of using higher dimensions to unify different forces remains an active area of research in particle physics.

Modified to include magnetic monopoles

Maxwell's equations provide for an electric charge, but posit no magnetic charge. Magnetic charge has never been seen and may not exist. Nevertheless, Maxwell's equations including magnetic charge (and magnetic current) is of some theoretical interest.

For one reason, Maxwell's equations can be made fully symmetric under interchange of electric and magnetic field by allowing for the possibility of magnetic charges with magnetic charge density ρ_m and currents with magnetic current density $\mathbf{J}_m$. The extended Maxwell's equations (in cgs-Gaussian units) are:

Name	Without magnetic monopoles	With magnetic monopoles (hypothetical)
Gauss's law:	$\nabla \cdot \mathbf{E} = 4\pi\rho_e$	$\nabla \cdot \mathbf{E} = 4\pi\rho_e$
Gauss's law for magnetism:	$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 4\pi\rho_m$
Maxwell–Faraday equation (Faraday's law of induction):	$-\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$	$-\nabla \times \mathbf{E} = \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_m$
Ampère's law (with Maxwell's extension):	$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_e$	$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}_e$

If magnetic charges do not exist, or if they exist but not in the region studied, then the new variables are zero, and the symmetric equations reduce to the conventional equations of electromagnetism such as $\nabla \cdot \mathbf{B} = 0$. Further if every particle has the same ratio of electric to magnetic charge then an E and a B field can be defined that obeys the normal Maxwell's equation (having no magnetic charges or currents) with its own charge and current densities.

Boundary conditions using Maxwell's equations

Like all sets of differential equations, Maxwell's equations cannot be uniquely solved without a suitable set of boundary conditions and initial conditions.

For example, consider a region with no charges and no currents. One particular solution that satisfies all of Maxwell's equations in that region is that both **E** and **B** = 0 everywhere in the region. This solution is obviously false if there is a charge just outside of the region. In this particular example, all of the electric and magnetic fields in the interior are due to the charges outside of the volume. Different charges outside of the volume produce different fields on the surface of that volume and therefore have a different boundary conditions. In general, knowing the appropriate boundary conditions for a given region along with the currents and charges in that region allows one to solve for all the fields everywhere within that region. An example of this type is a an electromagnetic scattering problem, where an electromagnetic wave originating outside the scattering region is scattered by a target, and the scattered electromagnetic wave is analyzed for the information it contains about the target by virtue of the interaction with the target during scattering.

In some cases, like waveguides or cavity resonators, the solution region is largely isolated from the universe, for example, by metallic walls, and boundary conditions at the walls define the fields with influence of the outside world confined to the input/output ends of the structure. In other cases, the universe at large sometimes is approximated by an artificial absorbing boundary, or, for example for radiating antennas or communication satellites, these boundary conditions can take the form of *asymptotic limits* imposed upon the solution. In addition, for example in an optical fiber or thin-film optics, the solution region often is broken up into subregions with their own simplified properties, and the solutions in each subregion must be joined to each other across the subregion interfaces using boundary conditions. A particular example of this use of boundary conditions is the replacement of a material with a volume polarization by a charged surface layer, or of a material with a volume magnetization by a surface current, as described in the section Bound charge and current.

Following are some links of a general nature concerning boundary value problems: Examples of boundary value problems, Sturm–Liouville theory, Dirichlet boundary condition, Neumann boundary condition, mixed boundary condition, Cauchy boundary condition, Sommerfeld radiation condition. Needless to say, one must choose the boundary conditions appropriate to the problem being solved.

Gaussian units

Gaussian units is a popular electromagnetism variant of the centimetre gram second system of units (cgs). In gaussian units, Maxwell's equations are:

$$\nabla \cdot \mathbf{D} = 4\pi\rho_{\mathrm{f}}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\mathrm{f}}$$

where *c* is the speed of light in a vacuum. The microscopic equations are:

$$\nabla \cdot \mathbf{E} = 4\pi\rho_{\mathrm{tot}}$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\mathrm{tot}}.$$

The relation between electric displacement field, electric field and polarization density is:

$$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}.$$

And likewise the relation between magnetic induction, magnetic field and total magnetization is:

$$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}.$$

In the linear approximation, the electric susceptibility and magnetic susceptibility are defined so that:

$$\mathbf{P} = \chi_{\mathrm{e}}\mathbf{E}, \quad \mathbf{M} = \chi_{\mathrm{m}}\mathbf{H}.$$

(Note: although the susceptibilities are dimensionless numbers in both cgs and SI, they differ in value by a factor of 4π.) The permittivity and permeability are:

$$\epsilon = 1 + 4\pi\chi_{\mathrm{e}}, \quad \mu = 1 + 4\pi\chi_{\mathrm{m}},$$

so that

$$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}.$$

In vacuum, $\varepsilon = \mu = 1$, therefore **D** = **E**, and **B** = **H**.

The force exerted upon a charged particle by the electric field and magnetic field is given by the Lorentz force equation:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right),$$

where q is the charge on the particle and **v** is the particle velocity. This is slightly different from the SI-unit expression above. For example, the magnetic field **B** has the same units as the electric field **E**.

Alternative formulations of Maxwell's equations

Special relativity motivated a compact mathematical formulation of Maxwell's equations, in terms of covariant tensors. Quantum mechanics also motivated other formulations.

For example, consider a conductor moving in the field of a magnet. In the frame of the magnet, that conductor experiences a *magnetic* force. But in the frame of a conductor moving relative to the magnet, the conductor experiences a force due to an *electric* field. The following formulation shows how Maxwell's equations take the same form in any inertial coordinate system.

Covariant formulation of Maxwell's equations

In special relativity, in order to more clearly express the fact that Maxwell's ('microscopic') equations take the same form in any inertial coordinate system, Maxwell's equations are written in terms of four-vectors and tensors in the "manifestly covariant" form. The purely spatial components of the following are in SI units.

One ingredient in this formulation is the electromagnetic tensor, a rank-2 covariant antisymmetric tensor combining the electric and magnetic fields:

$$F_{\alpha\beta} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & B_z & -B_y \\ \frac{E_y}{c} & -B_z & 0 & B_x \\ \frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

and the result of raising its indices

$$F^{\mu\nu} \stackrel{\text{def}}{=} \eta^{\mu\alpha} \, F_{\alpha\beta} \, \eta^{\beta\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ \frac{-E_x}{c} & 0 & B_z & -B_y \\ \frac{-E_y}{c} & -B_z & 0 & B_x \\ \frac{-E_z}{c} & B_y & -B_x & 0 \end{pmatrix}.$$

The other ingredient is the four-current:

$$J^\alpha = (c\rho, \vec{J})$$

where ρ is the charge density and **J** is the current density.

With these ingredients, Maxwell's equations can be written:

$$\mu_0 \, J^\beta \; = \; \frac{\partial F^{\beta\alpha}}{\partial x^\alpha} \stackrel{\text{def}}{=} \partial_\alpha F^{\beta\alpha} \stackrel{\text{def}}{=} {F^{\beta\alpha}}_{,\alpha}$$

and

$$0 = \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma} \stackrel{\text{def}}{=} F_{\alpha\beta,\gamma} + F_{\gamma\alpha,\beta} + F_{\beta\gamma,\alpha}.$$

The first tensor equation is an expression of the two inhomogeneous Maxwell's equations, Gauss's law and Ampere's law with Maxwell's correction. The second equation is an expression of the two homogeneous equations, Faraday's law of induction and Gauss's law for magnetism. The second equation is equivalent to

$$0 = \epsilon^{\delta\alpha\beta\gamma} F_{\beta\gamma,\alpha}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the contravariant version of the Levi-Civita symbol, and

$$\frac{\partial}{\partial x^\alpha} \stackrel{\text{def}}{=} \partial_\alpha \stackrel{\text{def}}{=} {}_{,\alpha} \stackrel{\text{def}}{=} \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

is the 4-gradient. In the tensor equations above, repeated indices are summed over according to Einstein summation convention. We have displayed the results in several common notations. Upper and lower components of a vector, v^α and v^α respectively, are interchanged with the fundamental tensor g, e.g., $g = \eta = \mathrm{diag}(-1, +1, +1, +1)$.

Potential formulation

In advanced classical mechanics and in quantum mechanics (where it is necessary) it is sometimes useful to express Maxwell's equations in a 'potential formulation' involving the electric potential (also called scalar potential), φ, and the magnetic potential, **A**, (also called vector potential). These are defined such that:

$$\begin{aligned}\mathbf{E} &= -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla \times \mathbf{A}.\end{aligned}$$

With these definitions, the two homogeneous Maxwell's equations (Faraday's Law and Gauss's law for magnetism) are automatically satisfied and the other two (inhomogeneous) equations give the following equations (for "Maxwell's microscopic equations"):

$$\begin{aligned}&\nabla^2\varphi + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\varepsilon_0} \\ &\left(\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) - \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t}\right) = -\mu_0 \mathbf{J}.\end{aligned}$$

These equations, taken together, are as powerful and complete as Maxwell's equations. Moreover, if we work only with the potentials and ignore the fields, the problem has been reduced somewhat, as the electric and magnetic fields each have three components which need to be solved for (six components altogether), while the electric and magnetic potentials have only four components altogether.

Many different choices of **A** and φ are consistent with a given **E** and **B**, making these choices physically equivalent – a flexibility known as gauge freedom. Suitable choice of **A** and φ can simplify these equations, or can adapt them to suit a particular situation.

Four-potential

In the Lorenz gauge, the two equations that represent the potentials can be reduced to one manifestly Lorentz invariant equation, using four-vectors: the four-current defined by

$$j^\mu = (\rho c, \mathbf{j})$$

formed from the current density **j** and charge density ρ, and the electromagnetic four-potential defined by

$$A^\mu = (\varphi, \mathbf{A}c)$$

formed from the vector potential **A** and the scalar potential φ. The resulting single equation, due to Arnold Sommerfeld, a generalization of an equation due to Bernhard Riemann and known as the Riemann–Sommerfeld equation or the covariant form of the Maxwell equations, is:

$$\Box A^\mu = \mu_0 j^\mu,$$

where $\Box = \partial^2 = \partial_\alpha \partial^\alpha$ is the d'Alembertian operator, or four-Laplacian, $\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)$, sometimes written $\Box^2$, or $\Box \cdot \Box$, where $\Box$ is the four-gradient.

Differential geometric formulations

In free space, where $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ are constant everywhere, Maxwell's equations simplify considerably once the language of differential geometry and differential forms is used. In what follows, cgs-Gaussian units, not SI units are used. The electric and magnetic fields are now jointly described by a 2-form **F** in a 4-dimensional spacetime manifold. Maxwell's equations then reduce to the Bianchi identity

$$\mathrm{d}\mathbf{F} = 0$$

where d denotes the exterior derivative — a natural coordinate and metric independent differential operator acting on forms — and the source equation

$$\mathrm{d} * \mathbf{F} = \mathbf{J}$$

where the (dual) Hodge star operator * is a linear transformation from the space of 2-forms to the space of (4-2)-forms defined by the metric in Minkowski space (in four dimensions even by any metric conformal to this metric), and the fields are in natural units where $1/4\pi\varepsilon_0 = 1$. Here, the 3-form **J** is called the *electric current form* or *current 3-form* satisfying the continuity equation

$$\mathrm{d}\mathbf{J} = 0.$$

The current 3-form can be integrated over a 3-dimensional space-time region. The physical interpretation of this integral is the charge in that region if it is spacelike, or the amount of charge that flows through a surface in a certain amount of time if that region is a spacelike surface cross a timelike interval. As the exterior derivative is defined on any manifold, the differential form version of the Bianchi identity makes sense for any 4-dimensional manifold, whereas the source equation is defined if the manifold is oriented and has a Lorentz metric. In particular the differential form version of the Maxwell equations are a convenient and intuitive formulation of the Maxwell equations in general relativity.

In a linear, macroscopic theory, the influence of matter on the electromagnetic field is described through more general linear transformation in the space of 2-forms. We call

$$C : \Lambda^2 \ni \mathbf{F} \mapsto \mathbf{G} \in \Lambda^{(4-2)}$$

the constitutive transformation. The role of this transformation is comparable to the Hodge duality transformation. The Maxwell equations in the presence of matter then become:

$$\begin{aligned}\mathrm{d}\mathbf{F} &= 0\\ \mathrm{d}\mathbf{G} &= \mathbf{J}\end{aligned}$$

where the current 3-form **J** still satisfies the continuity equation d**J** = 0.

When the fields are expressed as linear combinations (of exterior products) of basis forms θ^p,

$$\mathbf{F} = \frac{1}{2}F_{pq}\theta^p \wedge \theta^q.$$

the constitutive relation takes the form

$$G_{pq} = C_{pq}^{mn}F_{mn}$$

where the field coefficient functions are antisymmetric in the indices and the constitutive coefficients are antisymmetric in the corresponding pairs. In particular, the Hodge duality transformation leading to the vacuum equations discussed above are obtained by taking

$$C_{pq}^{mn} = \frac{1}{2}g^{ma}g^{nb}\epsilon_{abpq}\sqrt{-g}$$

which up to scaling is the only invariant tensor of this type that can be defined with the metric.

In this formulation, electromagnetism generalises immediately to any 4-dimensional oriented manifold or with small adaptations any manifold, requiring not even a metric. Thus the expression of Maxwell's equations in terms of differential forms leads to a further notational and conceptual simplification. Whereas Maxwell's Equations could be written as two tensor equations instead of eight scalar equations, from which the propagation of electromagnetic disturbances and the continuity equation could be derived with a little effort, using differential forms leads to an even simpler derivation of these results.

Conceptual insight from this formulation

On the conceptual side, from the point of view of physics, this shows that the second and third Maxwell equations should be grouped together, be called the homogeneous ones, and be seen as geometric *identities* expressing nothing else than: the *field* **F** derives from a more "fundamental" *potential* **A**. While the first and last one should be seen as the dynamical *equations of motion*, obtained via the Lagrangian principle of least action, from the "interaction term" **A J** (introduced through gauge covariant derivatives), coupling the field to matter.

Often, the time derivative in the third law motivates calling this equation "dynamical", which is somewhat misleading; in the sense of the preceding analysis, this is rather an artifact of breaking relativistic covariance by choosing a preferred time direction. To have physical degrees of freedom propagated by these field equations, one must include a kinetic term **F *F** for **A**; and take into account the non-physical degrees of freedom which can be removed by gauge transformation **A** → **A'** = **A** − dα.

Geometric Algebra (GA) formulation

In geometric algebra, Maxwell's equations are reduced to a single equation,

$$\left(\frac{1}{c}\partial_t + \boldsymbol{\nabla}\right) F = \mu_0 c J,$$

where *F* and *J* are multivectors

$$F = \mathbf{E} + Ic\mathbf{B}$$

and

$$J = c\rho + \mathbf{J}.$$

with the unit pseudoscalar $I^2 = -1$

The GA spatial gradient operator ∇ acts on a vector field, such that

$$\boldsymbol{\nabla}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} + I\boldsymbol{\nabla}\times\mathbf{F},$$

In spacetime algebra using the same geometric product the equation is simply

$$\nabla F = \mu_0 c J,$$

the spacetime derivative of the electromagnetic field is its source. Here the (non-bold) spacetime gradient

$$\nabla = \gamma^\mu \partial_\mu$$

is a four vector, as is the current density

$$J = \gamma_\mu J^\mu = \gamma_0 c\rho + J^k \gamma_k = (c\rho + \mathbf{J})\gamma_0.$$

Classical electrodynamics as the curvature of a line bundle

An elegant and intuitive way to formulate Maxwell's equations is to use complex line bundles or principal bundles with fibre U(1). The connection ∇ on the line bundle has a curvature $\mathbf{F} = \nabla^2$ which is a two-form that automatically satisfies $\mathrm{d}\mathbf{F} = 0$ and can be interpreted as a field-strength. If the line bundle is trivial with flat reference connection *d* we can write $\nabla = \mathrm{d} + \mathbf{A}$ and **F** = d**A** with **A** the 1-form composed of the electric potential and the magnetic vector potential.

In quantum mechanics, the connection itself is used to define the dynamics of the system. This formulation allows a natural description of the Aharonov-Bohm effect. In this experiment, a static magnetic field runs through a long magnetic wire (e.g., an iron wire magnetized longitudinally). Outside of this wire the magnetic induction is zero, in contrast to the vector potential, which essentially depends on the magnetic flux through the cross-section of the wire and does not vanish outside. Since there is no electric field either, the Maxwell tensor **F = 0** throughout the space-time region outside the tube, during the experiment. This means by definition that the connection ∇ is flat there.

However, as mentioned, the connection depends on the magnetic field through the tube since the holonomy along a non-contractible curve encircling the tube is the magnetic flux through the tube in the proper units. This can be detected quantum-mechanically with a double-slit electron diffraction experiment on an electron wave traveling around the tube. The holonomy corresponds to an extra phase shift, which leads to a shift in the diffraction pattern.

Curved spacetime

Traditional formulation

Matter and energy generate curvature of spacetime. This is the subject of general relativity. Curvature of spacetime affects electrodynamics. An electromagnetic field having energy and momentum also generates curvature in spacetime. Maxwell's equations in curved spacetime can be obtained by replacing the derivatives in the equations in flat spacetime with covariant derivatives. (Whether this is the appropriate generalization requires separate investigation.) The sourced and source-free equations become (cgs-Gaussian units):

$$\frac{4\pi}{c}j^{\beta} = \partial_{\alpha}F^{\alpha\beta} + {\Gamma^{\alpha}}_{\mu\alpha}F^{\mu\beta} + {\Gamma^{\beta}}_{\mu\alpha}F^{\alpha\mu} \overset{\text{def}}{=} D_{\alpha}F^{\alpha\beta} \overset{\text{def}}{=} {F^{\alpha\beta}}_{;\alpha}$$

and

$$0 = \partial_{\gamma}F_{\alpha\beta} + \partial_{\beta}F_{\gamma\alpha} + \partial_{\alpha}F_{\beta\gamma} = D_{\gamma}F_{\alpha\beta} + D_{\beta}F_{\gamma\alpha} + D_{\alpha}F_{\beta\gamma}.$$

Here,

$${\Gamma^{\alpha}}_{\mu\beta}$$

is a Christoffel symbol that characterizes the curvature of spacetime and D_{γ} is the covariant derivative.

Formulation in terms of differential forms

The formulation of the Maxwell equations in terms of differential forms can be used without change in general relativity. The equivalence of the more traditional general relativistic formulation using the covariant derivative with the differential form formulation can be seen as follows. Choose local coordinates x^{α} which gives a basis of 1-forms dx^{α} in every point of the open set where the coordinates are defined. Using this basis and cgs-Gaussian units we define

- The antisymmetric infinitesimal field tensor $F_{\alpha\beta}$, corresponding to the field 2-form **F**

$$\mathbf{F} := \frac{1}{2}F_{\alpha\beta}\,\mathrm{d}\,x^{\alpha} \wedge \mathrm{d}\,x^{\beta}.$$

- The current-vector infinitesimal 3-form **J**

$$\mathbf{J} := \frac{4\pi}{c}j^{\alpha}\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}\mathrm{d}\,x^{\beta} \wedge \mathrm{d}\,x^{\gamma} \wedge \mathrm{d}\,x^{\delta}.$$

Here g is as usual the determinant of the metric tensor $g_{\alpha\beta}$. A small computation that uses the symmetry of the Christoffel symbols (i.e., the torsion-freeness of the Levi Civita connection) and the covariant constantness of the Hodge star operator then shows that in this coordinate neighborhood we have:

- the Bianchi identity

$$\mathrm{d}\mathbf{F} = 2(\partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} + \partial_\alpha F_{\beta\gamma})\mathrm{d}\,x^\alpha \wedge \mathrm{d}\,x^\beta \wedge \mathrm{d}\,x^\gamma = 0,$$

- the source equation

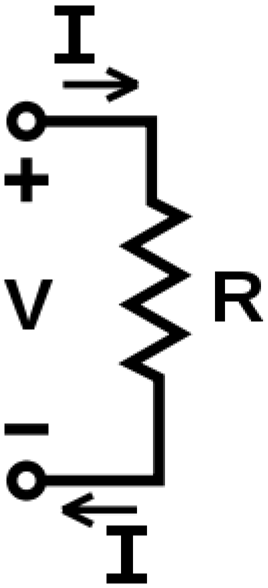
$$\mathrm{d} * \mathbf{F} = F^{\alpha\beta}{}_{;\alpha}\sqrt{-g}\,\epsilon_{\beta\gamma\delta\eta}\mathrm{d}\,x^\gamma \wedge \mathrm{d}\,x^\delta \wedge \mathrm{d}\,x^\eta = \mathbf{J},$$

- the continuity equation

$$\mathrm{d}\mathbf{J} = \frac{4\pi}{c} j^\alpha{}_{;\alpha}\sqrt{-g}\,\epsilon_{\alpha\beta\gamma\delta}\mathrm{d}\,x^\alpha \wedge \mathrm{d}\,x^\beta \wedge \mathrm{d}\,x^\gamma \wedge \mathrm{d}\,x^\delta = 0.$$

Chapter- 6

Ohm's Law



V, I, and R, the parameters of Ohm's law.

Ohm's law states that the current through a conductor between two points is directly proportional to the potential difference or voltage across the two points, and inversely proportional to the resistance between them.

The mathematical equation that describes this relationship is:

$$I = \frac{V}{R}$$

where I is the current through the conductor in units of amperes, V is the potential difference measured *across* the conductor in units of volts, and R is the resistance of the conductor in units of ohms. More specifically, Ohm's law states that the R in this relation is constant, independent of the current.

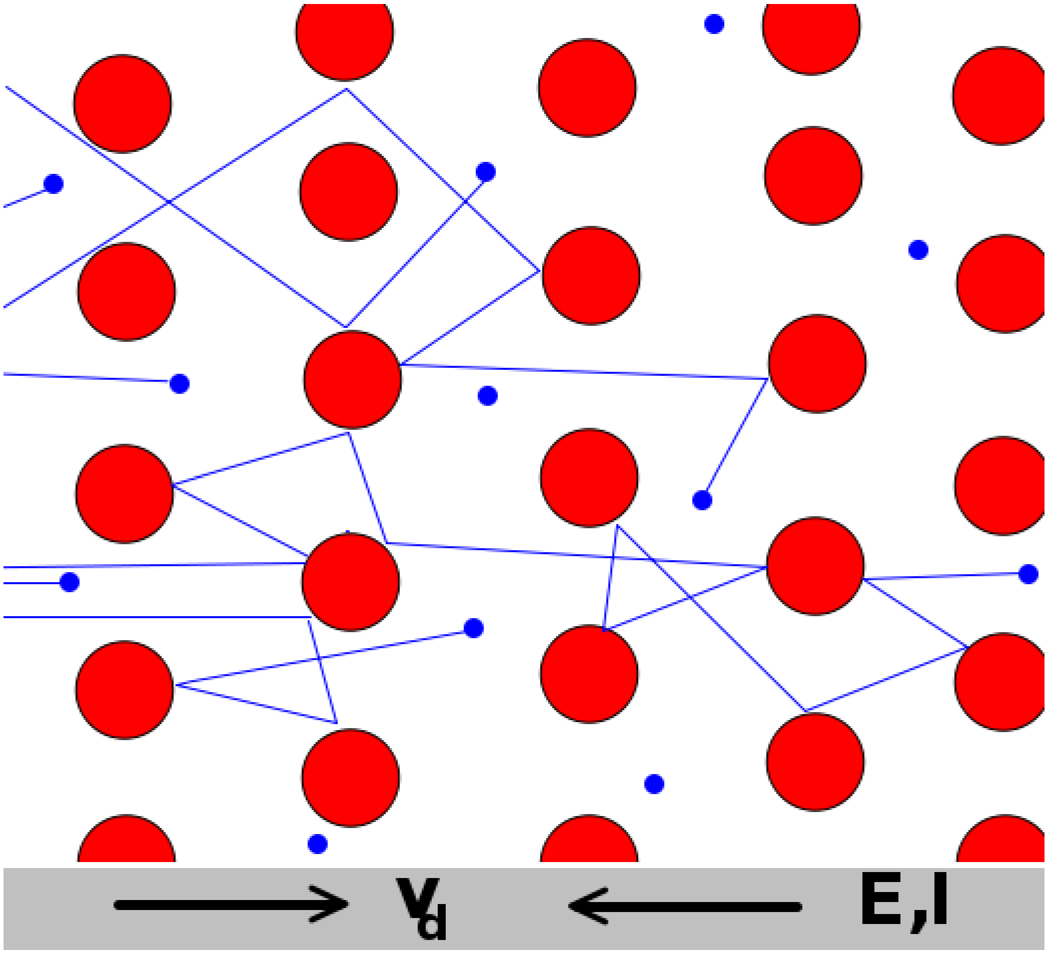
The law was named after the German physicist Georg Ohm, who, in a treatise published in 1827, described measurements of applied voltage and current through simple electrical circuits containing various lengths of wire. He presented a slightly more complex equation than the one above to explain his experimental results. The above equation is the modern form of Ohm's law.

In physics, the term *Ohm's law* is also used to refer to various generalizations of the law originally formulated by Ohm. The simplest example of this is:

$$\boldsymbol{J} = \sigma \boldsymbol{E},$$

where $\boldsymbol{J}$ is the current density at a given location in a resistive material, $\boldsymbol{E}$ is the electric field at that location, and σ is a material dependent parameter called the conductivity. This reformulation of Ohm's law is due to Gustav Kirchhoff.

Microscopic origins of Ohm's law



Drude Model electrons (shown here in blue) constantly bounce between heavier, stationary crystal ions (shown in red).

The dependence of the current density on the applied electric field is essentially quantum mechanical in nature. A qualitative description leading to Ohm's law can be based upon classical mechanics using the Drude model developed by Paul Drude in 1900.

The Drude model treats electrons (or other charge carriers) like pinballs bouncing between the ions that make up the structure of the material. Electrons will be accelerated in the opposite direction to the electric field by the average electric field at their location. With each collision, though, the electron is deflected in a random direction with a velocity that is much larger than the velocity gained by the electric field. The net result is that electrons take a tortuous path due to the collisions, but generally drift in a direction opposing the electric field.

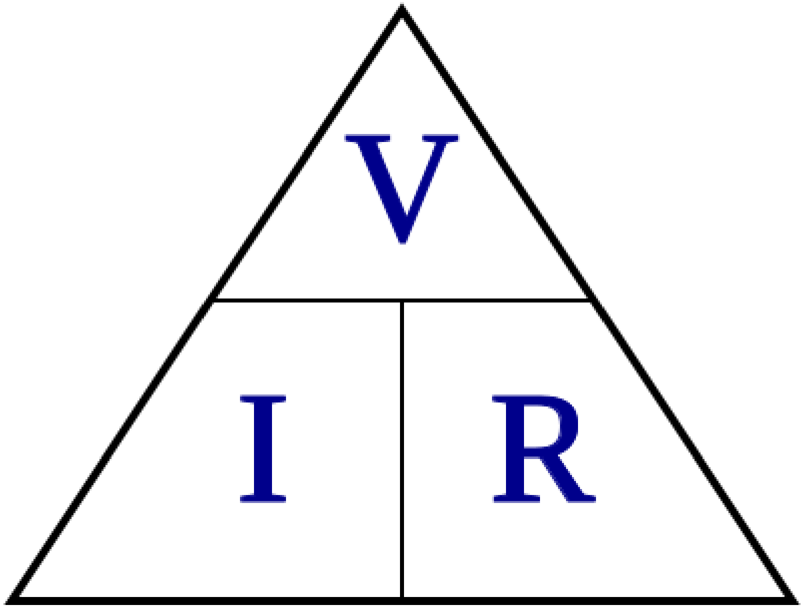
The drift velocity then determines the electric current density and its relationship to $\boldsymbol{E}$ and is independent of the collisions. Drude calculated the average drift velocity from $\boldsymbol{p} = -e\boldsymbol{E}\tau$ where $\boldsymbol{p}$ is the average momentum, $-e$ is the charge of the electron and τ is the average time between the collisions. Since both the momentum and the current density are proportional to the drift velocity, the current density becomes proportional to the applied electric field; this leads to Ohm's law.

Hydraulic analogy

A hydraulic analogy is sometimes used to describe Ohm's Law. Water pressure, measured by pascals (or PSI), is the analog of voltage because establishing a water pressure difference between two points along a (horizontal) pipe causes water to flow. Water flow rate, as in liters per second, is the analog of current, as in coulombs per second. Finally, flow restrictors — such as apertures placed in pipes between points where the water pressure is measured — are the analog of resistors. We say that the rate of water flow through an aperture restrictor is proportional to the difference in water pressure across the restrictor. Similarly, the rate of flow of electrical charge, that is, the electric current, through an electrical resistor is proportional to the difference in voltage measured across the resistor.

Flow and pressure variables can be calculated in fluid flow network with the use of the hydraulic ohm analogy. The method can be applied to both steady and transient flow situations. In the linear laminar flow region, Poiseuille's law describes the hydraulic resistance of a pipe, but in the turbulent flow region the pressure–flow relations become nonlinear.

Circuit analysis



Ohm's law triangle

In circuit analysis, three equivalent expressions of Ohm's law are used interchangeably:

$$I = \frac{V}{R} \quad \text{or} \quad V = IR \quad \text{or} \quad R = \frac{V}{I}.$$

Each equation is quoted by some sources as the defining relationship of Ohm's law, or all three are quoted, or derived from a proportional form, or even just the two that do not correspond to Ohm's original statement may sometimes be given.

The interchangeability of the equation may be represented by a triangle, where V (voltage) is placed on the top section, the I (current) is placed to the left section, and the R (resistance) is placed to the right. The line that divides the left and right sections indicate multiplication, and the divider between the top and bottom sections indicates division (hence the division bar).

Resistive circuits

Resistors are circuit elements that impede the passage of electric charge in agreement with Ohm's law, and are designed to have a specific resistance value *R*. In a schematic diagram the resistor is shown as a zig-zag symbol. An element (resistor or conductor) that behaves according to Ohm's law over some operating range is referred to as an *ohmic device* (or an *ohmic resistor*) because Ohm's law and a single value for the resistance suffice to describe the behavior of the device over that range.

Ohm's law holds for circuits containing only resistive elements (no capacitances or inductances) for all forms of driving voltage or current, regardless of whether the driving voltage or current is constant (DC) or time-varying such as AC. At any instant of time Ohm's law is valid for such circuits.

Reactive circuits with time-varying signals

When reactive elements such as capacitors, inductors, or transmission lines are involved in a circuit to which AC or time-varying voltage or current is applied, the relationship between voltage and current becomes the solution to a differential equation, so Ohm's law (as defined above) does not directly apply since that form contains only resistances having value R, not complex impedances which may contain capacitance ("C") or inductance ("L").

Equations for time-invariant AC circuits take the same form as Ohm's law, however, the variables are generalized to complex numbers and the current and voltage waveforms are complex exponentials.

In this approach, a voltage or current waveform takes the form Ae^{st}, where t is time, s is a complex parameter, and A is a complex scalar. In any linear time-invariant system, all of the currents and voltages can be expressed with the same s parameter as the input to the system, allowing the time-varying complex exponential term to be canceled out and the system described algebraically in terms of the complex scalars in the current and voltage waveforms.

The complex generalization of resistance is impedance, usually denoted Z; it can be shown that for an inductor,

$$Z = sL$$

and for a capacitor,

$$Z = \frac{1}{sC}.$$

We can now write,

$$\boldsymbol{V} = \boldsymbol{I} \cdot \boldsymbol{Z}$$

where $\boldsymbol{V}$ and $\boldsymbol{I}$ are the complex scalars in the voltage and current respectively and $\boldsymbol{Z}$ is the complex impedance.

This form of Ohm's law, with Z taking the place of R, generalizes the simpler form. When Z is complex, only the real part is responsible for dissipating heat.

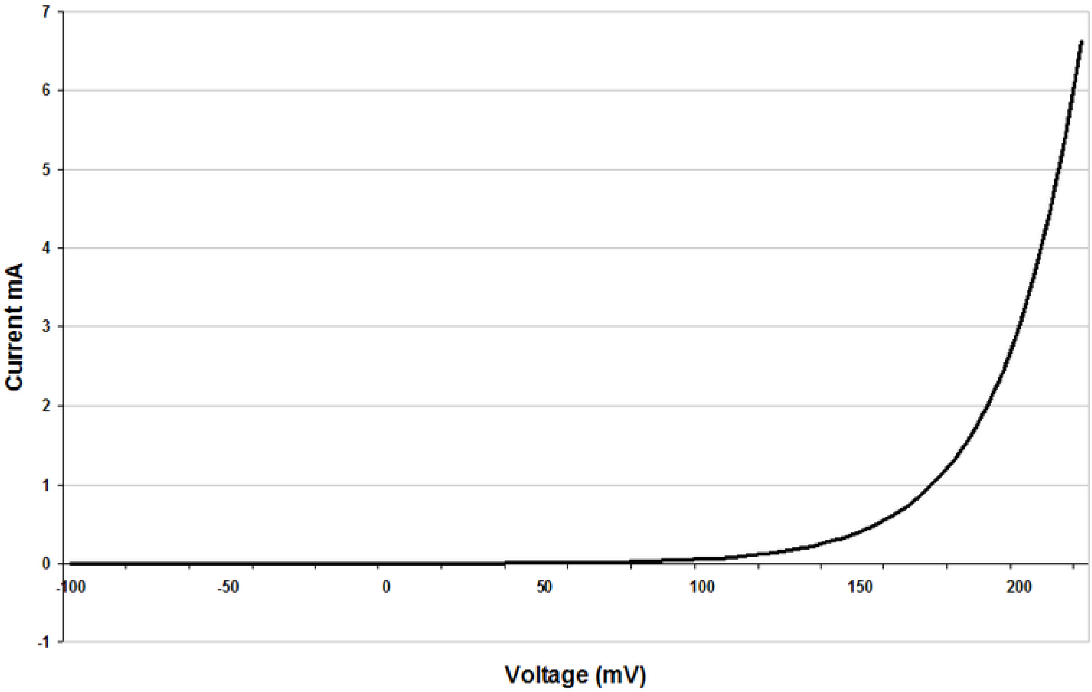
In the general AC circuit, *Z* varies strongly with the frequency parameter *s*, and so also will the relationship between voltage and current.

For the common case of a steady sinusoid, the *s* parameter is taken to be $j\omega$, corresponding to a complex sinusoid $Ae^{j\omega t}$. The real parts of such complex current and voltage waveforms describe the actual sinusoidal currents and voltages in a circuit, which can be in different phases due to the different complex scalars.

Linear approximations

Ohm's law is one of the basic equations used in the analysis of electrical circuits. It applies to both metal conductors and circuit components (resistors) specifically made for this behaviour. Both are ubiquitous in electrical engineering. Materials and components that obey Ohm's law are described as "ohmic" which means they produce the same value for resistance (R = V/I) regardless of the value of V or I which is applied and whether the applied voltage or current is DC (direct current) of either positive or negative polarity or AC (alternating current).

In a true ohmic device, the same value of resistance will be calculated from R = V/I regardless of the value of the applied voltage V. That is, the ratio of V/I is constant, and when current is plotted as a function of voltage the curve is *linear* (a straight line). If voltage is forced to some value V, then that voltage V divided by measured current I will equal R. Or if the current is forced to some value I, then the measured voltage V divided by that current I is also R. Since the plot of I versus V is a straight line, then it is also true that for any set of two different voltages V_1 and V_2 applied across a given device of resistance R, producing currents $I_1 = V_1/R$ and $I_2 = V_2/R$, that the ratio $(V_1-V_2)/(I_1-I_2)$ is also a constant equal to R. The operator "delta" (Δ) is used to represent a difference in a quantity, so we can write $\Delta V = V_1-V_2$ and $\Delta I = I_1-I_2$. Summarizing, for any truly ohmic device having resistance R, $V/I = \Delta V/\Delta I = R$ for any applied voltage or current or for the difference between any set of applied voltages or currents.



Plot of I–V curve of an ideal p-n junction diode at 1µA reverse leakage current. Failure of the device to follow Ohm's law is clearly shown since the curve is not a straight line.

There are, however, components of electrical circuits which do not obey Ohm's law; that is, their relationship between current and voltage (their I–V curve) is *nonlinear*. An example is the p-n junction diode (curve at right). As seen in the figure, the current does not increase linearly with applied voltage for a diode. One can determine a value of current (I) for a given value of applied voltage (V) from the curve, but not from Ohm's law, since the value of "resistance" is not constant as a function of applied voltage. Further, the current only increases significantly if the applied voltage is positive, not negative. The ratio *V*/*I* for some point along the nonlinear curve is sometimes called the *static*, or *chordal*, or DC, resistance , but as seen in the figure the value of total *V* over total *I* varies depending on the particular point along the nonlinear curve which is chosen. This means the "DC resistance" V/I at some point on the curve is not the same as what would be determined by applying an AC signal having peak amplitude ΔV volts or ΔI amps centered at that same point along the curve and measuring ΔV/ΔI. However, in some diode applications, the AC signal applied to the device is small and it is possible to analyze the circuit in terms of the *dynamic*, *small-signal*, or *incremental* resistance, defined as the one over the slope of the V–I curve at the average value (DC operating point) of the voltage (that is, one over the derivative of current with respect to voltage). For sufficiently small signals, the dynamic resistance allows the Ohm's law small signal resistance to be calculated as approximately one over the slope of a line drawn tangentially to the V-I curve at the DC operating point.

Temperature effects

Ohm's law has sometimes been stated as, "for a conductor in a given state, the electromotive force is proportional to the current produced." That is, that the resistance, the ratio of the applied electromotive force (or voltage) to the current, "does not vary with the current strength ." The qualifier "in a given state" is usually interpreted as meaning "at a constant temperature," since the resistivity of materials is usually temperature dependent. Because the conduction of current is related to Joule heating of the conducting body, according to Joule's first law, the temperature of a conducting body may change when it carries a current. The dependence of resistance on temperature therefore makes resistance depend upon the current in a typical experimental setup, making the law in this form difficult to directly verify. Maxwell and others worked out several methods to test the law experimentally in 1876, controlling for heating effects.

Relation to heat conductions

Ohm's principle predicts the flow of electrical charge (i.e. current) in electrical conductors when subjected to the influence of voltage differences; Jean-Baptiste-Joseph Fourier's principle predicts the flow of heat in heat conductors when subjected to the influence of temperature differences.

The same equation describes both phenomena, the equation's variables taking on different meanings in the two cases. Specifically, solving a heat conduction (Fourier) problem with *temperature* (the driving "force") and *flux of heat* (the rate of flow of the driven "quantity", i.e. heat energy) variables also solves an analogous electrical conduction (Ohm) problem having *electric potential* (the driving "force") and *electric current* (the rate of flow of the driven "quantity", i.e. charge) variables.

The basis of Fourier's work was his clear conception and definition of thermal conductivity. He assumed that, all else being the same, the flux of heat is strictly proportional to the gradient of temperature. Although undoubtedly true for small temperature gradients, strictly proportional behavior will be lost when real materials (e.g. ones having a thermal conductivity that is a function of temperature) are subjected to large temperature gradients.

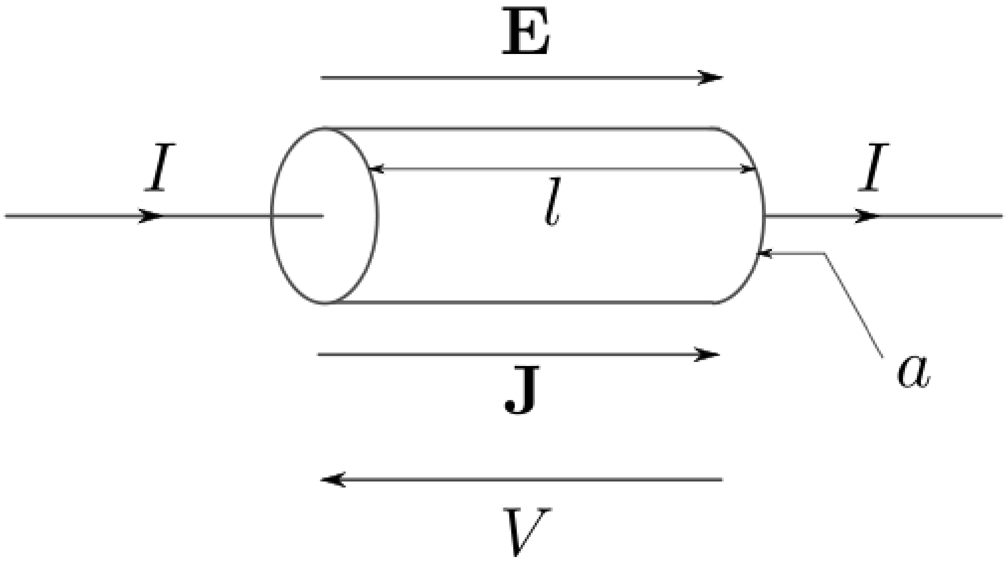
A similar assumption is made in the statement of Ohm's law: other things being alike, the strength of the current at each point is proportional to the gradient of electric potential. The accuracy of the assumption that flow is proportional to the gradient is more readily tested, using modern measurement methods, for the electrical case than for the heat case.

Other versions of Ohm's law

Ohm's law, in the form above, is an extremely useful equation in the field of electrical/electronic engineering because it describes how voltage, current and resistance are interrelated on a "macroscopic" level, that is, commonly, as circuit elements in an electrical circuit. Physicists who study the electrical properties of matter at the microscopic level use a closely related and more general vector equation, sometimes also referred to as Ohm's law, having variables that are closely related to the V, I, and R scalar variables of Ohm's law, but are each functions of position within the conductor. Physicists often use this continuum form of Ohm's Law:

$$\mathbf{E} = \rho \mathbf{J}$$

where "**E**" is the electric field vector with units of volts per meter (analogous to "V" of Ohm's law which has units of volts), "**J**" is the current density vector with units of amperes per unit area (analogous to "I" of Ohm's law which has units of amperes), and "ρ" (Greek "rho") is the resistivity with units of ohm·meters (analogous to "R" of Ohm's law which has units of ohms). The above equation is sometimes written as **J** = σ**E** where "σ" is the conductivity which is the reciprocal of ρ.



Current flowing through a uniform cylindrical conductor (such as a round wire) with a uniform field applied.

The potential difference between two points is defined as:

$$\Delta V = -\int \mathbf{E} \cdot d\mathbf{l}$$

with $d\mathbf{l}$ the element of path along the integration of electric field vector **E**. If the applied **E** field is uniform and oriented along the length of the conductor as shown in the figure, then defining the voltage V in the usual convention of being opposite in direction to the field (see figure), and with the understanding that the voltage V is measured differentially across the length of the conductor allowing us to drop the Δ symbol, the above vector equation reduces to the scalar equation:

$$V = El \quad \text{or} \quad E = \frac{V}{l}.$$

Since the **E** field is uniform in the direction of wire length, for a conductor having uniformly consistent resistivity ρ, the current density **J** will also be uniform in any cross-sectional area and oriented in the direction of wire length, so we may write:

$$J = \frac{I}{a}.$$

Substituting the above 2 results (for E and J respectively) into the continuum form shown at the beginning of this section:

$$\frac{V}{l} = \frac{I}{a}\rho \qquad \text{or} \qquad V = I\rho\frac{l}{a}.$$

The electrical resistance of a uniform conductor is given in terms of resistivity by:

$$R = \rho\frac{l}{a}$$

where l is the length of the conductor in SI units of meters, a is the cross-sectional area (for a round wire $a = \pi r^2$ if r is radius) in units of meters squared, and ρ is the resistivity in units of ohm·meters.

After substitution of R from the above equation into the equation preceding it, the continuum form of Ohm's law for a uniform field (and uniform current density) oriented along the length of the conductor reduces to the more familiar form:

$$V = IR.$$

A perfect crystal lattice, with low enough thermal motion and no deviations from periodic structure, would have no resistivity, but a real metal has crystallographic defects, impurities, multiple isotopes, and thermal motion of the atoms. Electrons scatter from all of these, resulting in resistance to their flow.

The more complex generalized forms of Ohm's law are important to condensed matter physics, which studies the properties of matter and, in particular, its electronic structure. In broad terms, they fall under the topic of constitutive equations and the theory of transport coefficients.

Magnetic effects

The continuum form of the equation is only valid in the reference frame of the conducting material. If the material is moving at velocity **v** relative to a magnetic field **B**, a term must be added as follows:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{J}\rho.$$

History

In January 1781, before Georg Ohm's work, Henry Cavendish experimented with Leyden jars and glass tubes of varying diameter and length filled with salt solution. He measured the current by noting how strong a shock he felt as he completed the circuit with his body. Cavendish wrote that the "velocity" (current) varied directly as the "degree of electrification" (voltage). He did not communicate his results to other scientists at the time, and his results were unknown until Maxwell published them in 1879.

Ohm did his work on resistance in the years 1825 and 1826, and published his results in 1827 as the book *Die galvanische Kette, mathematisch bearbeitet* (The galvanic Circuit investigated mathematically). He drew considerable inspiration from Fourier's work on heat conduction in the theoretical explanation of his work. For experiments, he initially used voltaic piles, but later used a thermocouple as this provided a more stable voltage source in terms of internal resistance and constant potential difference. He used a galvanometer to measure current, and knew that the voltage between the thermocouple terminals was proportional to the junction temperature. He then added test wires of varying length, diameter, and material to complete the circuit. He found that his data could be modeled through the equation

$$x = \frac{a}{b + l},$$

where x was the reading from the galvanometer, l was the length of the test conductor, a depended only on the thermocouple junction temperature, and b was a constant of the entire setup. From this, Ohm determined his law of proportionality and published his results.

Ohm's law was probably the most important of the early quantitative descriptions of the physics of electricity. We consider it almost obvious today. When Ohm first published his work, this was not the case; critics reacted to his treatment of the subject with hostility. They called his work a "web of naked fancies" and the German Minister of Education proclaimed that "a professor who preached such heresies was unworthy to teach science." The prevailing scientific philosophy in Germany at the time asserted that experiments need not be performed to develop an understanding of nature because nature is so well ordered, and that scientific truths may be deduced through reasoning alone.

Also, Ohm's brother Martin, a mathematician, was battling the German educational system. These factors hindered the acceptance of Ohm's work, and his work did not become widely accepted until the 1840s. Fortunately, Ohm received recognition for his contributions to science well before he died.

In the 1850s, Ohm's law was known as such, and was widely considered proved, and alternatives such as "Barlow's law" discredited, in terms of real applications to telegraph system design, as discussed by Samuel F. B. Morse in 1855.

While the old term for electrical conductance, the mho (the inverse of the resistance unit ohm), is still used, a new name, the siemens, was adopted in 1971, honoring Ernst Werner von Siemens. The siemens is preferred in formal papers.

In the 1920s, it was discovered that the current through an ideal resistor actually has statistical fluctuations, which depend on temperature, even when voltage and resistance are exactly constant; this fluctuation, now known as Johnson–Nyquist noise, is due to the discrete nature of charge. This thermal effect implies that measurements of current and voltage that are taken over sufficiently short periods of time will yield ratios of V/I that fluctuate from the value of R implied by the time average or ensemble average of the measured current; Ohm's law remains correct for the average current, in the case of ordinary resistive materials.

Ohm's work long preceded Maxwell's equations and any understanding of frequency-dependent effects in AC circuits. Modern developments in electromagnetic theory and circuit theory do not contradict Ohm's law when they are evaluated within the appropriate limits.